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EXISTENCE AND UNIQUENESS OF SQUARE-MEAN PSEUDO ALMOST AUTOMORPHIC SOLUTION FOR FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY G-BROWNIAN MOTION

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Abstract. In this paper, we will discuss existence of solution of square-mean pseudo almost automorphic solution for fractional stochastic evolution equations driven by G-Brownian motion which is given as

$${}_0^cD_\rho^\alpha \Psi_\rho = \mathcal{A}(\rho)\Psi_\rho d\rho + \Phi(\rho, \Psi_\rho)d\rho + \Upsilon(\rho, \Psi_\rho)d\left\langle \aleph \right\rangle_\rho + \chi(\rho, \Psi_\rho)d\aleph_\rho, \rho \in R.$$

Furthermore, we also prove that solution of the above equation is unique by using Lipschitz conditions and Cauchy-Schwartz inequality. Moreover, examples demonstrate the validity of the obtained main result and we obtain the solution for an equation, and proved that this solution is unique.

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1. Introduction

Some results on the problem of existence and uniqueness of the square-mean pseudo almost automorphic mild solutions for fractional differential equation and stochastic evolution equations driven by G-Brownian motion have been discussed by some authors which can be found in [2, 5, 6, 1, 4]. The main purpose of this article is to discuss the existence and uniqueness of the square-mean pseudo almost automorphic mild solutions for the following stochastic evolution equation of fractional order driven by G-Brownianmotion (G-SEEF, in short)

$${}_0^cD_\rho^\alpha \Psi_\rho = \mathcal{A}(\rho)\Psi_\rho d\rho + \Phi(\rho, \Psi_\rho)d\rho + \Upsilon(\rho, \Psi_\rho)d\left\langle \aleph \right\rangle_\rho + \chi(\rho, \Psi_\rho)d\aleph_\rho, \rho \in R. \quad (1)$$

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where $\mathcal{A}(\rho) : D(\mathcal{A}(\rho)) \subset \mathcal{L}^2_\Upsilon(\Omega) \to \mathcal{L}^2_\Upsilon(\Omega)$ is densely closed linear operator, and satisfies Acquistapace-Terrani conditions $\aleph_\rho$ is a one dimensional G-Brownian motion, the functions Φ, Υ and $\chi : \mathcal{L}^2_\Upsilon(\Omega) \to \mathcal{L}^2_\Upsilon(\Omega)$ are jointly continuous.

Definition 1.1. Riemann–Liouville definition [7, 9, 8, 3]: For $\alpha \in [n-1, n)$ the α - derivative of f is

$$D^\alpha_a f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n x}{dt^n} \int_a^\alpha \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Definition 1.2. Caputo definition[7, 9, 8, 3]: For $\alpha \in (n-1, n)$ the α - derivative of f is

$${}^C_a D^\alpha_t f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

2. Preliminaries

Proposition 2.1. *If $\eta_\rho \in \mathcal{M}^2_\Upsilon(0, \mathcal{T})$, then*

$$\mathcal{E}\left(\int_0^{\mathcal{T}} \eta_\rho d\aleph_\rho\right)^2 = \mathcal{E}\left(\int_0^{\mathcal{T}} \eta_\rho^2 d\left\langle \aleph \right\rangle_\rho\right) \leq \iota^{-2} \mathcal{E}\left(\int_0^{\mathcal{T}} \eta_\rho^2 d\rho\right).$$

Lemma 2.2. *Let $\phi(\rho,\nu) \in SPAA(R \times \mathcal{L}^2_\Upsilon(\Omega), \mathcal{L}^2_\Upsilon(\Omega))$, and $\nu, \upsilon \in \mathcal{L}^2_\Upsilon(\Omega))$. If there exists a positive number $\mathcal{L}$ such that,*

$$\mathcal{E} \parallel \phi(\rho,\nu) - \phi(\rho,\upsilon) \parallel^2 \leq \mathcal{L} \parallel \nu - \upsilon \parallel^2, \rho \in R$$

then $\phi(\rho,\nu) \in SPAA(R, \mathcal{L}^2_\Upsilon(\Omega))$ for any $\nu \in SPAA(R, \mathcal{L}^2_\Upsilon(\Omega))$.

Definition 2.3. An $\mathcal{F}_\rho$ progressively measurable process $\{\Psi_\rho\}_{\rho \in R}$ is called a mild solution of the stochastic evolution equation of fractional order driven by G-Brownianmotion, if it satisfies the equation

$$\begin{aligned}\Psi_\rho = \psi(\rho, s)\Psi_\varsigma &+ \frac{1}{\Gamma(-\alpha-n)} \int_\varsigma^\rho \frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_\varsigma^\rho \frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\left\langle \aleph \right\rangle_r \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_\varsigma^\rho \frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r\end{aligned} \tag{2}$$

for any $\rho \geq \varsigma$ and $\varsigma \in R$.

For our convenience and further use we consider following assumptions:
$(H1)$ The evolution family $\psi(\rho,\varsigma)$ generated by $\mathcal{A}(\rho)$, and there exist $\mathcal{M} \geq 1$ and $\delta > 0$ such that

$$\mid \psi(\rho, r) \mid \leq \mathcal{M} e^{-\delta(\rho-r)}, \rho \geq r.$$

$(H2)$ The functions $\Phi(\rho,\Psi) : R \times \mathcal{L}^2_\Upsilon(\Omega) \to \mathcal{L}^2_\Upsilon(\Omega)$,$\Upsilon(\rho,\Psi) : R \times \mathcal{L}^2_\Upsilon(\Omega) \to \mathcal{L}^2_\Upsilon(\Omega)$ and $\chi(\rho,\Psi) : R \times \mathcal{L}^2_\Upsilon(\Omega) \to \mathcal{L}^2_\Upsilon(\Omega)$ are square-mean pseudo almost automorphic process in $\rho \in R$. Moreover, Φ, Υ and χ satisfy the following Lipschitz conditions for some positive constants $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ such that

$$\mathcal{E} \mid \Phi(\rho,\nu) - \Phi(\rho,\upsilon) \mid^2 \leq \mathcal{L}_1 \mathbb{E} \mid \nu - \upsilon \mid^2,$$

$$\mathcal{E} \mid \Upsilon(\rho,\nu) - \Upsilon(\rho,\upsilon) \mid^2 \leq \mathcal{L}_2 \mathbb{E} \mid \nu - \upsilon \mid^2,$$

and

$$\mathcal{E} \mid \chi(\rho,\nu) - \chi(\rho,\upsilon) \mid^2 \leq \mathcal{L}_3 \mathcal{E} \mid \nu - \upsilon \mid^2,$$

for $\nu, \upsilon \in \mathcal{L}^2_{\Upsilon}(\Omega)$ and $\rho \in R$.

3. Main results

Theorem 3.1. *If the assumptions* $(H1)$ *and* $(H2)$ *be satisfied then the equation G-SEEF (1) has a unique square-mean pseudo almost automorphic mild solution* $\Psi \in SPAA(R; \mathcal{L}^2_{\Upsilon}(\Omega))$ *provided that*

$$\frac{3\mathcal{M}^2\mathcal{L}_1}{\delta^2} + \frac{3\sigma^{-4}\mathcal{M}^2\mathcal{L}_2}{\delta^2} + \frac{3\sigma^{-2}\mathcal{M}^2\mathcal{L}_2}{2\delta} < 1.$$

which can be explicitly expressed as follows:

$$\begin{aligned}\Psi_\rho = &\frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r,\end{aligned}$$

for each $\rho \in R$.

Proof. As $\psi(\rho,r) = \psi(\rho,\varsigma)\psi(\varsigma,r)$, and hence we have

$$\begin{aligned}\Psi_\rho = &\frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r,\end{aligned} \tag{3}$$

$$\begin{aligned}\Rightarrow \Psi_\rho = &\psi(\rho,\varsigma)\Psi_\varsigma + \frac{1}{\Gamma(-\alpha-n)} \int_{\varsigma}^{\rho} \frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{\varsigma}^{\rho} \frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{\varsigma}^{\rho} \frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r\end{aligned}$$

$\forall\rho \geq \varsigma$ and $\varsigma \in R$, and hence Ψ_ρ given by (3) is a mild solution to (1).
Now for any $\Psi_\rho \in SPAA(R;\mathcal{L}^2_\Upsilon(\Omega))$, define

$$\begin{aligned}\daleth\Psi_\rho &= \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{t}\frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r.\end{aligned}$$

Claim-I: $\daleth\Psi_\rho \in SPAA(R;\mathcal{L}^2_\Upsilon(\Omega))$.
From $(H2)$, for any $\Psi,\mathcal{Y} \in \mathcal{L}^2_\Upsilon(\Omega))$ and all $\rho \in R$, by using the Lemma 2.2, we conclude that $\Phi(\rho,\Psi_\rho),\Upsilon(\rho,\Psi_\rho),\chi(\rho,\Psi_\rho) \in SPAA(R,\mathcal{L}^2_\Upsilon(\Omega))$. So, there exist $\phi,\kappa,\zeta \in SAA(R,\mathcal{L}^2_\Upsilon(\Omega))$ and $\gamma,\beta,\varphi \in SBC_0(R,\mathcal{L}^2_\Upsilon(\Omega))$ such that $\Phi = \phi+\gamma, \Upsilon = \kappa+\beta, \chi = \zeta+\varphi$. Hence,

$$\begin{aligned}\daleth\Psi_\rho &= \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\phi^{(n)}(r,\Psi_r)+\gamma^{(n)}(r,\Psi_r))}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\kappa^{(n)}(r,\Psi_r)+\beta^{(n)}(r,\Psi_r))}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\zeta^{(n)}(r,\Psi_r)+\varphi^{(n)}(r,\Psi_r))}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\\ &=\Big(\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\Big)\\ &+\Big(\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\gamma^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\beta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\varphi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\Big)\end{aligned}$$

Define

$$\begin{aligned}\Pi\Psi_\rho &= \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\end{aligned}$$

$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)h^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r,$$

$$\begin{aligned}\Lambda\Psi_\rho &= \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\gamma^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\beta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\ &+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\varphi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r.\end{aligned}$$

Claim 2: If Ψ_ρ then $\daleth\Psi_\rho$ is square-mean pseudo almost automorphic.
For this it is sufficient to prove that $\Pi\Psi_\rho \in SAA(R,\mathcal{L}^2_\Upsilon(\Omega))$ and
$\Lambda\Psi_\rho \in SBC_0(R,\mathcal{L}^2_\Upsilon(\Omega))$. Thus, the following verification procedure is divided into three steps.

Step 1. If $\rho \in R$, then we have

$$\begin{aligned}&\mathcal{E}\mid \Pi\Psi_\rho-\Pi\Psi_{\rho_0}\mid^2=\mathcal{E}\mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\
&+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\
&+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\
&-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\
&-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\mid^2\\
&\le 3\mathcal{E}\mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\\
&-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr\mid^2\\
&+3\mathcal{E}\mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\\
&-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r\mid^2\\
&+3\mathcal{E}\mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\\
&-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho_0}\frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r\mid^2 .\end{aligned}$$

Then, let $\theta = r - \rho + \rho_0$, we can get the following relation

$$\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \mid^2$$

$$= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\theta)\phi^{(n)}(\theta+\rho+\rho_0,\Psi_{\theta+\rho+\rho_0})}{(\rho-r)^{-\alpha+1-n}} d\theta$$

$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \mid^2$$

$$= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\theta)(\phi^{(n)}(\theta+\rho+\rho_0,\Psi_{\theta+\rho+\rho_0}) - \phi^{(n)}(r,\Psi_r))}{(\rho-r)^{-\alpha+1-n}} dr \mid^2 .$$

Let $\{\rho_n\}$ be an arbitrary sequence of real numbers, with $\rho_n \to \rho_0$ as $n \to \infty$. As $\phi \in SBC(R, \mathcal{L}^2_\Upsilon(\Omega))$ then we have

$$\mid \psi(\rho_0,r) \mid \mathcal{E} \mid \phi^{(n)}(r+\rho_n-\rho_0, \Psi_{r+\rho_n-\rho_0}) - \phi^{(n)}(r,\Psi_r) \mid^2 \to 0, n \to \infty,$$

$$\Rightarrow \mid \psi(\rho_0,r) \mid \mathcal{E}\phi^{(n)}(r+\rho_n-\rho_0, \Psi_{r+\rho_n-\rho_0}) - \phi^{(n)}(r,\Psi_r) \mid^2$$
$$\leq \mid \psi(\rho_0,r) \mid (1+ \parallel \phi^{(n)} \parallel_\infty),$$

for every n sufficiently large. It is easy to note that

$$\int_{-\infty}^{\rho_0} \mid \psi(\rho,r) \mid (1+ \parallel \phi^{(n)} \parallel_\infty) dr \frac{1}{\xi^*} < +\infty.$$

Then, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r) \mid \mathcal{E} \mid \phi^{(n)}(r+\rho_n-\rho_0, \Psi_{r+\rho_n-\rho_0}) - \phi^{(n)}(r,\Psi_r) \mid^2 = 0.$$

As $\{\rho_n\}$, be arbitrary sequence then we have

$$\lim_{\rho\to\rho_0} \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r) \mid \mathcal{E} \mid \phi^{(n)}(r+\rho-\rho_0, \Psi_{r+\rho-\rho_0}) - \phi^{(n)}(r,\Psi_r) \mid^2 = 0.$$

Together with all above estimations, we obtain

$$\lim_{\rho\to\rho_0} \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr$$
$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \mid^2 = 0.$$

Let $\left\langle \tilde{\aleph} \right\rangle_\iota = \langle \aleph \rangle_{\iota+\rho-\rho_0} - \langle \aleph \rangle_{\rho-\rho_0}$ for each $\iota \in R$. Then it is easy to verify that $\left\langle \tilde{\aleph} \right\rangle_\iota$ is the quadratic variation process of the G-Brownian motion and identically distributed like $\langle\aleph\rangle_\iota$. As $\iota = r - \rho + \rho_0$ and by using Proposition 2.1 we have

$$\begin{aligned}
&\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\
&- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \mid^2 \\
&= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\iota)\kappa^{(n)}(\iota+\rho-\rho_0,\Psi_{\iota+\rho-\rho_0})}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_{\iota+\rho-\rho_0} \\
&- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(t_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \mid^2 \\
&= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\iota)\kappa^{(n)}(\iota+\rho-\rho_0,\Psi_{\iota+\rho-\rho_0})}{(\rho-r)^{-\alpha+1-n}} d\left\langle\tilde{\aleph}\right\rangle_\iota \\
&- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \mid^2 \\
&= \mathcal{E} \mid \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)(\kappa^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) - \kappa^{(n)}(r,\Psi_r))}{\Gamma(-\alpha-n)(\rho-r)^{-\alpha+1-n}} d\left\langle\tilde{\aleph}\right\rangle_r \mid^2 \\
&\leq \iota^{-4}\mathcal{E} \mid \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)(\kappa^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) - \kappa^{(n)}(r,\Psi_r))}{\Gamma(-\alpha-n)(\rho-r)^{-\alpha+1-n}} dr \mid^2 .
\end{aligned}$$

The Cauchy–Schwarz inequality shows that

$$\begin{aligned}
&\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\
&- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \mid^2 \\
&\leq \iota^{-4} \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r) \mid dr \frac{1}{\xi^*}. \int_{-\infty}^{t_0} \mid \psi(\rho_0,r) \mid \mathcal{E} \mid \kappa^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) \\
&- \kappa^{(n)}(r,\Psi_r) \mid^2 dr \frac{1}{\xi^*}.
\end{aligned}$$

By an analogous arguments performed as above, we can show that

$$\begin{aligned}
&\lim_{\rho\to\rho_0} \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r) \mid dr \frac{1}{\xi^*}. \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r) \mid \mathcal{E} \mid \kappa^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) \\
&- \kappa^{(n)}(r,\Psi_r) \mid^2 dr \frac{1}{\xi^*} = 0.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\lim_{\rho\to\rho_0} \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\
&- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\kappa^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\rho \langle\aleph\rangle_r \mid^2 = 0.
\end{aligned}$$

As $\iota = r - \rho + \rho_0$ then by using the proposition 2.1, we get

$$\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r \mid^2$$

$$= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\iota)\zeta^{(n)}(\sigma+\rho-\rho_0,\Psi_{\iota+\rho-\rho_0})}{(\rho-r)^{-\alpha+1-n}} d\aleph_{\iota+\rho-\rho_0} - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r \mid^2$$

$$= \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,\iota)\zeta^{(n)}(\iota+\rho-\rho_0,\Psi_{\iota+\rho-\rho_0})}{(\rho-r)^{-\alpha+1-n}} d\tilde{\aleph}_\iota - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r \mid^2$$

$$= \mathcal{E} \mid \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)(\zeta^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) - \zeta^{(n)}(r,\Psi_r))}{\Gamma(-\alpha-n)(\rho-r)^{-\alpha+1-n}} d\tilde{\aleph}_r \mid^2$$

$$= \mathcal{E} \int_{-\infty}^{\rho_0} \frac{\mid \psi(\rho_0,r)\mid^2 \mid (\zeta^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) - \zeta^{(n)}(r,\Psi_r))\mid^2}{\mid \Gamma(-\alpha-n)(t-r)^{-\alpha+1-n}\mid^2} d\left\langle \tilde{\aleph}\right\rangle_r$$

$$\leq \sigma^{-2}\mathcal{E}\Big(\int_{-\infty}^{\rho_0} \frac{\mid \psi(\rho_0,r)\mid^2 \mid (\zeta^{(n)}(r+\rho-\rho_0,\Psi_{\iota+\rho-\rho_0}) - \zeta^{(n)}(r,\Psi_r))\mid^2}{\mid \Gamma(-\alpha-n)(t-r)^{-\alpha+1-n}\mid^2} dr\Big).$$

$$\leq \iota^{-2} \int_{-\infty}^{\rho_0} \mid \psi(\rho_0,r)\mid^2 \mathcal{E} \mid (\zeta^{(n)}(r+\rho-\rho_0,\Psi_{r+\rho-\rho_0}) - \zeta^{(n)}(r,\Psi_r))\mid^2 dr \frac{1}{\xi^*}.$$

For shake of simplicity Let $\Gamma(-\alpha-n)(\rho-r)^{-\alpha+1-n} = \xi$ and $\mid \Gamma(-\alpha-n)(\rho-r)^{-\alpha+1-n}\mid^2 = \xi^*$.
By an analogous arguments performed as above, we can show that

$$\lim_{\rho\to\rho_0} \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho_0} \frac{\psi(\rho_0,r)\zeta^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r \mid^2 = 0.$$

Hence,

$$\lim_{t\to\rho_0} \mathcal{E} \mid \Pi\Psi_\rho - \Pi\Psi_{\rho_0}\mid^2 = 0,$$

which implies that $\Pi\Psi_\rho$ is a stochastic continuous process.
Step 2. As $\phi,\kappa,\zeta \in SAA(R,\mathcal{L}^2_\Upsilon(\Omega))$then for every sequence of real numbers $\{r^{\shortmid}_n\}_{n\in N}$ a subsequence $\{r_n\}$ such that, for some stochastic processes $\tilde{\phi},\tilde{\kappa},\tilde{\zeta}$: $R \to \mathcal{L}^2_\Upsilon(\Omega))$, we have

$$\lim_{n\to\infty} \mathcal{E} \mid \phi^{(n)}(\rho+r_n,\Psi_{\rho+r_n}) - \tilde{\phi^{(n)}}(\rho,\Psi_\rho)\mid^2 = 0$$

and

$$\lim_{n\to\infty} \mathcal{E} \mid \tilde{\phi^{(n)}}(\rho + r_n, \Psi_{\rho+r_n}) - \phi^{(n)}(\rho, \Psi_\rho) \mid^2 = 0,$$

$$\lim_{n\to\infty} \mathcal{E} \mid \kappa^{(n)}(\rho + r_n, \Psi_{\rho+r_n}) - \tilde{\kappa^{(n)}}(\rho, \Psi_\rho) \mid^2 = 0$$

$$\lim_{n\to\infty} \mathcal{E} \mid \tilde{\kappa^{(n)}}(\rho + r_n, \Psi_{\rho+r_n}) - \kappa^{(n)}(\rho, \Psi_\rho) \mid^2 = 0,$$

$$\lim_{n\to\infty} \mathcal{E} \mid \zeta^{(n)}(\rho + r_n, \Psi_{\rho+r_n}) - \tilde{\zeta^{(n)}}(\rho, \Psi_\rho) \mid^2 = 0$$

and

$$\lim_{n\to\infty} \mathcal{E} \mid \tilde{\zeta^{(n)}}(\rho + r_n, \Psi_{\rho+r_n}) - \zeta^{(n)}(\rho, \Psi_\rho) \mid^2 = 0,$$

$\forall \rho \in R$ and $\Psi(.) \in \mathcal{L}^2_\Upsilon(\Omega))$. Let

$$\begin{aligned}\tilde{\Pi}\Psi_\rho &= \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\tilde{\phi^{(n)}}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} dr \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\tilde{\kappa^{(n)}}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\langle\aleph\rangle_r \\ &+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)\tilde{\zeta^{(n)}}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}} d\aleph_r.\end{aligned}$$

Claim 3. $\Lambda\Psi_\rho \in SBC_0(R, \mathcal{L}^2_\Upsilon(\Omega))$.
By Step 1, we have $\Lambda\Psi_\rho$ is stochastically continuous process. As $\gamma, \beta, \varphi \in SBC_0(R, \mathcal{L}^2_\Upsilon(\Omega))$ and the exponential dissipation property of $\psi(\rho, r)$, it gives $\Lambda\Psi_\rho$ is stochastically bounded and therefore $\Lambda\Psi_\rho \in SBC(R, \mathcal{L}^2_\Upsilon(\Omega))$. Hence, it is sufficient to prove that

$$\lim_{\mathcal{T}\to\infty} \frac{1}{2\mathcal{T}} \int_{-\mathcal{T}}^{\mathcal{T}} \mathbb{E} \mid \Lambda\Psi_\rho \mid^2 d\rho \frac{1}{\xi^*} = 0.$$

By the Cauchy–Schwarz inequality and Lipschitz conditions in $(H2)$, we evaluate the first term of the last inequality, as follows,

$$\begin{aligned}&\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} \frac{\psi(\rho,r)(\Phi^{(n)}(r,\Psi_r) - \Phi^{(n)}(r,\mathcal{Y}_r))}{(\rho-r)^{-\alpha+1-n}} dr \mid^2 \\ &\leq \mathcal{M}^2 \mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\rho} e^{-\delta(\rho-r)} \frac{(\Phi^{(n)}(r,\Psi_r) - \Phi^{(n)}(r,\mathcal{Y}_r))}{(\rho-r)^{-\alpha+1-n}} dr \mid^2 \\ &\leq \mathcal{M}^2 \int_{-\infty}^{\rho} \mid e^{-\delta(\rho-r)} \mid dr \frac{1}{\xi^*}. \int_{-\infty}^{\rho} \mid e^{-\delta(\rho-r)} \mid \mathcal{E} \mid \Phi^{(n)}(r,\Psi_r) - \Phi^{(n)}(r,\mathcal{Y}_r) \mid^2 dr \\ &\leq \frac{\mathcal{M}^2 \mathcal{L}_1}{\delta} \int_{-\infty}^{\rho} e^{-\delta(\rho-r)} \mathcal{E} \mid \Psi_r - \mathcal{Y}_r \mid^2 dr \frac{1}{\xi^*}.\end{aligned}$$

According to

$$\int_{-\infty}^{\rho} e^{-\delta(\rho-r)} dr \frac{1}{\xi^*} = \frac{1}{\delta},$$

we have

$$\begin{aligned}&\mathcal{E}\mathbb{E} \mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\Phi^{(n)}(r,\Psi_r)-\Phi^{(n)}(r,\mathcal{Y}_r))}{(\rho-r)^{-\alpha+1-n}}dr \mid^2\\&\leq \frac{\mathcal{M}^2\mathcal{L}_1}{\delta^2}\sup_{r\in R}\mathcal{E}\mid \Psi_r-\mathcal{Y}_r\mid^2 \frac{1}{\xi^*}.\end{aligned} \tag{4}$$

Moreover according to

$$\int_{-\infty}^{\rho} e^{-\delta(\rho-r)}dr\frac{1}{\xi^*}=\frac{1}{\delta},$$

we get,

$$\begin{aligned}&\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\Upsilon^{(n)}(r,\Psi_r)-\Upsilon^{(n)}(r,\mathcal{Y}_r))}{(\rho-r)^{-\alpha+1-n}}d\left\langle\aleph\right\rangle_r \mid^2\\&\leq \frac{\iota^{-4}\mathcal{M}^2\mathcal{L}_2}{\delta^2}\sup_{r\in R}\mathcal{E}\mid \Psi_r-\mathcal{Y}_r\mid^2 \frac{1}{\xi^*}.\end{aligned} \tag{5}$$

Additionally according to

$$\int_{-\infty}^{\rho} e^{-2\delta(\rho-r)}dr\frac{1}{\xi^*}=\frac{1}{2\delta},$$

we have

$$\begin{aligned}&\mathcal{E} \mid \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)(\chi^{(n)}(r,\Psi_r)-\chi^{(n)}(r,\mathcal{Y}_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r \mid^2\\&\leq \frac{\iota^{-2}\mathcal{M}^2\mathcal{L}_3}{2\delta}\sup_{r\in R}\mathcal{E}\mid \Psi_r-\mathcal{Y}_r\mid^2 \frac{1}{\xi^*}.\end{aligned} \tag{6}$$

By equations (4), (5), and (6), it follows that,

$$\mathcal{E}\mid \daleth\Psi_\rho-\daleth\mathcal{Y}_\rho\mid^2\leq\left(\frac{\mathcal{M}^2\mathcal{L}_1}{\delta}+\frac{\iota^{-4}\mathcal{M}^2\mathcal{L}_2}{\delta^2}+\frac{\iota^{-2}\mathcal{M}^2\mathcal{L}_3}{2\delta}\right)\sup_{r\in R}\mathcal{E}\mid \Psi_r-\mathcal{Y}_r\mid^2 \frac{1}{\xi^*},$$

and, therefore,

$$\mathcal{E}\mid \daleth\Psi_\rho-\daleth Y_\rho\mid^2_{SPAA}\leq\left(\frac{\mathcal{M}^2\mathcal{L}_1}{\delta}+\frac{\iota^{-4}\mathcal{M}^2\mathcal{L}_2}{\delta^2}+\frac{\iota^{-2}\mathcal{M}^2\mathcal{L}_3}{2\delta}\right)\mid \Psi-\mathcal{Y}\mid^2_{SPAA} \frac{1}{\xi^*}.$$

Consequently, when

$$\frac{3\mathcal{M}^2\mathcal{L}_1}{\delta}+\frac{3\iota^{-4}\mathcal{M}^2\mathcal{L}_2}{\delta^2}+\frac{3\iota^{-2}\mathcal{M}^2\mathcal{L}_3}{2\delta}<1,$$

$\daleth$ has a unique fixed point Ψ_ρ in $SPAA(R, \mathcal{L}_G^2(\Omega))$ such that $\daleth\Psi_\rho = \Psi_\rho$. Namely,

$$\Psi_\rho = \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\Phi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}dr + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\Upsilon^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\langle\aleph\rangle_r + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\rho}\frac{\psi(\rho,r)\chi^{(n)}(r,\Psi_r)}{(\rho-r)^{-\alpha+1-n}}d\aleph_r,$$

for all $\rho \in R$, which shows that the G-SEEF (1) has a unique square-mean pseudo almost automorphic mild solution. □

4. Application

Now we will discuss the application of our main theorems. Let $\mathcal{U} \subset R^n$, which is a bounded subset, with boundary $\partial\mathcal{U} \in C^2$ and is locally on one side of $\mathcal{U}$. Consider the following system

$$\begin{aligned} {}_0^cD_\rho^\alpha\Psi(\rho,\varrho) = \mathcal{A}(\rho,\varrho)\Psi(\rho,\varrho)d\rho + \Phi(\rho,\Psi(\rho,\varrho))d\rho + \Upsilon(\rho,\Psi(\rho,\varrho))d\langle\aleph\rangle_\rho \\ + \chi(\rho,\Psi(\rho,\varrho))d\aleph_\rho, \rho \in R, \end{aligned} \tag{7}$$

$$\sum_{i,j=1}^{n} n_i(\varrho)\pi_{ij}(\rho,\varrho){}_0^cD_{\varrho_i}^\alpha\Psi(\rho,\varrho) = 0, \rho \in R, , \varrho \in \partial\mathcal{U}, \tag{8}$$

let $n(\varrho) = (n_1(\varrho), n_2(\varrho), ..., n_n(\varrho))$ be the outer unit normal vector. Then, we define the family of operators $A(\rho,\varrho)$ as follows,

$$\mathcal{A}(\rho,\varrho) = \sum_{i,j=1}^{n}\frac{\partial^\alpha}{\partial x_i^\alpha}\Big(\pi_{ij}(\rho,\varrho)\frac{\partial^\alpha}{\partial x_i^\alpha}\Big) + c(\rho,\varrho), \rho \in R, \varrho \in \mathcal{U},$$

$\aleph_\rho$ is a two-sided standard one-dimensional G-Brownian motion associated with the filtration $\mathcal{F}_\rho = \iota\{\aleph_u - \aleph_\varrho, u, \varrho \le \rho\}$, $\langle\aleph\rangle$ is the quadratic variation process of the G-Brownian motion $\aleph$. In addition, $\pi_{ij}(i,j = 1,2,....,n)$ and c satisfy the following assumptions.

$(H3)$

(1) As $i,j = 1,2,....,n$ then $\pi_{ij} = \pi_{ji}$ Moreover, $\pi_{ij} \in \mathcal{C}_b^\mu(R, \mathcal{L}_\Upsilon^2(\mathcal{C}(\overline{\mathcal{U}})) \cap \mathcal{C}_b(R, \mathcal{L}_\Upsilon^2(\mathcal{C}^1(\overline{\mathcal{U}})) \cap SPAA(R, \mathcal{L}_\Upsilon^2(\mathcal{L}^2(\mathcal{U})))$ for all $i,j = 1,2,....,n$,
$c \in \mathcal{C}_b^\mu(R, \mathcal{L}_\Upsilon^2(\mathcal{L}^2(\mathcal{U}))) \cap \mathcal{C}_b(R, \mathcal{L}_\Upsilon^2(\mathcal{C}(\overline{\mathcal{U}})) \cap SPAA(R, \mathcal{L}_\Upsilon^2(\mathcal{L}^1(\mathcal{U})))$ for some $\mu \in (\frac{1}{2}]$.

(2) As for $(\rho,\varrho) \in R \times \overline{\mathcal{U}}$ and $\eta \in R^n$ then $\exists$ a positive number δ, such that

$$\sum_{i,j=1}^{n}\pi_{ij}(\rho,\varrho)\eta_i\eta_j \ge \delta \mid \eta \mid^2,$$

If the above assumptions hold then $\psi(\rho,s)$ satisfying $(H1)$. For $\rho \in R$, we define an operator $\mathcal{A}(\rho)$ on $\mathcal{L}_\Upsilon^2(\mathcal{L}^2(\mathcal{U}))$ by

$$D(\mathcal{A}(\rho)) = \left\{ \Psi \in \mathcal{L}^2_{\Upsilon}(\mathcal{L}^2(\mathcal{U})) : \sum_{i,j=1}^{n} n_i(.)\pi_{ij}(\rho,.)^c_0 D^{\alpha}_{\varrho_i}\Psi(\rho,.) = 0, on\ \partial\mathcal{U} \right\}$$

and $\mathcal{A}(\rho)\Psi = \mathcal{A}(\rho,\varrho)\Psi(\varrho)$ for all $\Psi \in D(\mathcal{A}(\rho))$. Thus, assumptions $(H1)$–$(H3)$ are satisfy and hence system (7) -(8) has a unique square-mean pseudo almost automorphic mild solution, provided $\mathcal{M}$ is small enough.

Conflicts of interest : The authors declare that they have no competing interests.

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