Korean J. Math. **31** (2023), No. 3, pp. 295–303
https://dx.doi.org/10.11568/kjm.2023.31.3.295

GEOMETRIC APPLICATIONS AND KINEMATICS OF UMBRELLA MATRICES

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Abstract. This paper introduces a novel method for obtaining umbrella matrices, which are defined as orthogonal matrices with row sums of one, using skew-symmetric matrices and Cayley's Formula. This method is presented for the first time in this paper. We also investigate the kinematic properties and applications of umbrella matrices, demonstrating their usefulness as a tool in geometry and kinematics. Our findings provide new insights into the connections between matrix theory and geometric applications.

1. Intoduction

Kinematics is an important field that deals with the representation of angular displacement of rigid bodies in n-dimensional Euclidean space using proper solid transformations. Orthogonal matrices, which are widely used in kinematics, correspond to a solid transformation in Euclidean space. For example, the motion of a robot arm can be expressed in a three-dimensional coordinate system. This motion can be divided into two components: rotational motion and translational motion. Rotational motions, such as the rotation of the arm around its axis, can be expressed using a rotation matrix. This matrix is an orthogonal matrix and contains information about the rotation of the arm. That is, mathematically, these matrices provide information about how vectors will move in space, and this information can be used for planning and controlling robotic movements(see [3, 4]).

Throughout this paper, we focus on orthogonal matrices whose row sums are one and whose determinant is +1. We refer to these matrices as "umbrella matrices" and show that they are a transformation that leaves the $S = \begin{bmatrix}1 & 1 \dots & 1\end{bmatrix}^T \in \mathbb{R}^n_1$ axis constant. It is known as follows that an orthogonal matrix is obtained from a skew-symmetric matrix in n-dimensional Euclidean space $\mathbb{E}^n$ with Cayley's Formula(see [1–4]):

Let be any $n \times n$ orthogonal matrix one of the characteristic values of which is not -1. Then A can always be expressed as

$$A = (I_n - B)^{-1}(I_n + B),$$

Received April 7, 2023. Revised August 29, 2023. Accepted September 1, 2023.
2010 Mathematics Subject Classification: 53A17, 70B15, 15B10.
Key words and phrases: Umbrella matrix, Orthogonal matrix, Darboux matrix, Infinitesimal motion.
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where B is as $n \times n$ skew-symmetric matrix.

We demonstrate that an umbrella matrix can be obtained by applying Cayley's Formula to a skew-symmetric matrix whose row sums are zero. This method is novel and provides a comprehensive approach for generating umbrella matrices. There are some previous studies on these matrices. In [10–12], the geometry of umbrella matrices was studied. In [11], umbrella matrices were obtained for a particular case of the curvature matrix along a curve on a hypersurface. Moreover, the relationship between the Darboux matrix of the obtained umbrella matrix and the curvature matrix obtained along the curve is also given(see [11]). Furthermore, It is also known that there are some studies related to the umbrella motion(see [5,6]).

In the present study, we investigate the kinematic properties of umbrella matrices and explore their applications in geometry and kinematics. In particular, we obtain umbrella surfaces for some orbital curves (Fig.1 and 2) and provide a characterization for infinitesimal umbrella matrices in the theorem (4.1). Additionally, we establish the relationship between the Darboux matrix of instantaneous motion and the skew-symmetric matrix. Our findings provide new insights into the connections between matrix theory and geometric applications, and further enrich the study of umbrella matrices. For special cases, the study in [11] was obtained.

2. PRELIMINARIES

In this section, we will give some necessary definitions. A is an orthogonal matrix, for which -1 is not an eigenvalue, may be written as

$$A = (I_n - B)^{-1}(I_n + B)$$

in which B is skew-symmetric matrix; *Cayley Formula.* [8]

Let A orthogonal matrix. If $AS = S$, then A is called an *umbrella matrix* , where $S = \begin{bmatrix}1 & 1 \dots & 1\end{bmatrix}^T \in \mathbb{R}^n_1$. [10]

Let y and x be the position vectors, represented by column matrices, of a point P in the fixed space Σ^n and the moving space Σ^n, respectively. A continuous series of displacements, given by

$$y = Ax + b$$

where the orthogonal matrix A and the translation vector b are functions of a parameter t which may be identified with the time, is called a motion. Now we consider the rotational motion, given by

$$y(t) = A(t)x.$$

If derivative is taken

$$\dot{y} = \dot{A}x$$

and

$$x = A^{-1}y.$$

Thus we obtain $\dot{y} = \dot{A}A^{-1}y$. The matrix

$$W = \dot{A}A^{-1}$$

is called the *angular velocity* matrix or the *umbrella motion* of the motion (Ref. [8]). In this sitation, let A be an umbrella matrix. The motion generated by the transformation

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

or

$$y = Ax + b$$

is called an *umbrella motion* in E^n, where $x, y \in \mathbb{R}_1^n$ and $b \in \mathbb{R}_1^n$. [6]
Finally, we will talk about the definition of infinitesimal transformation. An *infinitesimal linear transformation* is defined as a transformation whose matrix is

$$A = I_n + \varepsilon[b_{ij}]$$

where $[b_{ij}]$ skew-symmetric matrix and ε denotes an infinitesimal quantity of the first order. [9]

Teorem 2.1. *Let M be a hypersurface in $\mathbb{E}^n$ and α curve on M. The derivative formulas of the natural frame field $\{X_1, \ldots, X_n\}$ are*

$$\begin{aligned} D_{X_1}X_i &= X_i' = -k_{(i-1)g}X_{i-1} + k_{ig}X_{i+1} + II(X_1, X_i)X_n \\ D_{X_1}X_n &= -II(X_1, X_1)X_1 - II(X_1, X_2)X_2 - \ldots - II(X_1, X_{n-1})X_n \end{aligned}$$

where $1 \le i \le n-1$ and $k_{0g} = k_{(n-1)g} = 0$. In the matrix form, these derivative formulas become

$$\begin{bmatrix} X_1' \\ X_2' \\ \vdots \\ X_{n-1}' \\ X_n' \end{bmatrix} = \begin{bmatrix} 0 & k_{1g} & 0 & \ldots & 0 & 0 & II(X_1, X_1) \\ -k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 & II(X_1, X_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -k_{(n-2)g} & 0 & II(X_1, X_{n-1}) \\ -II(X_1, X_1) & \ldots & \ldots & \ldots & \ldots & -II(X_1, X_{n-1}) & 0 \end{bmatrix}$$

or simply

$$X' = K(x)X.$$

The matrix $K(x)$ is called the (higher) curvature matrix of the pair (α, M) . [7]

3. Obtaining Umbrella Matrices and Kinematics of Umbrella Matrices

In this section, we present a theorem that establishes the possibility of obtaining umbrella matrices using Cayley's formula from skew-symmetric matrices with zero row sums. Furthermore, we investigate the kinematics of these umbrella matrices with another theorem.

Teorem 3.1. *Let B be an skew-symmetric matrix of type $n \times n$. If row sums of the skew-symmetric matrix B are zero, then orthogonal matrix*

$$A = (I_n - B)^{-1}(I_n + B)$$

is an umbrella matrix.

Proof. Suppose that the row sums of the skew-symmetric matrix B be zero. Hence, $BS = 0$ where $S = \begin{bmatrix} 1 & 1 \ldots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$. Therefore, we can write that

$$(I_n - B)S = S. \tag{1}$$

Since $det(I_n - B) \neq 0$, we have

$$(I_n - B)^{-1}S = S. \tag{2}$$

Thus, from Eq. (1) and Eq. (2), we obtain

$$\begin{aligned} AS &= (I_n - B)^{-1}(I_n + B)S \\ &= (I_n - B)^{-1}[I_nS + BS] \\ &= (I_n - B)^{-1}S \\ &= S. \end{aligned}$$

So, A orthogonal matrix is an umbrella matrix. This completes the proof. □

Teorem 3.2. *Let A be the umbrella matrix. Then, the matrix A leaves the Sp{(1,...,1)} line and the $x_1 + x_2 + \ldots + x_n = 0$ hyperplane invariant.*

Proof. Since A is an umbrella matrix, we have

$$AS = S,$$

where $S = \begin{bmatrix} 1 & 1 \ldots & 1 \end{bmatrix}^T \in \mathbb{R}^n_1$. Hence, A leaves the $S = \begin{bmatrix} 1 & 1 \ldots & 1 \end{bmatrix}^T$ invariant. Now, suppose that $y = Ax$, where $y = (y_1, y_2, \ldots, y_n)^T$. Thus, we obtain,

$$x = A^T y. \tag{3}$$

Using the hyperplane $x_1 + x_2 + \ldots + x_n = 0$,we can write,

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0. \tag{4}$$

Considering Eq.(3) and Eq.(4), we have,

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} A^T y = 0$$

and since $\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} A^T = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} y = 0. \tag{5}$$

Consequently, Eq.(5) may be written as

$$y_1 + y_2 + \ldots + y_n = 0.$$

Therefore, the matrix A leaves the hyperplane $x_1 + x_2 + \ldots + x_n = 0$ invariant.
This completes the proof. □

Example 3.1. *Let's consider an skew-symmetric matrix K whose row sums are zero, as shown in the following*

$$K = \begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix}.$$

If we calculate the orthogonal matrix A corresponding to the skew-symmetric matrix K by Cayley's formula, we find that

$$A = \frac{1}{3}\begin{pmatrix} 0 & 0 & 3 & 0 \\ -1 & 2 & 0 & 2 \\ 2 & -1 & 0 & 2 \\ 2 & 2 & 0 & -1. \end{pmatrix}.$$

From theroem (3.1), the orthogonal matrix A is an umbrella matrix. In addition, the umbrella matrix A leaves invariant that the line $Sp\{(1,1,1,1)\}$ and the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ due to the Theorem (3.2).

Special Case. *Let M be a surface, and α be a curve on this surface. For K(x), which is the curvature matrix of the α curve, according to Theorem (3.1) which is given in [11], the A orthogonal matrix obtained as*

$$b_1 = b_2 = \ldots = b_n = c$$

is an umbrella matrix.

Example 3.2. *The curvature matrix of an α curve in 3-dimensional Euclidean space $\mathbb{E}^3$ has the form as*

$$K(x) = \begin{pmatrix} 0 & \kappa_g & -\kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & -\tau_g & 0 \end{pmatrix}.$$

If we take $\kappa_g = \kappa_n = \tau_g = u$, we get

$$K(x) = \begin{pmatrix} 0 & u & -u \\ -u & 0 & u \\ u & -u & 0 \end{pmatrix}$$

and by using Cayley's formula, we obtain the orthogonal matrix

$$A = \frac{1}{1+3u^2}\begin{pmatrix} 1-u^2 & 2(u^2+u) & 2(u^2-u) \\ 2(u^2-u) & 1-u^2 & 2(u^2+u) \\ 2(u^2+u) & 2(u^2-u) & 1-u^2 \end{pmatrix}$$

which is an umbrella matrix. [11]

Let us consider the umbrella matrix A with the orbit curve $\alpha(v) = (cosv, v, 0)$, then we obtain the following surface

$$H(u,v) = (\frac{(1-u^2)cosv + (2u^2+2u)v)}{1+3u^2}, \frac{(2u^2-2u)cosv + (1-u^2)v)}{1+3u^2}, \frac{(2u^2+2u)cosv + (2u^2-2u)v)}{1+3u^2}).$$

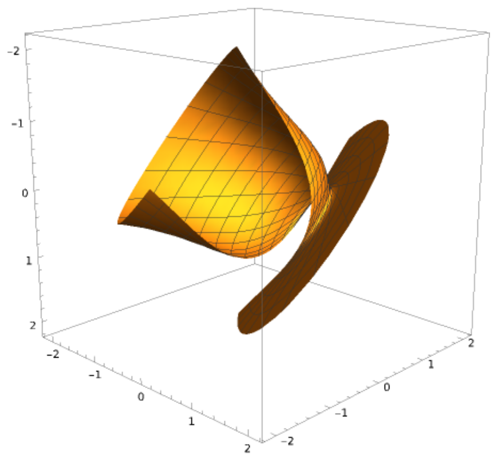

Fig. 1. H Umbrella Surface

The picture of the surface of H is rendered in Figure 1. In addition, considering the orbital curve $\beta(v) = (v^2, v, 0)$, we obtain the following surface

$$G(u,v) = (\frac{(1-u^2)v^2 + (2u^2+2u)v)}{1+3u^2}, \frac{(2u^2-2u)v^2 + (1-u^2)v)}{1+3u^2}, \frac{(2u^2+2u)v^2 + (2u^2-2u)v)}{1+3u^2}).$$

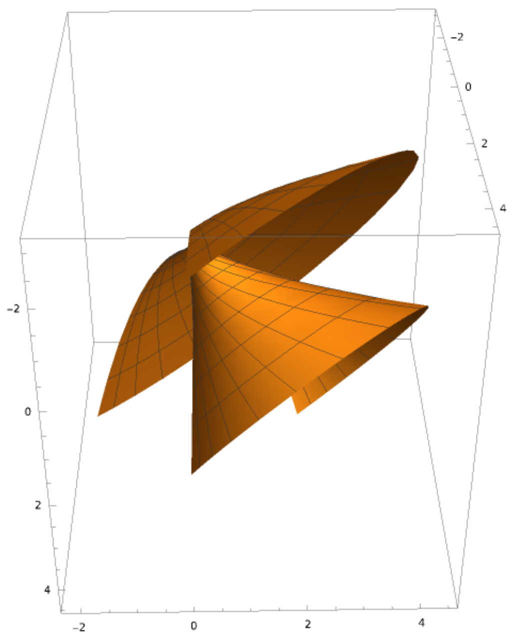

Fig. 2. G Umbrella Surface

The picture of the surface of H is rendered in Figure 2.

Let $A(t)$ be the umbrella matrix obtained by Cayley's formula from the skew-symmetric matrix B, the row sums of which are zero. For the Darboux matrix $W(A)$ of the 1-parameter umbrella motion which is defined by the matrix $A(t)$, we can give the relationship between B and $W(A)$ by the following theorem.

Teorem 3.3. *Let be the skew-symmetric matrix B, the row sums of which are zero, and the umbrella matrix corresponding to this matrix can be given by*

$$A = (I_n - B)^{-1}(I_n + B). \tag{6}$$

Then,

$$W(A) = 2(I_n - B)^{-1}B'(I_n + B),$$

where W(A) is the Darboux matrix of the umbrella motion.

Proof. Differentiating Eq.(6) with respect to t, we have

$$A' = [(I_n - B)^{-1}]'(I_n + B) + (I_n - B)^{-1}(I_n + B)'. \tag{7}$$

For $[(I_n - B)^-1]'$, by differantiating $(I_n - B)^{-1}(I_n - B) = I_n$ with respect to t, we have

$$[(I_n - B)^{-1}]'(I_n - B) + (I_n - B)^{-1}(I_n - B)' = 0. \tag{8}$$

Then, considering $(I_n - B)' = -B'$ with Eq.(8), we obtain

$$[(I_n - B)^-1]' = (I_n - B)^{-1}B'(I_n - B)^{-1}. \tag{9}$$

If we combine Eq.(9) with Eq.(7) this gives us,

$$\begin{aligned}
A' &= (I_n - B)^{-1}B'(I_n - B)^{-1}(I_n + B) + (I_n - B)^{-1}B' \\
&= (I_n - B)^{-1}B'[(I_n - B)^{-1}(I_n + B) + I_n] \\
&= (I_n - B)^{-1}B'[(I_n - B)^{-1}(I_n + B) + (I_n - B)^{-1}(I_n - B)] \\
&= (I_n - B)^{-1}B'(I_n - B)^{-1}[I_n + B + I_n - B] \\
&= 2(I_n - B)^{-1}B'(I_n - B)^{-1}.
\end{aligned}$$

Finally, considering $A' = 2(I_n - B)^{-1}B'(I_n - B)^{-1}$ with $W(A) = A'A^T$, we can write,

$$W(A) = 2(I_n - B)^{-1}B'(I_n + B)^{-1}.$$

This completes the proof. □

Corollory 3.1. *The Darboux matrix of the umbrella motion obtained by Cayley's formula from the skew-symmetric matrix B, the row sums of which are zero, can be obtained with the help of the skew-symmetric matrix B without calculating the umbrella matrix.*

Proposition 3.1. *The Darboux matrix obtained in Theorem*(3.3) *is a skew-symmetric matrix whose row sums are zero.*

Proof. Let W be obtained with the help of a skew-symmetric matrix B, the row sums of which are zero. Hence, we have

$$BS = 0, \tag{10}$$

where $S = \begin{bmatrix}1 & 1\dots & 1\end{bmatrix}^T \in \mathbb{R}_1^n$, then differantiating Eq.(10) with respect to t, we get

$$B'S = 0.$$

Therefore, we can write

$$\begin{aligned}
WS &= 2(I_n - B)^{-1}B'(I_n + B)^{-1}S \\
&= 2(I_n - B)^{-1}B'S \\
&= 0
\end{aligned}$$

Thus, the Darboux matrix W is a skew-symmetric matrix whose row sums are zero. □

EXAMPLE 3.3. *Let umbrella matrix A obtained by Cayley's formula from the skew-symmetric matrix B whose row sums are zero, as shown in the following*

$$B = \begin{pmatrix} 0 & 1 & -1 & t & -t \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -t & t \\ -t & 0 & t & 0 & 0 \\ t & 0 & -t & 0 & 0 \end{pmatrix}.$$

The Darboux matrix of this umbrella motion W(A) according to Theorem(3.3), *can be written as,*

$$W(A) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{t^2+1} & -\frac{1}{t^2+1} \\ 0 & 0 & 0 & -\frac{1}{t^2+1} & \frac{1}{t^2+1} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{t^2+1} & \frac{1}{t^2+1} & 0 & 0 & 0 \\ \frac{1}{t^2+1} & -\frac{1}{t^2+1} & 0 & 0 & 0 \end{pmatrix}.$$

4. Infinitisemal Motions and Umbrella Matrices

The infinitesimal matrix A can be given as,

$$A = I_n + \varepsilon B$$

where B is a skew-symmetric matrix and ε is an infinitesimal quantity. By the following theorem, we will show that when B is a skew-symmetric matrix with zero row sums, the matrix A is an infinitesimal umbrella matrix.

TEOREM 4.1. *Let B be a skew-symmetric matrix with row sums are zero. Then,*

$$A = I_n + \varepsilon B$$

is an infinitesimal matrix, where ε is infinitesimal quantity.

Proof. At first, we must show the orthogonality of the matrix A. Thus, we need to prove the following equation

$$AA^T = I_n. \tag{11}$$

For the left side of Eq.(11), we can write,

$$\begin{aligned} (I_n + \varepsilon B)(I_n - \varepsilon B) &= I_n^2 - \varepsilon^2 B^2 \\ &= I_n. \end{aligned}$$

Hence, the Eq.(11) is proved. Secondly, we must show following equation,

$$AS = S$$

where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$. Therefore, considering $BS = 0$ with the matrix A, we get

$$\begin{aligned} (I_n + \varepsilon B).S &= I_n S + \varepsilon BS \\ &= S. \end{aligned}$$

Consequently, the infinitesimal matrix A is an umbrella matrix. □

5. Conclusion

In this study, we have given a method for obtaining umbrella matrices from skew-symmetric matrices with row sums of zero. Furthermore, we have also shown that the Darboux matrix can be obtained with a 1-parameter umbrella motion. One theorem is also given for the infinitisemal motion. The Lie group structures and Lie algebra applications of the umbrella motion will be discussed in a further study.

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