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A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH BESSEL FUNCTIONS

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Abstract. In this article, we are presenting and examining a subclass of Meromorphic univalent functions as stated by the Bessel function. We get disparities in terms of coefficients, properties of distortion, closure theorems, Hadamard product. Finally, for the class $\Sigma^*(\wp, \ell, \hbar, \tau, c)$, we obtain integral transformations.

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1. Introduction

Let Σ signify the $\imath(\ss)$ function class of the type

$$\imath(\ss) = \frac{1}{\ss} + \sum_{\zeta=1}^{\infty} \varrho_\zeta \ss^\zeta,\ \zeta \in N = \{1, 2, 3, \cdots\} \tag{1}$$

in the punctured unit disc, which is regular $\Upsilon^* = \{\ss \in C : 0 < |\ss| < 1\} = \Upsilon \backslash \{0\}$.

Mathematically, it is said that the function $\imath \in \Sigma$ given by (1) is meromorphically star-like in the order ℓ if it complies with the following:

$$\Re\left\{-\left(\frac{\ss\imath'(\ss)}{\imath(\ss)}\right)\right\} > \ell,\ (\ss \in \Upsilon)$$

for some $\ell (0 \leq \ell < 1)$. We say $\imath \in \Sigma^*(\ell)$ class of functions like this.

Similarly, it is said that the $\imath \in \Sigma$ function provided by (1) is meromorphically convex. If it fulfills the following, ℓ is ordered:

$$\Re\left\{-\left(1 + \frac{\ss\imath''(\ss)}{\imath'(\ss)}\right)\right\} > \ell,\ (\ss \in \Upsilon)$$

for some $\ell (0 \leq \ell < 1)$. We say $\imath$ belongs to the $\Sigma_\zeta(\ell)$ class of functions.

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The $\Sigma^*(\ell)$ class and numerous other subclasses of Σ were examined by Miller [9], Pommerenke [10], Royster [11], Akgul [2, 3], and Venkateswarlu et al. [14, 15, 16] numerous times.

For the $\imath \in \Sigma$ function specified by (1) and $g \in \Sigma$ specified by

$$g(\text{ß}) = \frac{1}{\text{ß}} + \sum_{\zeta=1}^{\infty} b_\zeta \text{ß}^\zeta,$$

we indicate the Hadamard product of $\imath$ and g by

$$(\imath * g)(\text{ß}) = \frac{1}{\text{ß}} + \sum_{\zeta=1}^{\infty} \varrho_\zeta b_\zeta \text{ß}^\zeta.$$

Geometric function theory (GFT) is the area of complex analysis which deals with the geometric characterization of analytic functions, established around the turn of the twentieth century [5].

Bessel functions are essential in many branches of mathematics and applied mathematics. Recently, there has been a clear interest on Bessel and hypergeometric functions from the point of view of geometric function theory.

It is a known fact that the study of special functions plays a significant role in GFT. One reason is that the solutions of extremal problems can be frequently written in terms of special function. Another reason is that some important conformal mappings are given by special function.

In recent times, the solution of the Bieberbach conjecture by de Branges is obtained with the help of special functions [1]. Bessel function is one of the most significant special functions. It is therefore important for solving many problems in engineering, physics, and mathematics (see [7, 8]).

The first-order generalized Bessel function γ (see [6]) denoted by γ is recalled here

$$w(\text{ß}) = \sum_{\zeta=0}^{\infty} \frac{(-c)^\zeta}{\zeta!\Gamma(\gamma+\zeta+\frac{b+1}{2})} \left(\frac{\text{ß}}{2}\right)^{2\zeta+\gamma}, (\text{ß} \in \Upsilon).$$

The special solution to the second linear homogeneous order of the differential equation (see [17]) for more details)

$$\text{ß}^2 w''(\text{ß}) + b\text{ß}w'(\text{ß}) + [c\text{ß}^2 - \gamma^2 + (1-b)\gamma]w(\text{ß}) = 0$$

where $c, \gamma, b \in C$.

With respect to the generalized Bessel function w, we are bringing the formula φ that is supported by

$$\varphi(\text{ß}) = 2^\gamma \Gamma\left(\gamma + \frac{b+1}{2}\right) \text{ß}^{-\left(1+\frac{\gamma}{2}\right)} w(\sqrt{\text{ß}}).$$

Using a symbol of Pochhammer, mentioned by $(l)_\epsilon$, for $l \in C$ and in terms of $l \in C$, role of Euler gamma, by

$$(l)_\epsilon = \frac{\Gamma(l+\epsilon)}{\Gamma(l)} = \begin{cases} 1, & (\epsilon = 0); \\ l(l+1)(l+2)\cdots(l+\zeta-1), & (\epsilon = \zeta \in N = \{1,2,3\cdots\}). \end{cases}$$

For the function $\varphi(ß)$, the appropriate sequence specification is obtained:

$$\varphi(ß) = \frac{1}{ß} + \sum_{\zeta=0}^{\infty} \frac{(-c)^{\zeta+1}}{4^{\zeta+1}(\zeta+1)!(\tau)_{\zeta+1}} ß^\zeta \left(\tau = \gamma + \frac{b+1}{2} \notin Z_0^- = \{0,-1,-2,\cdots\}\right).$$

Connecting to the φ function, the Bessel operator $\mathscr{S}_\tau{}^c$ is characterized by the following Hadamard product

$$\begin{aligned} \mathscr{S}_\tau{}^c \imath(ß) = (\varphi * \imath)(ß) &= \frac{1}{ß} + \sum_{\zeta=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{\zeta+1}}{(\zeta+1)!(\tau)_{\zeta+1}} \varrho_\zeta ß^\zeta \\ &= \frac{1}{ß} + \sum_{\zeta=1}^{\infty} \phi_\zeta(\tau,c) \varrho_\zeta ß^\zeta, \end{aligned} \tag{2}$$

where $\phi_\zeta(\tau,c) = \frac{\left(\frac{-c}{4}\right)^\zeta}{(\zeta)!(\tau)_\zeta}$.

We now describe a new subclass inspired by Sivaprasad Kumar et al. [13] and Venkateswarlu et al. [14], $\Sigma^*(\wp,\ell,\hbar,\tau,c)$ of Σ.

Definition 1.1. For $0 \le \wp < 1,\ \ell \ge 0,\ 0 \le \hbar < \frac{1}{2}$, let's say $\Sigma^*(\wp,\ell,\hbar,\tau,c)$ be the Σ subclass containing of (1) functions and fulfilling the analytic requirement

$$\begin{aligned} &-\Re\left(\frac{ß(\mathscr{S}_\tau{}^c\imath(ß))' + \hbar ß^2(\mathscr{S}_\tau{}^c\imath(ß))''}{(1-\hbar)\mathscr{S}_\tau{}^c\imath(ß) + \hbar ß(\mathscr{S}_\tau{}^c\imath(ß))'} + \wp\right) \\ &> \ell\left|\frac{ß(\mathscr{S}_\tau{}^c\imath(ß))' + \hbar ß^2(\mathscr{S}_\tau{}^c\imath(ß))''}{(1-\hbar)\mathscr{S}_\tau{}^c\imath(ß) + \hbar ß(\mathscr{S}_\tau{}^c\imath(ß))'} + 1\right|. \end{aligned} \tag{3}$$

We need the corresponding lemmas to prove our findings, [4].

Lemma 1.2. *If d is a real number and ω is a complex number then*

$$\Re(\omega) \ge d \Leftrightarrow |\omega + (1-d)| - |\omega - (1+d)| \ge 0.$$

Lemma 1.3. *If ω is a complex number and d,k are real numbers then*

$$-\Re(\omega) \ge k|\omega+1| + d \Leftrightarrow -\Re\left(\omega(1+ke^{i\theta}) + ke^{i\theta}\right) \ge d, -\pi \le \ \theta \le \pi.$$

The main goal of this study is to examine some common geometric function theory properties, such as coefficient boundaries, distortion properties, closure theorems, Hadamard product and integral transformations for the class.

2. Coefficient estimates

We obtain a required and proper condition in this section for a function $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$.

Theorem 2.1. *Let $\imath \in \Sigma$ be given by* (1). *Then* $\imath \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$

$$\Leftrightarrow \sum_{\zeta=1}^{\infty}[(1+(\zeta-1)\hbar)][\zeta(\ell+1)+(\ell+\wp)]\phi_\zeta(\tau,c)\varrho_\zeta \leq (1-\wp)(1-2\hbar). \quad (4)$$

Proof. Let $\imath \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$. Then by Definition 1.1 and using Lemma 1.3, it is enough to demonstrate that

$$-\Re\left\{\frac{ß(\mathscr{S}_\tau{}^c\imath(ß))' + \hbar ß^2(\mathscr{S}_\tau{}^c\imath(ß))''}{(1-\hbar)\mathscr{S}_\tau{}^c\imath(ß) + \hbar ß(\mathscr{S}_\tau{}^c\imath(ß))'}(1+\ell e^{i\theta}) + \ell e^{i\theta}\right\} \geq \wp, -\pi \leq \theta \leq \pi. \quad (5)$$

For convenience

$$\begin{aligned} C(ß) =& -\left[ß(\mathscr{S}_\tau{}^c\imath(ß))' + \hbar ß^2(\mathscr{S}_\tau{}^c\imath(ß))''\right](1+\ell e^{i\theta}) \\ & - \ell e^{i\theta}\left[(1-\hbar)\mathscr{S}_\tau{}^c\imath(ß) + \hbar ß(\mathscr{S}_\tau{}^c\imath(ß))'\right] \\ D(ß) =& (1-\hbar)\mathscr{S}_\tau{}^c\imath(ß) + \hbar ß(\mathscr{S}_\tau{}^c\imath(ß))'. \end{aligned}$$

Then (5) is equivalent to

$$-\Re\left(\frac{C(ß)}{D(ß)}\right) \geq \wp.$$

We only need to prove that with regard to Lemma 1.2,

$$|C(ß) + (1-\wp)D(ß)| - |C(ß) - (1+\wp)D(ß)| \geq 0.$$

Therefore

$$\begin{aligned} & |C(ß) + (1-\wp)D(ß)| \\ \geq & (2-\wp)(1-2\lambda)\frac{1}{|ß|} - \sum_{\zeta=1}^{\infty}[\zeta-(1-\wp)][1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta \\ & - \ell\sum_{\zeta=1}^{\infty}(\zeta+1)[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta \end{aligned}$$

and $|C(ß) - (1+\wp)D(ß)|$

$$\begin{aligned} \leq & \wp(1-2\hbar)\frac{1}{|ß|} + \sum_{\zeta=1}^{\infty}[\zeta+(1+\wp)][1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta \\ & + \ell\sum_{\zeta=1}^{\infty}(\zeta+1)[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta. \end{aligned}$$

It is to explain that

$$\begin{aligned} & |C(ß) + (1-\wp)D(ß)| - |C(ß) - (1+\wp)D(ß)| \\ \geq & 2(1-\wp)(1-2\hbar)\frac{1}{|ß|} - 2\sum_{\zeta=1}^{\infty}[(\zeta+\wp)(1+(\zeta-1)\hbar)]\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta \\ & - 2\ell\sum_{\zeta=1}^{\infty}(\zeta+1)(1+(\zeta-1)\hbar)\phi_\zeta(\tau,c)\varrho_\zeta|ß|^\zeta \end{aligned}$$

$\geq 0,$ by the condition given (4).

Conversely suppose $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$. Then by Lemma 1.2, we have (5).

Having chosen the ß value on the positive real axis reduces the inequality of (5) to

$$\Re\left\{\frac{[(1-2\hbar)(1-\wp)(1+\ell e^{i\theta})]\frac{1}{ß^2}+\sum\limits_{\zeta=1}^{\infty}\{\zeta+\ell e^{i\theta}(\zeta+1)+\wp\}[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)ß^{\zeta-1}}{(1-2\hbar)\frac{1}{ß^2}+\sum\limits_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta ß^{\zeta-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq -|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{[(1-2\hbar)(1-\wp)(1+\ell e^{i\theta})]\frac{1}{r^2}+\sum\limits_{\zeta=1}^{\infty}\{\zeta+\ell(\zeta+1)+\wp\}[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)\varrho_\zeta r^{\zeta-1}}{(1-2\hbar)\frac{1}{r^2}+\sum\limits_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)]\phi_\zeta(\tau,c)r^{\zeta-1}}\right\}\geq 0.$$

We have obtained the inequality (4) by letting $r\to 1^-$ and by the mean value theorem □

Corollary 2.2. *If* $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ *then*

$$a_\zeta \leq \frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\wp+\ell)]\phi_\zeta(\tau,c)}. \tag{6}$$

For the function, the estimate is sharp

$$\imath(ß)=\frac{1}{ß}+\frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\wp+\ell)]\phi_\zeta(\tau,c)}ß^\zeta. \tag{7}$$

We get the following corollary by using $\hbar=0$ in the 2.1 theorem.

Corollary 2.3. *If* $\imath \in \Sigma^*(\wp,\ell,\tau,c)$ *then*

$$a_\zeta \leq \frac{1-\wp}{[\zeta(1+\ell)+(\wp+\ell)]\phi_\zeta(\tau,c)}. \tag{8}$$

3. Distortion theorem

Theorem 3.1. *If* $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ *then for* $0<|ß|=r<1$,

$$\frac{1}{r}-\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}\ r\leq |\imath(ß)|\leq \frac{1}{r}+\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}r. \tag{9}$$

This estimate is sharp

$$\imath(ß)=\frac{1}{ß}+\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}\ ß. \tag{10}$$

Proof. Since $\imath(ß)=\frac{1}{ß}+\sum\limits_{\zeta=1}^{\infty}\varrho_\zeta ß^\zeta$, we have

$$|\imath(ß)|=\frac{1}{r}+\sum_{\zeta=1}^{\infty}\varrho_\zeta r^\zeta\leq \frac{1}{r}+r\sum_{\zeta=1}^{\infty}\varrho_\zeta. \tag{11}$$

Since $\zeta \geq 1$, $(2\ell+\wp+1)\phi_1(\tau,c) \leq [1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)$, using Theorem 2.1, we have

$$\begin{aligned}(2\ell+\wp+1)\phi_1(\tau,c)\sum_{\zeta=1}^{\infty}\varrho_\zeta &\leq \sum_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)\\ &\leq (1-\wp)(1-2\hbar)\\ \Rightarrow \sum_{\zeta=1}^{\infty}\varrho_\zeta &\leq \frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}.\end{aligned}$$

Using the above inequality in (11), we have

$$\begin{aligned}|\imath(\text{ß})| &\leq \frac{1}{r}+\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}r\\ \text{and } |\imath(\text{ß})| &\geq \frac{1}{r}-\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}r.\end{aligned}$$

The estimate is sharp $\imath(\text{ß})=\frac{1}{\text{ß}}+\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}\text{ß}$. □

The evidence for the following corollary is analogous to that of the 3.1 theorem, so we omit the evidence.

Corollary 3.2. *If* $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ *then*

$$\frac{1}{r^2}-\frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)} \leq |\imath'(\text{ß})| \leq \frac{1}{r^2}+\frac{((1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}.$$

The estimate is sharp assigned by (10).

4. Closure theorems

Let $\imath_j$ be specified, for $j=1,2,\cdots,m$, by

$$\imath_j(\text{ß})=\frac{1}{\text{ß}}+\sum_{\zeta=1}^{\infty}\varrho_{\zeta,j}\text{ß}^\zeta,\ \varrho_{\zeta,j}\geq 0. \tag{12}$$

Theorem 4.1. *Let* $\imath_j, j=1,2,\cdots,m$ *specified by* (12) *in* $\Sigma^*(\wp,\ell,\hbar,\tau,c)$. *Then the function* h *indicate by*

$$h(\text{ß})=\frac{1}{\text{ß}}+\sum_{\zeta=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}\varrho_{\zeta,j}\right)\text{ß}^\zeta \tag{13}$$

also in $\Sigma^*(\wp,\ell,\hbar,\tau,c)$.

Proof. Since $\imath_j \in \Sigma^*(\wp,\ell,\hbar,\tau,c), j=1,2,\cdots,m$, it implies from Theorem 2.1, that

$$\sum_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)\varrho_{\zeta,j} \leq (1-\wp)(1-2\hbar),$$

$\forall\ j = 1, 2, \cdots, m$. Hence

$$\begin{aligned}
&\sum_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)\left(\frac{1}{m}\sum_{j=1}^{m}\text{ọ}_{\zeta,j}\right)\\
&=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)\text{ọ}_{\zeta,j}\right)\\
&\le(1-\wp)(1-2\hbar).
\end{aligned}$$

From Theorem (4), it concludes that $h \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$.

Hence the proof. □

Theorem 4.2. *The Class under convex linear combinations, $\Sigma^*(\wp, \ell, \hbar, \tau, c)$ is closed.*

Proof. Let $\imath_j \in \Sigma^*(\wp, \ell, \hbar, \tau, c), j = 1, 2, \cdots, m$, defined by (12). Then

$$h(\text{ß}) = \varsigma\imath_1(\text{ß}) + (1-\varsigma)\imath_2(\text{ß}),\ 0 \le \varsigma \le 1 \tag{14}$$

in $\Sigma^*(\wp, \ell, \hbar, \tau, c)$. Since for $0 \le \varsigma \le 1$,

$$h(\text{ß}) = \frac{1}{\text{ß}} + \sum_{\zeta=1}^{\infty}[\varsigma\text{ọ}_{\zeta,1} + (1-\varsigma)\text{ọ}_{\zeta,1}]\text{ß}^{\zeta}. \tag{15}$$

We have, with the help of the 2.1 theorem,

$$\begin{aligned}
&\sum_{\zeta=1}^{\infty}[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)[\varsigma\text{ọ}_{\zeta,1}+(1-\varsigma)\text{ọ}_{\zeta,1}]\\
\le&\varsigma(1-\wp)(1-2\hbar)+(1-\varsigma)(1-\wp)(1-2\hbar)\\
=&(1-\wp)(1-2\hbar),
\end{aligned}$$

which implies that $h \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$. □

Theorem 4.3. *Let $\xi \ge 0$. Then $\Sigma^{*\xi}(\wp, \ell, \hbar, \tau, c) \subseteq N(\ell, \xi)$, where*

$$\xi = 1 - \frac{2(1-\wp)(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)+(1-\wp)(1-2\hbar)}. \tag{16}$$

Proof. If $\imath \in \Sigma^{*\xi}(\wp, \ell, \hbar, \tau, c)$ then

$$\sum_{\zeta=1}^{\infty}\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\text{ọ}_\zeta \le 1.$$

The value of ξ needs to be found to be such that

$$\sum_{\zeta=1}^{\infty}\frac{[\zeta(1+\ell)+(\ell+\xi)]\phi_\zeta(\tau,c)}{1-\xi}\text{ọ}_\zeta \le 1.$$

It is thus enough to demonstrate that

$$\frac{[\zeta(1+\ell)+(\ell+\xi)]\phi_\zeta(\tau,c)}{1-\xi} \leq \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}.$$

Then

$$\xi \leq 1-\frac{(\zeta+1)(1-\wp)(1-2\hbar)(1+\ell)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]+(1-\wp)(1-2\hbar)}.$$

Since

$$D(\zeta)=1-\frac{(\zeta+1)(1-\wp)(1-2\hbar)(1+\ell)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]+(1-\wp)(1-2\hbar)}$$

is an increasing function of $\zeta,\ \zeta\geq 1$, we get

$$\xi \leq D(1)=1-\frac{2(1-\wp)(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)+(1-\wp)(1-2\hbar)}.$$

□

Theorem 4.4. *Let* $\imath_0(ß)=\frac{1}{ß}$ *and*

$$\imath_\zeta(ß)=\frac{1}{ß}+\sum_{\zeta=1}^{\infty}\frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}ß^\zeta,\ \zeta\geq 1. \tag{17}$$

Then $\imath\in\Sigma^*(\wp,\ell,\hbar,\tau,c) \Leftrightarrow$ *the form can be described*

$$\imath(ß)=\sum_{\zeta=0}^{\infty}\omega_\zeta \imath_\zeta(ß), \tag{18}$$

where $\omega_\zeta\geq 0$ *and* $\sum\limits_{\zeta=0}^{\infty}\omega_\zeta=1$.

Proof. Assume that

$$\begin{aligned}\imath(ß)&=\sum_{\zeta=0}^{\infty}\omega_\zeta \imath_\zeta(ß)\\ &=\frac{1}{ß}+\sum_{\zeta=1}^{\infty}\frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}ß^\zeta.\end{aligned}$$

Then it implies that

$$\begin{aligned}&\sum_{\zeta=1}^{\infty}\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\\ &\times\frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}ß^\zeta\\ &=\sum_{\zeta=1}^{\infty}\omega_\zeta=1-\omega_0\leq 1\end{aligned}$$

which implies that $\imath\in\Sigma^*(\wp,\ell,\hbar,\tau,c)$.

Make the assumption, on the other hand, that the $\imath$ function described by (1) is in the class $\imath \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$. Then

$$\varrho_\zeta \leq \frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau, c)}.$$

Setting

$$\omega_\zeta = \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau, c)}{(1-\wp)(1-2\hbar)}\varrho_\zeta,$$

where

$$\omega_0 = 1 - \sum_{\zeta=0}^{\infty}\omega_\zeta.$$

It can be said that $\imath$ can be expressed as (18). □

Corollary 4.5. *The class's extreme points $\Sigma^*(\wp, \ell, \hbar, \tau, c)$ are the functions $\imath_0(ß) = \frac{1}{ß}$ and*

$$\imath_\zeta(ß) = \frac{1}{ß} + \frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau, c)}ß^\zeta. \tag{19}$$

5. Modified Hadamard products

Let $\imath_j(j = 1, 2)$ indicate by (12). The Hadamard product of $\imath_1$ and $\imath_2$ is characterize by

$$(\imath_1 * \imath_2)(ß) = \frac{1}{ß} + \sum_{\zeta=1}^{\infty}\varrho_{\zeta,1}\varrho_{\zeta,2}ß^\zeta = (\imath_2 * \imath_1)(ß). \tag{20}$$

Theorem 5.1. *Let $\imath j \in \Sigma^*(\wp, \ell, \hbar, \tau, c), j = 1, 2$ be a function specified by (12). Then $\imath_1 * \imath_2 \in \Sigma^*(\wp, \ell, \hbar, \tau, c)$, where*

$$\varphi = 1 - \frac{2(1-\wp)^2(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)^2\phi_1(\tau, c)+(1-\wp)^2(1-2\hbar)}. \tag{21}$$

The estimate is sharp for the functions $\imath_j(j = 1, 2)$ provided by

$$\imath_j(ß) = \frac{1}{ß} + \frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau, c)}ß, \ (j = 1, 2). \tag{22}$$

Proof. Using the approach previously used by Schild and Silverman[12],We ought to find a parameter called φ, which is the biggest real parameter, so that

$$\sum_{\zeta=1}^{\infty}\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\varphi)]\phi_\zeta(\tau, c)}{(1-\varphi)(1-2\hbar)}\varrho_{\zeta,1}\varrho_{\zeta,2} \leq 1. \tag{23}$$

Since $\imath_j \in \Sigma^*(\wp, \ell, \hbar, \tau, c), \ j = 1, 2$, we get

$$\sum_{\zeta=1}^{\infty}\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]\phi_\zeta(\tau, c)}{(1-\wp)(1-2\hbar)}\varrho_{\zeta,1} \leq 1$$

and

$$\sum_{\zeta=1}^{\infty} \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)} \underset{\cdot}{\mathrm{o}}_{\zeta,2} \leq 1.$$

By Cauchy- Schwarz inequality, we have

$$\sum_{\zeta=1}^{\infty} \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)} \sqrt{\underset{\cdot}{\mathrm{o}}_{\zeta,1}\underset{\cdot}{\mathrm{o}}_{\zeta,2}} \leq 1. \tag{24}$$

Then it is enough to demonstrate that

$$\sum_{\zeta=1}^{\infty} \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\varphi)]\phi_\zeta(\tau,c)}{(1-\varphi)(1-2\hbar)} \underset{\cdot}{\mathrm{o}}_{\zeta,1}\underset{\cdot}{\mathrm{o}}_{\zeta,2}$$
$$\leq \sum_{\zeta=1}^{\infty} \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)} \sqrt{\underset{\cdot}{\mathrm{o}}_{\zeta,1}\underset{\cdot}{\mathrm{o}}_{\zeta,2}}$$

or equivalently that

$$\sqrt{\underset{\cdot}{\mathrm{o}}_{\zeta,1}\underset{\cdot}{\mathrm{o}}_{\zeta,2}} \leq \frac{[\zeta(1+\ell)+(\ell+\wp](1-\varphi)}{[\zeta(1+\ell)+(\ell+\varphi](1-\wp)}.$$

Hence, it light of the inequality (24), it is enough to demonstrate that

$$\frac{(1-\wp)(1-2\hbar)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]\phi_\zeta(\tau,c)} \leq \frac{[\zeta(1+\ell)+(\ell+\wp](1-\varphi)}{[\zeta(1+\ell)+(\ell+\varphi](1-\wp)}. \tag{25}$$

From (25), it implements that

$$\varphi \leq 1 - \frac{(1-\wp)^2(1-2\hbar)(1+\ell)(\zeta+1)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]^2\phi_\zeta(\tau,c)+(1-\wp)^2(1-2\hbar)}.$$

Now $E(\zeta)$,is the function specified by

$$E(\zeta) = 1 - \frac{(1-\wp)^2(1-2\hbar)(1+\ell)(\zeta+1)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp]^2\phi_\zeta(\tau,c)+(1-\wp)^2(1-2\hbar)}.$$

We see that $E(\zeta)$ is an increasing of $\zeta,\ \zeta \geq 1$. Therefore, we conclude that

$$\varphi \leq E(\zeta) = 1 - \frac{2(1-\wp)^2(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)^2\phi_1(\tau,c)+(1-\wp)^2(1-2\hbar)},$$

hence the proof.

□

Utilizing considerations similar to those in the proof theorem of 5.1, we get the following theorem.

Theorem 5.2. *Let* $\imath_1 \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ *be a function described by* (12). *Suppose also that* $\imath_2 \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ *function described by* (12). *Then* $\imath_1 * \imath_2 \in \Sigma^*(\Upsilon,\wp,\ell,\hbar,\tau,c)$, *where*

$$\Upsilon = 1 - \frac{2(1-\wp)(1-\rho)(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)(2\ell+\rho+1)\phi_1(\tau,c)+(1-\wp)(1-\rho)(1-2\hbar)}. \tag{26}$$

The estimate is sharp for the functions $\imath_j (j = 1, 2)$ *given by*

$$\imath_1(\text{ß}) = \frac{1}{\text{ß}} + \frac{(1-\wp)(1-2\hbar)}{(2\ell+\wp+1)\phi_1(\tau,c)}\text{ß}$$

and

$$\imath_2(\text{ß}) = \frac{1}{\text{ß}} + \frac{(1-\rho)(1-2\hbar)}{(2\ell+\rho+1)\phi_1(\tau,c)}\text{ß}.$$

Theorem 5.3. *Let* $\imath_j \in \Sigma^*(\wp, \ell, \hbar, \tau, c), j = 1, 2$ *be a function specified by* (12). *Then the function*

$$h(\text{ß}) = \frac{1}{\text{ß}} + \sum_{\zeta=1}^{\infty}(\varrho_{\zeta,1}^2 + \varrho_{\zeta,2}^2)\text{ß}^{\zeta} \tag{27}$$

belongs to the class $\Sigma^*(\varepsilon, \wp, \ell, \hbar, \tau, c)$, *where*

$$\varepsilon = 1 - \frac{4(1-\wp)^2(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)^2\phi_1(\tau,c) + 2(1-\wp)^2(1-2\hbar)}. \tag{28}$$

The estimate is sharp for the functions $\imath_j (j = 1, 2)$ *given by* (22).

Proof. By Theorem 2.1, we get

$$\begin{aligned}&\sum_{\zeta=1}^{\infty}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\right\}^2 \varrho_{\zeta,1}^2 \\ \leq&\sum_{\zeta=1}^{\infty}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\varrho_{\zeta,1}\right\}^2 \leq 1\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\sum_{\zeta=1}^{\infty}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\right\}^2 \varrho_{\zeta,2}^2 \\ \leq&\sum_{\zeta=1}^{\infty}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\varrho_{\zeta,2}\right\}^2 \leq 1.\end{aligned} \tag{30}$$

It follows from (29) and (30) that

$$\sum_{\zeta=1}^{\infty}\frac{1}{2}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\right\}^2(\varrho_{\zeta,1}^2+\varrho_{\zeta,2}^2) \leq 1.$$

We need to find the biggest ε, so that

$$\begin{aligned}&\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\varepsilon)]\phi_\zeta(\tau,c)}{(1-\varepsilon)(1-2\hbar)} \\ \leq&\frac{1}{2}\left\{\frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)}\right\}^2,\end{aligned}$$

that is

$$\varepsilon \leq 1 - \frac{2(1-\wp)^2(1-2\hbar)(1+\ell)(\zeta+1)}{1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]^2\phi_\zeta(\tau,c)+2(1-\wp)^2(1-2\hbar)}.$$

Since

$$G(\zeta) = 1 - \frac{2(1-\wp)^2(1-2\hbar)(1+\ell)(\zeta+1)}{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]^2\phi_\zeta(\tau,c)+2(1-\wp)^2(1-2\hbar)}$$

is an increasing function of $\zeta,\ \zeta \geq 1$, we obtain

$$\varepsilon \leq G(1) = \frac{4(1-\wp)^2(1-2\hbar)(1+\ell)}{(2\ell+\wp+1)^2\phi_1(\tau,c)+2(1-\wp)^2(1-2\hbar)}$$

and hence the proof. $\square$

6. Integral operators

Theorem 6.1. *Let $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$ given by (1). Then the integral operator*

$$F(ß) = \nu \int_0^1 u^\nu \imath(uß)du,\ 0 < u \leq 1,\ \nu > 0 \tag{31}$$

is in the class $\Sigma^*(\wp,\ell,\hbar,\tau,c)$, where

$$\xi = 1 - \frac{2\nu(1-\wp)(1+\ell)}{(\nu+2)(2\ell+\wp+1)+\nu(1-\wp)}. \tag{32}$$

The estimate is sharp for the function $\imath$ provided by (10).

Proof. Let $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$. Then

$$\begin{aligned} F(ß) =& \nu \int_0^1 u^\nu \imath(uß)du \\ =& \frac{1}{ß} + \sum_{\zeta=1}^{\infty} \frac{\nu}{\zeta+\nu+1} ϙ_\zeta ß^\zeta. \end{aligned}$$

Thus it is to present that

$$\sum_{\zeta=1}^{\infty} \frac{\nu[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\xi)]\phi_\zeta(\tau,c)}{(\zeta+\nu+1)(1-\xi)(1-2\hbar)} ϙ_\zeta \leq 1. \tag{33}$$

Since $\imath \in \Sigma^*(\wp,\ell,\hbar,\tau,c)$, then

$$\sum_{\zeta=1}^{\infty} \frac{[1+\hbar(\zeta-1)][\zeta(1+\ell)+(\ell+\wp)]\phi_\zeta(\tau,c)}{(1-\wp)(1-2\hbar)} ϙ_\zeta \leq 1. \tag{34}$$

From (33) and (34), we have

$$\frac{[\zeta(1+\ell)+(\ell+\xi)]}{(\zeta+\nu+1)(1-\xi)} \leq \frac{[\zeta(1+\ell)+(\ell+\wp)]}{(1-\wp)}.$$

Then

$$\xi \leq 1 - \frac{\nu(1-\wp)(\zeta+1)(1+\ell)}{(\zeta+\nu+1)[\zeta(1+\ell)+(\ell+\wp)]+\nu(1-\wp)}.$$

Since

$$Y(\zeta) = 1 - \frac{\nu(1-\wp)(\zeta+1)(1+\ell)}{(\zeta+\nu+1)[\zeta(1+\ell)+(\ell+\wp)]+\nu(1-\wp)}$$

is an increasing function of $\zeta,\ \zeta \geq 1$, we attain

$$\xi \leq Y(1) = 1 - \frac{2\nu(1-\wp)(1+\ell)}{(\nu+2)(2\ell+\wp+1)+\nu(1-\wp)}$$

and hence the proof. □

7. Conclusions

This research has introduced study a new subclass of meromorphic functions associated with Bessels functions and studied some basic properties of geometric function theory. Accordingly, some results to coefficient estimates, distortion theorem, closure theorems, and Hadamard product results and integral transforms have also been considered, inviting future research for this field of study.

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