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CANAL HYPERSURFACES GENERATED BY NON-NULL CURVES IN LORENTZ-MINKOWSKI 4-SPACE

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Abstract. In the present paper, firstly we obtain the general expression of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in E_1^4 and give their some geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures. Also, we give some results about their flatness and minimality conditions and Weingarten canal hypersurfaces. Also, we obtain these characterizations for tubular hypersurfaces in E_1^4 by taking constant radius function and finally, we construct some examples and visualize them with the aid of Mathematica.

1. General information and basic concepts

Canal surfaces, which are the class of surfaces formed by sweeping a sphere, have been investigated by Monge in 1850. Thus, a canal surface can be seen as the envelope of a moving sphere with varying radius, defined by the trajectory $\beta(s)$ of its centers and a radius function $r(s)$. If the radius function is constant, then the canal surface is called a tubular surface or pipe surface. Canal surfaces (especially tubular surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and they are useful to represent various objects such as pipe, hose, rope or intestine of a body [13,28]. In this context, canal and tubular (hyper)surfaces have been studied by many mathematicians in different three, four or higher dimensional spaces. For instance, in Euclidean 3-space, some interesting and important relations about Gaussian curvature, mean curvature and second Gaussian curvature of canal surface have been found and also principal curvatures and principal curvature lines on canal surfaces have been determined in [18] and [9], respectively. In Minkowski 3-space, some different

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geometric characterizations of canal surfaces which are obtained as the envelope of a family of pseudospheres, pseudo hyperbolic spheres or lightlike cones have been obtained in [8], [26] and [28]. Furthermore, the general expressions of canal hypersurfaces in Euclidean n-space have been given and especially canal hypersurfaces in Euclidean 4-space have been studied in [17]. For further studies about canal and tubular (hyper)surfaces in different spaces, we refer to [1, 5, 7–9, 11, 12, 14–16, 20, 21, 24–28, 30, 31], and etc.

Since the extrinsic differential geometry of submanifolds in Lorentz-Minkowski 4-space E_1^4 is of special interest in Relativity Theory, lots of studies about curves and (hyper)surfaces have been done in this space and this motivated us to construct the canal hypersurfaces in E_1^4. Now, let us recall some basic concepts for curves and hypersurfaces in E_1^4.

If $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ and $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ are three vectors in E_1^4, then the inner product and vector product are defined by

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 \tag{1}$$

and

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}, \tag{2}$$

respectively. Here e_i $(i = 1, 2, 3, 4)$ are standard basis vectors.

A vector $\overrightarrow{x} \in E_1^4 - \{0\}$ is called spacelike if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle > 0$; timelike if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle < 0$ and lightlike (null) if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle = 0$. In particular, the vector $\overrightarrow{x} = 0$ is spacelike. Also, the norm of the vector $\overrightarrow{x}$ is $\|\overrightarrow{x}\| = \sqrt{|\langle \overrightarrow{x}, \overrightarrow{x} \rangle|}$. A curve $\beta(s)$ in E_1^4 is spacelike, timelike or lightlike (null) if all its velocity vectors $\beta'(s)$ are spacelike, timelike or lightlike, respectively, and a non-null (i.e., timelike or spacelike) curve has unit speed if $\langle \beta', \beta' \rangle = \mp 1$ [22].

If $\{F_1, F_2, F_3, F_4\}$ is the moving Frenet frame along the timelike or spacelike curve $\beta(s)$ in E_1^4, where we will call F_1, F_2, F_3 and F_4 are unit tangent vector field, principal normal vector field, binormal vector field and trinormal vector field, respectively, then the Frenet equations can be given according to the causal characters of non-null Frenet vector fields F_1, F_2, F_3 and F_4 as follows [29]:

If the curve $\beta(s)$ is timelike, i.e., $\langle F_1, F_1 \rangle = -1$, $\langle F_2, F_2 \rangle = \langle F_3, F_3 \rangle = \langle F_4, F_4 \rangle = 1$, then

$$\begin{bmatrix} F_1' \\ F_2' \\ F_3' \\ F_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \tag{3}$$

if the curve $\beta(s)$ is spacelike with timelike principal normal vector field F_2, i.e., $\langle F_2, F_2\rangle = -1$, $\langle F_1, F_1\rangle = \langle F_3, F_3\rangle = \langle F_4, F_4\rangle = 1$, then

$$\begin{bmatrix} F_1' \\ F_2' \\ F_3' \\ F_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \tag{4}$$

if the curve $\beta(s)$ is spacelike with timelike binormal vector field F_3, i.e., $\langle F_3, F_3\rangle = -1$, $\langle F_1, F_1\rangle = \langle F_2, F_2\rangle = \langle F_4, F_4\rangle = 1$, then

$$\begin{bmatrix} F_1' \\ F_2' \\ F_3' \\ F_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \tag{5}$$

if the curve $\beta(s)$ is spacelike with timelike trinormal vector field F_4, i.e., $\langle F_4, F_4\rangle = -1$, $\langle F_1, F_1\rangle = \langle F_2, F_2\rangle = \langle F_3, F_3\rangle = 1$, then

$$\begin{bmatrix} F_1' \\ F_2' \\ F_3' \\ F_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}, \tag{6}$$

where k_1, k_2, k_3 are the first, second and third curvatures of the non-null curve $\beta(s)$.

Furthermore, if p is a fixed point in E_1^4 and r is a positive constant, then the pseudo-Riemannian hypersphere is defined by

$$S_1^3(p, r) = \{x \in E_1^4 : \langle x - p, x - p\rangle = r^2\},$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_0^3(p, r) = \{x \in E_1^4 : \langle x - p, x - p\rangle = -r^2\},$$

the pseudo-Riemannian null hypercone is defined by

$$Q_1^3 = \{x \in E_1^4 : \langle x - p, x - p\rangle = 0\}.$$

In the present study, we construct the canal hypersurfaces in E_1^4 as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres or null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields.

On the other hand, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has been a popular topic for geometers, recently ([2–4,6,17,19], and etc.). In this context, let Γ be a hypersurface in E_1^4 given by $\Gamma : U \subset E^3 \to E_1^4$, $(u_1, u_2, u_3) \to \Gamma(u_1, u_2, u_3) = (\Gamma_1(u_1, u_2, u_3), \Gamma_2(u_1, u_2, u_3), \Gamma_3(u_1, u_2, u_3), \Gamma_4(u_1, u_2, u_3))$. Then the Gauss map (i.e., the unit normal vector field), the matrix forms of the first and second fundamental forms are

$$N_\Gamma = \frac{\Gamma_{u_1} \times \Gamma_{u_2} \times \Gamma_{u_3}}{\|\Gamma_{u_1} \times \Gamma_{u_2} \times \Gamma_{u_3}\|}, \tag{7}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{8}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{9}$$

respectively. Here $g_{ij} = \left\langle \Gamma_{u_i}, \Gamma_{u_j} \right\rangle$, $h_{ij} = \left\langle \Gamma_{u_i u_j}, N_\Gamma \right\rangle$, $\Gamma_{u_i} = \frac{\partial \Gamma}{\partial u_i}$, $\Gamma_{u_i u_j} = \frac{\partial^2 \Gamma}{\partial u_i u_j}$, $i, j \in \{1, 2, 3\}$.

Also, the matrix of shape operator of the hypersurface Γ is

$$S = [g_{ij}]^{-1} \cdot [h_{ij}], \tag{10}$$

where $[g_{ij}]^{-1}$ is the inverse matrix of $[g_{ij}]$.

With the aid of (8)-(10), the Gaussian curvature and mean curvature of a hypersurface in E_1^4 are given by

$$K = \varepsilon \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{11}$$

and

$$3\varepsilon H = tr(S), \tag{12}$$

respectively. Here, $\varepsilon = \langle N_\Gamma, N_\Gamma \rangle$. We say that a hypersurface is flat or minimal, if it has zero Gaussian or zero mean curvature, respectively. For more details about hypersurfaces in E_1^4, we refer to [10, 23], and etc.

In Section 2, we obtain the general expression of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in E_1^4 and give some characterizations for them. In Section 3, we obtain these results for tubular hypersurfaces by taking constant radius function and in Section 4, we construct some examples and visualize them with the aid of Mathematica.

2. Canal hypersurfaces generated by non-null curves

In this section, firstly we construct the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in E_1^4. After that, we obtain some important geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures of these canal hypersurfaces in general form. Also, we give some results for flat, minimal and Weingarten canal hypersurfaces in E_1^4.

2.1. Construction of canal hypersurfaces

Here, we prove a theorem which gives us the general parametric expressions of 11 different types of canal hypersurfaces obtained by pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in E_1^4. Also, we write the parametric expressions of these canal hypersurfaces explicitly.

Theorem 2.1. *The canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a non-null curve $\beta(s)$ with non-null Frenet vector fields F_i in E_1^4 can be parametrized by*

$$\mathfrak{C}^{\{j;\lambda\}}(s,t,w) = \beta(s) - \lambda\varepsilon_1 r(s) r'(s) F_1(s) \mp r(s)\sqrt{r'(s)^2 - \lambda\varepsilon_1}\left(\textstyle\sum_{i=2}^{4}\mathfrak{a}_{ji}(s,t,w)F_i(s)\right), \tag{13}$$

where

(i) $g(F_j,F_j) = -1 = \varepsilon_j$ *and for* $i \neq j$, $g(F_i,F_i) = 1 = \varepsilon_i$, $i,j \in \{1,2,3,4\}$;

(ii) $$\begin{cases} \mathfrak{a}_{12}(s,t,w) = \cos t \cos w, \\ \mathfrak{a}_{13}(s,t,w) = \sin t \cos w, \\ \mathfrak{a}_{14}(s,t,w) = \sin w, \\ \mathfrak{a}_{22}(s,t,w) = \mathfrak{a}_{33}(s,t,w) = \mathfrak{a}_{44}(s,t,w) = \cosh t \cosh w, \\ \mathfrak{a}_{23}(s,t,w) = \mathfrak{a}_{34}(s,t,w) = \mathfrak{a}_{42}(s,t,w) = \sinh w, \\ \mathfrak{a}_{24}(s,t,w) = \mathfrak{a}_{32}(s,t,w) = \mathfrak{a}_{43}(s,t,w) = \sinh t \cosh w, \end{cases}$$

also, we suppose $r'(s)^2 > \lambda\varepsilon_1$ and if the canal hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ or $\lambda = -1$, respectively.

Furthermore, the canal hypersurfaces that are formed as the envelope of a family of null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields F_i in E_1^4 can be parametrized by

$$\mathfrak{C}^{\{j;0\}}(s,t,w) = \beta(s) \mp \left(\textstyle\sum_{i=2}^{4} a_i(s,t,w)F_i(s)\right), \quad j = 2,3,4, \tag{14}$$

where $a_i(s,t,w)$ are arbitrary functions satisfying

$$\textstyle\sum_{i=2}^{4}\varepsilon_i a_i^2(s,t,w) = 0.$$

It is obvious from this expression that, the canal hypersurface $\mathfrak{C}^{\{1;0\}}$ cannot be defined.

Here, the center curve $\beta(s)$ which generates the canal hypersurface is timelike or spacelike if $j = 1$ or $j = 2,3,4$, respectively.

Proof. Let $\beta : I \subseteq \mathbb{R} \to E_1^4$ be a spacelike or timelike center curve parametrized by arc-length with non-zero curvature and $g(F_j,F_j) = \varepsilon_j = -1$, where $F_1(s)$, $F_2(s)$, $F_3(s)$, $F_4(s)$ are unit tangent, principal normal, binormal and trinormal vectors of $\beta(s)$, respectively. Then, the parametrization of the envelope of pseudo hyperspheres ($\lambda = 1$) (resp. pseudo hyperbolic hyperspheres ($\lambda = -1$)

or null hypercones ($\lambda = 0$)) defining the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}$ in E_1^4 can be given by

$$\begin{aligned}\mathfrak{C}^{\{j;\lambda\}}(s,t,w) - \beta(s) = {} & a_1(s,t,w)F_1(s) + a_2(s,t,w)F_2(s) \\ & + a_3(s,t,w)F_3(s) + a_4(s,t,w)F_4(s),\end{aligned} \tag{15}$$

where $a_i(s,t,w)$ are differentiable functions of s,t,w on the interval I. Furthermore, since $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ lies on the pseudo hyperspheres (resp. pseudo hyperbolic hyperspheres or null hypercones), we have

$$g(\mathfrak{C}^{\{j;\lambda\}}(s,t,w) - \beta(s), \mathfrak{C}^{\{j;\lambda\}}(s,t,w) - \beta(s)) = \lambda r^2(s) \tag{16}$$

which leads to from (15) that

$$\varepsilon_1 a_1^2 + \varepsilon_2 a_2^2 + \varepsilon_3 a_3^2 + \varepsilon_4 a_4^2 = \lambda r^2 \tag{17}$$

and

$$\varepsilon_1 a_1 a_{1_s} + \varepsilon_2 a_2 a_{2_s} + \varepsilon_3 a_3 a_{3_s} + \varepsilon_4 a_4 a_{4_s} = \lambda r r_s, \tag{18}$$

where $r(s)$ is the radius function; $r = r(s)$, $r_s = \frac{dr(s)}{ds}$, $a_i = a_i(s,t,w)$, $a_{i_s} = \frac{\partial a_i(s,t,w)}{\partial s}$.

Also from (3)-(6), the non-null Frenet vectors $F_i(s)$, $i \in \{1,2,3,4\}$, of the spacelike or timelike curve $\beta(s)$ satisfy the relations

$$\begin{cases} F_1'(s) = k_1(s)F_2(s), \\ F_2'(s) = \varepsilon_3\varepsilon_4 k_1(s)F_1(s) + k_2(s)F_3(s), \\ F_3'(s) = \varepsilon_1\varepsilon_4 k_2(s)F_2(s) + k_3(s)F_4(s), \\ F_4'(s) = \varepsilon_1\varepsilon_2 k_3(s)F_3(s). \end{cases} \tag{19}$$

Here, $\beta(s)$ is a timelike curve if $\varepsilon_1 = -1$. Also, $\beta(s)$ is a spacelike curve with timelike principal normal F_2 or timelike binormal F_3 or timelike trinormal F_4 if $\varepsilon_2 = -1$ or $\varepsilon_3 = -1$ or $\varepsilon_4 = -1$, respectively, where $g(F_i, F_i) = \varepsilon_i$, $i \in \{1,2,3,4\}$.

So, differentiating (15) with respect to s and using the Frenet formula (19), we get

$$\begin{aligned}\mathfrak{C}_s^{\{j;\lambda\}} = {} & (1 + \varepsilon_3\varepsilon_4 a_2 k_1 + a_{1_s}) F_1 + (a_1 k_1 + \varepsilon_1\varepsilon_4 a_3 k_2 + a_{2_s}) F_2 \\ & + (a_2 k_2 + \varepsilon_1\varepsilon_2 a_4 k_3 + a_{3_s}) F_3 + (a_3 k_3 + a_{4_s}) F_4,\end{aligned} \tag{20}$$

where $\mathfrak{C}_s^{\{j;\lambda\}} = \frac{\partial \mathfrak{C}_s^{\{j;\lambda\}}(s,t,w)}{\partial s}$. Furthermore, $\mathfrak{C}^{\{j;\lambda\}}(s,t,w) - \beta(s)$ is a normal vector to the canal hypersurfaces, which implies that

$$g(\mathfrak{C}^{\{j;\lambda\}}(s,t,w) - \beta(s), \mathfrak{C}_s^{\{j;\lambda\}}(s,t,w)) = 0 \tag{21}$$

and so, from (15), (20) and (21) we have

$$\begin{aligned} & \varepsilon_1 a_1 (1 + \varepsilon_3\varepsilon_4 a_2 k_1 + a_{1_s}) + \varepsilon_2 a_2 (a_1 k_1 + \varepsilon_1\varepsilon_4 a_3 k_2 + a_{2_s}) \\ & + \varepsilon_3 a_3 (a_2 k_2 + \varepsilon_1\varepsilon_2 a_4 k_3 + a_{3_s}) + \varepsilon_4 a_4 (a_3 k_3 + a_{4_s}) = 0.\end{aligned} \tag{22}$$

Using (18) in (22), we get

$$\varepsilon_1 a_1+\varepsilon_1 a_1 a_{1_s}+\varepsilon_2 a_2 a_{2_s}+\varepsilon_3 a_3 a_{3_s}+\varepsilon_4 a_4 a_{4_s}=0 \tag{23}$$

and thus, from (18) and the definition of ε_i, we obtain

$$a_1=-\varepsilon_1 \lambda r r_s. \tag{24}$$

Hence, using (24) in (17), we reach that

$$\varepsilon_1(\varepsilon_2 a_2^2+\varepsilon_3 a_3^2+\varepsilon_4 a_4^2)=-r^2(-\varepsilon_1\lambda+\lambda^2 r_s^2). \tag{25}$$

Here from (25);

(i) if $\lambda=1$ or $\lambda=-1$, then the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a non-null curve in E_1^4 can be parametrized by (13);

(ii) if $\lambda=0$, then the canal hypersurfaces $\mathfrak{C}^{\{j;0\}}$ that are formed as the envelope of a family of null hypercones whose centers lie on a non-null curve in E_1^4 can be parametrized by (14)

and this completes the proof. □

Thus, the explicit parametric expressions of the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ and $\mathfrak{C}^{\{j;0\}}(s,t,w)$ from Theorem 2.1 are as follows:

$$\left\{\begin{array}{l}
\mathfrak{C}^{\{1;1\}}=\beta+rr'F_1\mp r\sqrt{r'^2+1}\left(\cos t\cos wF_2+\sin t\cos wF_3+\sin wF_4\right),\\
\mathfrak{C}^{\{1;-1\}}=\beta-rr'F_1\mp r\sqrt{r'^2-1}\left(\cos t\cos wF_2+\sin t\cos wF_3+\sin wF_4\right),\\
\mathfrak{C}^{\{2;1\}}=\beta-rr'F_1\mp r\sqrt{r'^2-1}\left(\cosh t\cosh wF_2+\sinh wF_3+\sinh t\cosh wF_4\right),\\
\mathfrak{C}^{\{2;-1\}}=\beta+rr'F_1\mp r\sqrt{r'^2+1}\left(\cosh t\cosh wF_2+\sinh wF_3+\sinh t\cosh wF_4\right),\\
\mathfrak{C}^{\{3;1\}}=\beta-rr'F_1\mp r\sqrt{r'^2-1}\left(\sinh t\cosh wF_2+\cosh t\cosh wF_3+\sinh wF_4\right),\\
\mathfrak{C}^{\{3;-1\}}=\beta+rr'F_1\mp r\sqrt{r'^2+1}\left(\sinh t\cosh wF_2+\cosh t\cosh wF_3+\sinh wF_4\right),\\
\mathfrak{C}^{\{4;1\}}=\beta-rr'F_1\mp r\sqrt{r'^2-1}\left(\sinh wF_2+\sinh t\cosh wF_3+\cosh t\cosh wF_4\right),\\
\mathfrak{C}^{\{4;-1\}}=\beta+rr'F_1\mp r\sqrt{r'^2+1}\left(\sinh wF_2+\sinh t\cosh wF_3+\cosh t\cosh wF_4\right)
\end{array}\right. \tag{26}$$

and

$$\left\{\begin{array}{l}
\mathfrak{C}^{\{2;0\}}=\beta\mp\sqrt{a_3^2+a_4^2}F_2+a_3F_3+a_4F_4,\\
\mathfrak{C}^{\{3;0\}}=\beta+a_2F_2\mp\sqrt{a_2^2+a_4^2}F_3+a_4F_4,\\
\mathfrak{C}^{\{4;0\}}=\beta+a_2F_2+a_3F_3\mp\sqrt{a_2^2+a_3^2}F_4.
\end{array}\right. \tag{27}$$

Remark 2.2. As one can see in Theorem 2.1, we deal with the canal hypersurfaces satisfying $r'(s)^2>\lambda\varepsilon_1$. If we assume that $r'(s)^2<\lambda\varepsilon_1$, then the canal hypersurfaces $\mathfrak{C}^{\{2;1\}}$, $\mathfrak{C}^{\{3;1\}}$ and $\mathfrak{C}^{\{4;1\}}$ can be rewritten as

$$\left\{\begin{array}{l}
\mathfrak{C}^{\{2;1\}}=\beta-rr'F_1\mp r\sqrt{1-r'^2}\left(\cosh t\sinh wF_2+\cosh wF_3+\sinh t\sinh wF_4\right),\\
\mathfrak{C}^{\{3;1\}}=\beta-rr'F_1\mp r\sqrt{1-r'^2}\left(\sinh t\sinh wF_2+\cosh t\sinh wF_3+\cosh wF_4\right),\\
\mathfrak{C}^{\{4;1\}}=\beta-rr'F_1\mp r\sqrt{1-r'^2}\left(\cosh wF_2+\sinh t\sinh wF_3+\cosh t\sinh wF_4\right).
\end{array}\right. \tag{28}$$

2.2. Curvatures of canal hypersurfaces

At the beginning of this subsection, we must note that we'll obtain the following results by taking $r'(s)^2 > \lambda\varepsilon_1$ and "$\mp$" which is in (13) as "$+$". Similar characterizations can be obtained by taking $r'(s)^2 > \lambda\varepsilon_1$ and "$\mp$" as "$-$" (or $r'(s)^2 < \lambda\varepsilon_1$ and "$+$" or "$-$").

Here, we'll obtain some important geometric invariants of these canal hypersurfaces by proving the following theorem:

Theorem 2.3. *The unit normal vector fields* $N^{\{j;\lambda\}}$*, Gaussian curvatures* $K^{\{j;\lambda\}}$*, mean curvatures* $H^{\{j;\lambda\}}$ *and principal curvatures* $\mu_i^{\{j;\lambda\}}$*,* $i \in \{1,2,3\}$*, of the canal hypersurfaces* $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$*, given by* (13) *in* E_1^4*, are*

$$N^{\{j,\lambda\}} = -\varepsilon_3\varepsilon_4\lambda^j\left(-\lambda\varepsilon_1 r' F_1(s) + \sqrt{r'^2-\lambda\varepsilon_1}\left(\textstyle\sum_{i=2}^4 \mathfrak{a}_{ji}(s,t,w)F_i(s)\right)\right), \tag{29}$$

$$K^{\{j,\lambda\}} = \tfrac{\varepsilon_3\varepsilon_4\lambda^{j+1}P}{r^2Q}, \tag{30}$$

$$H^{\{j,\lambda\}} = \tfrac{\varepsilon_3\varepsilon_4\lambda^{j+1}(2Q+rP)}{3rQ}, \tag{31}$$

$$\mu_1^{\{j,\lambda\}} = \mu_2^{\{j,\lambda\}} = \tfrac{\varepsilon_3\varepsilon_4\lambda^j}{r}, \quad \mu_3^{\{j,\lambda\}} = \tfrac{\varepsilon_3\varepsilon_4\lambda^j P}{Q}, \tag{32}$$

where

$$\begin{aligned} P &= rk_1^2 f_j^2\left(r'^2-\lambda\varepsilon_1\right) + r''\left(r'^2-\lambda\varepsilon_1+rr''\right) \\ &\quad + \varepsilon_2\lambda k_1 f_j \sqrt{r'^2-\lambda\varepsilon_1}''\left(r'^2-\lambda\varepsilon_1+2rr''\right), \\ Q &= \left(r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda r k_1 f_j\sqrt{r'^2-\lambda\varepsilon_1}+rr''\right)^2 \end{aligned}$$

and

$$f_j = \begin{cases} \cos t\cos w; & \text{if } j=1, \\ \cosh t\cosh w; & \text{if } j=2, \\ \sinh t\cosh w; & \text{if } j=3, \\ \sinh w; & \text{if } j=4. \end{cases} \tag{33}$$

Proof. Firstly, with the aid of the first derivatives of (26) according to s, t and w, we obtain the normals of the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ from (7) as

$$\begin{cases} N^{\{1;1\}} = -r'(s)F_1(s) - \sqrt{r'(s)^2+1}\,(\cos t\cos wF_2 + \sin t\cos wF_3 + \sin wF_4), \\ N^{\{1;-1\}} = -r'(s)F_1(s) + \sqrt{r'(s)^2-1}\,(\cos t\cos wF_2 + \sin t\cos wF_3 + \sin wF_4), \\ N^{\{2;1\}} = r'(s)F_1(s) - \sqrt{r'(s)^2-1}\,(\cosh t\cosh wF_2 + \sinh wF_3 + \sinh t\cosh wF_4), \\ N^{\{2;-1\}} = -r'(s)F_1(s) - \sqrt{r'(s)^2+1}\,(\cosh t\cosh wF_2 + \sinh wF_3 + \sinh t\cosh wF_4), \\ N^{\{3;1\}} = -r'(s)F_1(s) + \sqrt{r'(s)^2-1}\,(\sinh t\cosh wF_2 + \cosh t\cosh wF_3 + \sinh wF_4), \\ N^{\{3;-1\}} = -\,r'(s)F_1(s) - \sqrt{r'(s)^2+1}\,(\sinh t\cosh wF_2 + \cosh t\cosh wF_3 + \sinh wF_4), \\ N^{\{4;1\}} = -r'(s)F_1(s) + \sqrt{r'(s)^2-1}\,(\sinh wF_2 + \sinh t\cosh wF_3 + \cosh t\cosh wF_4), \\ N^{\{4;-1\}} = r'(s)F_1(s) + \sqrt{r'(s)^2+1}\,(\sinh wF_2 + \sinh t\cosh wF_3 + \cosh t\cosh wF_4) \end{cases} \tag{34}$$

and so, we can write (29).

Now, the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathfrak{C}^{\{1;\lambda\}}$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a timelike curve in E_1^4 are

$$(35)\quad \left\{ \begin{array}{l} g_{11}^{\{1;\lambda\}} = \frac{r'^2+\lambda}{4} \left(r^2 \left(\begin{array}{l} 4k_2^2 \cos^2 w - 4k_2 k_3 \cos t \sin 2w \\ - \left(2\cos 2t \cos^2 w + \cos 2w - 3\right) k_3^2 \end{array} \right) - 4\lambda \right) \\ \qquad -\lambda r^2 k_1^2 \left(\cos^2 t \cos^2 w + \left(\lambda \cos^2 t \cos^2 w - \lambda\right) r'^2\right) - 2\lambda r r'' - \frac{\lambda r^2 r''^2}{\lambda + r'^2} \\ \qquad - \frac{2\lambda k_1 r \cos w}{\sqrt{\lambda + r'^2}} \left((\lambda k_2 r r' \cos t \sin t) \left(r'^2 + \lambda\right) + \lambda r r'' \cos t \right), \\ g_{12}^{\{1;\lambda\}} = g_{21}^{\{1;\lambda\}} = r^2 \left(\begin{array}{l} k_2 \left(r'^2 + \lambda\right) \cos w - \lambda k_1 r' \sqrt{r'^2 + \lambda} \sin t \\ -k_3 \left(r'^2 + \lambda\right) \cos t \sin w \end{array} \right) \cos w, \\ g_{13}^{\{1;\lambda\}} = g_{31}^{\{1;\lambda\}} = r^2 \left(k_3 \left(r'^2 + \lambda\right) \sin t - \lambda k_1 r' \sqrt{r'^2 + \lambda} \cos t \sin w\right), \\ g_{22}^{\{1;\lambda\}} = \left(r'^2 + \lambda\right) r^2 \cos^2 w, \;\; g_{23}^{\{1;\lambda\}} = g_{32}^{\{1;\lambda\}} = 0, \;\; g_{33}^{\{1;\lambda\}} = \left(r'^2 + \lambda\right) r^2; \end{array} \right.$$

$$(36)\quad \left\{ \begin{array}{l} h_{11}^{\{1;\lambda\}} = \frac{\lambda r \left(r'^2+\lambda\right)}{4} \left(\begin{array}{l} 4k_2^2 \cos^2 w - \left(\cos(2t) + 2\cos^2 t \cos(2w) - 3\right) k_3^2 \\ -4k_2 k_3 \cos t \sin(2w) \end{array} \right) \\ \qquad -\lambda k_1^2 r \left(\lambda \cos^2 t \cos^2 w + \left(\cos^2 t \cos^2 w - 1\right) r'^2\right) - r'' - \frac{r r''^2}{\lambda + r'^2} \\ \qquad - \frac{\lambda k_1 \cos w}{\sqrt{\lambda + r'^2}} \left((\cos t + 2\lambda k_2 r r' \sin t) (r'^2 + \lambda) + 2 r r'' \cos t \right), \\ h_{12}^{\{1;\lambda\}} = h_{21}^{\{1;\lambda\}} = \lambda r \left(\begin{array}{l} \cos w k_2 \left(r'^2 + \lambda\right) - \lambda k_1 \sin t r' \sqrt{r'^2 + \lambda} \\ - \cos t k_3 \sin w \left(r'^2 + \lambda\right) \end{array} \right) \cos w, \\ h_{13}^{\{1;\lambda\}} = h_{31}^{\{1;\lambda\}} = \lambda r \left(k_3 \left(r'^2 + \lambda\right) \sin t - \lambda k_1 r' \sqrt{r'^2 + \lambda} \cos t \sin w\right), \\ h_{22}^{\{1;\lambda\}} = \lambda r \left(r'^2 + \lambda\right) \cos^2 w, \;\; h_{23}^{\{1;\lambda\}} = h_{32}^{\{1;\lambda\}} = 0, \;\; h_{33}^{\{1;\lambda\}} = \lambda r \left(r'^2 + \lambda\right); \end{array} \right.$$

the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathfrak{C}^{\{2;\lambda\}}$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with the timelike principal normal vector in E_1^4 are

$$(37)\quad \left\{ \begin{array}{l} g_{11}^{\{2;\lambda\}} = \frac{1}{4} \left(4 - 4\lambda r'^2 + r^2 \left(r'^2 - \lambda\right) \left(\begin{array}{l} \left(\cosh 2t + 2\cosh^2 t \cosh 2w - 3\right) k_3^2 \\ + \left(\cosh 2t + 2\cosh 2w \sinh^2 t + 3\right) k_2^2 \\ -4k_2 k_3 \cosh^2 w \sinh 2t \end{array} \right) \right) \\ \qquad + r^2 k_1^2 \left(\left(\cosh^2 t \cosh^2 w - 1\right) r'^2 - \lambda \cosh^2 t \cosh^2 w \right) - 2\lambda r r'' - \frac{\lambda r^2 r''^2}{r'^2 - \lambda} \\ \qquad + \frac{2k_1 r}{\sqrt{r'^2 - \lambda}} \left((\cosh t \cosh w + \lambda k_2 r r' \sinh w) \left(r'^2 - \lambda\right) + r r'' \cosh t \cosh w \right), \\ g_{12}^{\{2;\lambda\}} = g_{21}^{\{2;\lambda\}} = r^2 \left(\begin{array}{l} \lambda \left(k_1 r' \sqrt{r'^2 - \lambda} - k_2 \lambda \left(r'^2 - \lambda\right) \sinh w\right) \sinh t \\ + k_3 \left(r'^2 - \lambda\right) \cosh t \sinh w \end{array} \right) \cosh w, \\ g_{13}^{\{2;\lambda\}} = g_{31}^{\{2;\lambda\}} = r^2 \left(\begin{array}{l} \lambda k_1 r' \sqrt{r'^2 - \lambda} \cosh t \sinh w \\ + k_2 \left(r'^2 - \lambda\right) \cosh t + k_3 \left(r'^2 - \lambda\right) \sinh t \end{array} \right), \\ g_{22}^{\{2;\lambda\}} = \left(r'^2 - \lambda\right) r^2 \cosh^2 w, \;\; g_{23}^{\{2;\lambda\}} = g_{32}^{\{2;\lambda\}} = 0, \;\; g_{33}^{\{2;\lambda\}} = r^2 \left(r'^2 - \lambda\right); \end{array} \right.$$

$$(38)\quad \begin{cases} h_{11}^{\{2;\lambda\}}=\frac{1}{4}\left(\begin{array}{l} r\left(r'^2-\lambda\right)\left(\begin{array}{l} \left(\cosh(2t)+2\cosh^2 t\cosh(2w)-3\right)k_3^2 \\ -4k_2k_3\cosh^2 w\sinh(2t) \\ +k_2^2\left(3+\cosh(2t)+2\sinh^2 t\cosh(2w)\right) \end{array}\right) \\ +k_1^2 r\left(\begin{array}{l} r'^2\left(\cosh(2t)+2\cosh^2 t\cosh(2w)-3\right) \\ -4\lambda\cosh^2 t\cosh^2 w \end{array}\right) \\ +\frac{4k_1}{\sqrt{r'^2-\lambda}}\left(\begin{array}{l} \left(\cosh t\cosh w+2\lambda k_2 rr'\sinh w\right)\left(r'^2-\lambda\right) \\ +2rr''\cosh t\cosh w \end{array}\right) \\ -4\lambda r''-\frac{4\lambda rr''^2}{r'^2-\lambda} \end{array}\right), \\ h_{12}^{\{2;\lambda\}}=h_{21}^{\{2;\lambda\}}=r\left(\begin{array}{l} \lambda k_1 r'\sqrt{r'^2-\lambda}\sinh t \\ +\left(r'^2-\lambda\right)\left(k_3\cosh t-k_2\sinh t\right)\sinh w \end{array}\right)\cosh w, \\ h_{13}^{\{2;\lambda\}}=h_{31}^{\{2;\lambda\}}=r\left(\begin{array}{l} \lambda k_1 r'\sqrt{r'^2-\lambda}\cosh t\sinh w \\ +\left(r'^2-\lambda\right)\left(k_2\cosh t-k_3\sinh t\right) \end{array}\right), \\ h_{22}^{\{2;\lambda\}}=r\left(r'^2-\lambda\right)\cosh^2 w,\quad h_{23}^{\{2;\lambda\}}=h_{32}^{\{2;\lambda\}}=0,\quad h_{33}^{\{2;\lambda\}}=r\left(r'^2-\lambda\right); \end{cases}$$

the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathfrak{C}^{\{3;\lambda\}}$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike center curve with the timelike binormal vector in E_1^4 are

$$(39)\quad \begin{cases} g_{11}^{\{3;\lambda\}}=\frac{r'^2-\lambda}{4}\left(r^2\left(\begin{array}{l} k_3^2\left(3+\cosh 2t+2\cosh 2w\sinh^2 t\right) \\ +4k_2^2\cosh^2 w-4k_2k_3\sinh t\sinh 2w \end{array}\right)-4\lambda\right) \\ \qquad +r^2k_1^2\left(r'^2\left(1+\sinh^2 t\cosh^2 w\right)-\lambda\sinh^2 t\cosh^2 w\right)-2\lambda rr''-\frac{\lambda r^2 r''^2}{r'^2-\lambda} \\ \qquad -\frac{2k_1 r\cosh w}{\sqrt{r'^2-\lambda}}\left(\left(r'^2-\lambda\right)\left(\lambda k_2 rr'\cosh t\sinh t\right)+rr''\sinh t\right), \\ g_{12}^{\{3;\lambda\}}=g_{21}^{\{3;\lambda\}}=r^2\left(\begin{array}{l} k_2\left(r'^2-\lambda\right)\cosh w-\lambda k_1 r'\sqrt{r'^2-\lambda}\cosh t \\ -k_3\left(r'^2-\lambda\right)\sinh t\sinh w \end{array}\right)\cosh^2 w, \\ g_{13}^{\{3;\lambda\}}=g_{31}^{\{3;\lambda\}}=r^2\left(k_3\left(r'^2-\lambda\right)\cosh t-\lambda k_1 r'\sqrt{r'^2-\lambda}\sinh t\sinh w\right), \\ g_{22}^{\{3;\lambda\}}=\left(r'^2-\lambda\right)r^2\cosh^2 w,\quad g_{23}^{\{3;\lambda\}}=g_{32}^{\{3;\lambda\}}=0,\quad g_{33}^{\{3;\lambda\}}=\left(r'^2-\lambda\right)r^2; \end{cases}$$

$$(40)\quad \begin{cases} h_{11}^{\{3;\lambda\}}=2k_1k_2rr'\sqrt{r'^2-\lambda}\cosh t\cosh w-\lambda k_2^2 r\left(r'^2-\lambda\right)\cosh^2 w \\ \qquad -\frac{\lambda k_3^2 r\left(r'^2-\lambda\right)}{4}\left(3+\cosh(2t)+2\sinh^2 t\cosh(2w)\right) \\ \qquad +\lambda k_2k_3 r\left(r'^2-\lambda\right)\sinh t\sinh(2w)+r''+\frac{rr''^2}{-\lambda+r'^2} \\ \qquad +\frac{k_1^2 r}{4}\left(4\cosh^2 w\sinh^2 t-\lambda r'^2\left(3+\cosh(2t)+2\sinh^2 t\cosh(2w)\right)\right) \\ \qquad +\frac{\lambda}{\sqrt{r'^2-\lambda}}\left(k_1\left(r'^2-\lambda+2rr''\right)\cosh w\sinh t\right), \\ h_{12}^{\{3;\lambda\}}=h_{21}^{\{3;\lambda\}}=r\left(\begin{array}{l} k_1 r'\sqrt{r'^2-\lambda}\cosh t \\ -\lambda\left(r'^2-\lambda\right)\left(k_2\cosh w+k_3\sinh t\sinh w\right) \end{array}\right)\cosh w, \\ h_{13}^{\{3;\lambda\}}=h_{31}^{\{3;\lambda\}}=r\left(k_1 r'\sqrt{r'^2-\lambda}\sinh t\sinh w-\lambda k_3\left(r'^2-\lambda\right)\cosh t\right), \\ h_{22}^{\{3;\lambda\}}=-\lambda r\left(r'^2-\lambda\right)\cosh^2 w,\quad h_{23}^{\{3;\lambda\}}=h_{32}^{\{3;\lambda\}}=0,\quad h_{33}^{\{3;\lambda\}}=-\lambda r\left(r'^2-\lambda\right) \end{cases}$$

and the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathfrak{C}^{\{4;\lambda\}}$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike center curve with the timelike trinormal vector in E_1^4 are

$$
(41)\quad \begin{cases}
g_{11}^{\{4;\lambda\}}=\frac{-\lambda+r'^2}{4}\left(r^2\left(\begin{array}{c} k_2^2\left(2\cosh 2t\cosh^2 w+\cosh 2w-3\right) \\ +4k_3^2\cosh^2 w+4k_2k_3\cosh t\sinh 2w \end{array}\right)-4\lambda\right) \\
\qquad +r^2k_1^2\left(r'^2\cosh^2 w-\lambda\sinh^2 w\right)-2\lambda rr''-\frac{\lambda r^2r''^2}{-\lambda+r'^2} \\
\qquad +\frac{2\lambda k_1 r}{\sqrt{r'^2-\lambda}}\left(k_2rr'\left(r'^2-\lambda\right)\cosh w\sinh t-\lambda\left(r'^2-\lambda+rr''\right)\sinh w\right), \\
g_{12}^{\{4;\lambda\}}=g_{21}^{\{4;\lambda\}}=r^2\left(r'^2-\lambda\right)\left(k_2\cosh t\sinh w+k_3\cosh w\right)\cosh w, \\
g_{13}^{\{4;\lambda\}}=g_{31}^{\{4;\lambda\}}=-r^2\left(\lambda k_1r'\sqrt{r'^2-\lambda}\cosh w+k_2\left(r'^2-\lambda\right)\sinh t\right), \\
g_{22}^{\{4;\lambda\}}=r^2\left(r'^2-\lambda\right)\cosh^2 w,\quad g_{23}^{\{4;\lambda\}}=g_{32}^{\{4;\lambda\}}=0,\quad g_{33}^{\{4;\lambda\}}=r^2\left(r'^2-\lambda\right)
\end{cases}
$$

$$
(42)\quad \begin{cases}
h_{11}^{\{4;\lambda\}}=\frac{-r\left(r'^2-\lambda\right)}{4}(k_2^2\left(\cosh(2t)+2\cosh^2 t\cosh(2w)-3\right) \\
\qquad +4k_3(k_3\cosh^2 w+k_2\cosh t\sinh(2w))) \\
\qquad +\lambda k_1^2 r\left(\sinh^2 w-\lambda r'^2\cosh^2 w\right)+\lambda r''+\frac{\lambda rr''^2}{r'^2-\lambda} \\
\qquad +\frac{k_1}{\sqrt{r'^2-\lambda}}\left(\left(\sinh w-2\lambda k_2rr'\sinh t\cosh w\right)\left(r'^2-\lambda\right)+2rr''\sinh w\right), \\
h_{12}^{\{4;\lambda\}}=h_{21}^{\{4;\lambda\}}=-r\left(k_3\cosh w+k_2\cosh t\sinh w\right)\left(r'^2-\lambda\right)\cosh w, \\
h_{13}^{\{4;\lambda\}}=h_{31}^{\{4;\lambda\}}=r\left(\lambda k_1r'\sqrt{r'^2-\lambda}\cosh w+k_2\left(r'^2-\lambda\right)\sinh t\right), \\
h_{22}^{\{4;\lambda\}}=-r\left(r'^2-\lambda\right)\cosh^2 w,\quad h_{23}^{\{4;\lambda\}}=h_{32}^{\{4;\lambda\}}=0,\quad h_{33}^{\{4;\lambda\}}=-r\left(r'^2-\lambda\right)
\end{cases}
$$

and these imply that

$$
(43)\quad \det g_{ij}^{\{j;\lambda\}}=-\lambda A^2r^4\left(r'^2-\lambda\varepsilon_1\right)\left(r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda k_1f_jr\sqrt{r'^2-\lambda\varepsilon_1}+rr''\right)^2,
$$

$$
\begin{aligned}
(44)\quad & \det h_{ij}^{\{j;\lambda\}} \\
&= -\lambda A^2r^2\left(r'^2-\lambda\varepsilon_1\right)\left[\varepsilon_3\varepsilon_4\lambda^j f_j^2k_1^2r\left(r'^2-\lambda\varepsilon_1\right)\right. \\
&\quad \left.+\varepsilon_3\varepsilon_4\lambda^j r''\left(r'^2-\lambda\varepsilon_1+rr''\right)+\varepsilon_2\varepsilon_3\varepsilon_4\lambda^{j+1}f_jk_1\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+2rr''\right)\right],
\end{aligned}
$$

where $A=\cos w$ for $j=1$ and $A=\cosh w$ for $j=2,3,4$. Thus, from (11), (43) and (44), the Gaussian curvatures are obtained as (30). Here, obtaining the inverse of the first fundamental forms and using these and second fundamental forms in (10), we obtain the components of the shape operators of canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}$ as

$$
(45)\quad \begin{cases}
S_{11}^{\{1;\lambda\}}=\frac{\lambda k_1^2r\left(r'^2+\lambda\right)\cos^2 t\cos^2 w+\lambda r''\left(r'^2+\lambda+rr''\right)+k_1\sqrt{r'^2+\lambda}\left(r'^2+\lambda+2rr''\right)\cos t\cos w}{\left(r'^2+\lambda+r\left(\lambda k_1\sqrt{r'^2+\lambda}\cos t\cos w+r''\right)\right)^2}, \\
S_{21}^{\{1;\lambda\}}=\frac{-\lambda\sqrt{r'^2+\lambda}\left(k_1r'\sin t\sec w-\lambda k_2\sqrt{r'^2+\lambda}+\lambda k_3\sqrt{r'^2+\lambda}\cos t\tan w\right)\sec w}{r\left(k_1r\sqrt{r'^2+\lambda}\cos t+\lambda(r'^2+\lambda+rr'')\sec w\right)}, \\
S_{31}^{\{1;\lambda\}}=\frac{\lambda\left(\lambda k_3\left(r'^2+\lambda\right)\sin t-k_1r'\sqrt{r'^2+\lambda}\cos t\sin w\right)}{r\left(1+\lambda r'^2+r\left(k_1\sqrt{\lambda+r'^2}\cos t\cos w+\lambda r''\right)\right)}, \\
S_{22}^{\{1;\lambda\}}=S_{33}^{\{1;\lambda\}}=\frac{\lambda}{r}, \\
S_{12}^{\{1;\lambda\}}=S_{13}^{\{1;\lambda\}}=S_{23}^{\{1;\lambda\}}=S_{32}^{\{1;\lambda\}}=0;
\end{cases}
$$

$$(46)\quad \begin{cases} S_{11}^{\{2;\lambda\}}=\frac{k_1^2 r\left(r'^2-\lambda\right)\cosh^2 t\cosh^2 w+r''\left(r'^2-\lambda+rr''\right)-\lambda k_1\sqrt{r'^2-\lambda}\left(r'^2-\lambda+2rr''\right)\cosh t\cosh w}{\left(r'^2-\lambda+r\left(-\lambda k_1\sqrt{r'^2-\lambda}\cosh t\cosh w+r''\right)\right)^2}, \\ S_{21}^{\{2;\lambda\}}=\frac{\left(\lambda k_1 r'\sqrt{r'^2-\lambda}\sinh t\sec hw+(k_3\cosh t-k_2\sinh t)\sqrt{r'^2-\lambda}\tan w\right)\sec hw}{r\left((r'^2-\lambda+rr'')\sec hw-\lambda k_1 r\sqrt{r'^2-\lambda}\cosh t\right)}, \\ S_{31}^{\{2;\lambda\}}=\frac{-k_1 r'\sqrt{r'^2-\lambda}\cosh t\sinh w-\lambda k_2\left(r'^2-\lambda\right)\cosh t+\lambda k_3\left(r'^2-\lambda\right)\sinh t}{r\left(1-\lambda r'^2+r\left(k_1\sqrt{r'^2-\lambda}\cosh t\cosh w-\lambda r''\right)\right)}, \\ S_{22}^{\{2;\lambda\}}=S_{33}^{\{2;\lambda\}}=\frac{1}{r}, \\ S_{12}^{\{2;\lambda\}}=S_{13}^{\{2;\lambda\}}=S_{23}^{\{2;\lambda\}}=S_{32}^{\{2;\lambda\}}=0; \end{cases}$$

$$(47)\quad \begin{cases} S_{11}^{\{3;\lambda\}}=-\lambda\frac{k_1^2 r\left(r'^2-\lambda\right)\sinh^2 t\cosh^2 w+r''\left(r'^2-\lambda+rr''\right)+\lambda k_1\sqrt{r'^2-\lambda}\left(r'^2-\lambda+2rr''\right)\sinh t\cosh w}{\left(r'^2-\lambda+r\left(\lambda k_1\sqrt{r'^2-\lambda}\sinh t\cosh w+r''\right)\right)^2}, \\ S_{21}^{\{3;\lambda\}}=\frac{\sec hw\left(k_1 r'\sqrt{r'^2-\lambda}\cosh t\sec hw-\lambda(k_2-k_3\sinh t\tanh w)\left(r'^2-\lambda\right)\right)}{r\left(\lambda k_1 r\sqrt{r'^2-\lambda}\sinh t+(-\lambda+r'^2+rr'')\sec hw\right)}, \\ S_{31}^{\{3;\lambda\}}=\frac{k_1 r'\sqrt{r'^2-\lambda}\sinh t\sinh w-\lambda k_3\left(r'^2-\lambda\right)\cosh t}{r\left(r'^2-\lambda+r\left(\lambda k_1\sqrt{-\lambda+r'^2}\sinh t\cosh w+r''\right)\right)}, \\ S_{22}^{\{3;\lambda\}}=S_{33}^{\{3;\lambda\}}=-\frac{\lambda}{r}, \\ S_{12}^{\{3;\lambda\}}=S_{13}^{\{3;\lambda\}}=S_{23}^{\{3;\lambda\}}=S_{32}^{\{3;\lambda\}}=0; \end{cases}$$

$$(48)\quad \begin{cases} S_{11}^{\{4;\lambda\}}=\frac{-k_1^2 r\left(r'^2-\lambda\right)\sinh^2 w-r''\left(r'^2-\lambda+rr''\right)-\lambda k_1\sqrt{r'^2-\lambda}\left(r'^2-\lambda+2rr''\right)\sinh w}{\left(r'^2-\lambda+r\left(\lambda k_1\sqrt{-\lambda+r'^2}\sinh w+r''\right)\right)^2}, \\ S_{21}^{\{4;\lambda\}}=-\lambda\frac{(k_3+k_2\tanh w\cosh t)\left(r'^2-\lambda\right)}{r\left(\lambda r'^2-1+r\left(k_1\sqrt{r'^2-\lambda}\sinh w+\lambda r''\right)\right)}, \\ S_{31}^{\{4;\lambda\}}=\frac{k_1 r'\sqrt{r'^2-\lambda}\cosh w+\lambda k_2\left(r'^2-\lambda\right)\sinh t}{r\left(\lambda r'^2-1+r\left(k_1\sqrt{r'^2-\lambda}\sinh w+\lambda r''\right)\right)}, \\ S_{22}^{\{4;\lambda\}}=S_{33}^{\{4;\lambda\}}=-\frac{1}{r}, \\ S_{12}^{\{4;\lambda\}}=S_{13}^{\{4;\lambda\}}=S_{23}^{\{4;\lambda\}}=S_{32}^{\{4;\lambda\}}=0. \end{cases}$$

From (12) and (45)-(48), we obtain the mean curvatures of the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}$ as (31).

Finally, from (45)-(48), we get

$$(49)\quad \det(S-\mu I) = \left(\mu-\frac{\varepsilon_3\varepsilon_4\lambda^j}{r}\right)^2\left(\frac{\left(\begin{array}{c}\varepsilon_3\varepsilon_4\lambda^j\left(rk_1^2f_j^2\left(r'^2-\lambda\varepsilon_1\right)+r''\left(r'^2-\lambda\varepsilon_1+rr''\right)\right)\\ +\varepsilon_2\varepsilon_3\varepsilon_4\lambda^{j+1}k_1f_j\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+2rr''\right)\end{array}\right)}{\left(r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda rk_1f_j\sqrt{r'^2-\lambda\varepsilon_1}+rr''\right)^2}-\mu\right)$$

and solving the equation of $\det(S-\mu I)=0$ we have (32). So, the proof is completed. □

Here, from (34) or (43), we can state the following proposition:

Proposition 2.4. *The canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ in E_1^4 are spacelike or timelike if $\lambda=-1$ or $\lambda=1$, respectively.*

2.3. Some geometric characterizations for canal hypersurfaces

From (30) and (31), we can obtain the following important relation between the Gaussian and mean curvatures of the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}$:

Theorem 2.5. *The Gaussian and mean curvatures of the canal hypersurfaces* $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$*, given by* (13) *in* E_1^4*, satisfy*

$$3H^{\{j;\lambda\}}r - K^{\{j;\lambda\}}r^3 - 2\varepsilon_3\varepsilon_4\lambda^{j+1} = 0. \tag{50}$$

Theorem 2.6. *The canal hypersurfaces* $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$*, given by* (13) *in* E_1^4*, are flat if and only if* $k_1 = 0$ *and* $r(s) = as + b$*, where* $a, b \in \mathbb{R}$*,* $a \neq \mp 1$*.*

Proof. Firstly, let us suppose that the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ in E_1^4 are flat. Then from (30) we get

$$\begin{aligned} &rk_1^2\left(r'^2 - \lambda\varepsilon_1\right)f_j^2 + \varepsilon_2\lambda k_1\sqrt{r'^2 - \lambda\varepsilon_1}\left(r'^2 - \lambda\varepsilon_1 + 2rr''\right)f_j \\ &+ r''\left(r'^2 - \lambda\varepsilon_1 + rr''\right) = 0. \end{aligned} \tag{51}$$

Since $\{f_j, f_j^2, 1\}$ are linearly independent, we have

$$\begin{aligned} rk_1^2\left(r'^2 - \lambda\varepsilon_1\right) &= \varepsilon_2\lambda k_1\sqrt{r'^2 - \lambda\varepsilon_1}\left(r'^2 - \lambda\varepsilon_1 + 2rr''\right) \\ &= r''\left(r'^2 - \lambda\varepsilon_1 + rr''\right) = 0. \end{aligned} \tag{52}$$

From the (52), we get $k_1 = 0$ and $r''\left(r'^2 - \lambda\varepsilon_1 + rr''\right) = 0$. When $k_1 = 0$, from (30) we have $r'^2 - \lambda\varepsilon_1 + rr'' \neq 0$ and so it must be $r'' = 0$.

Conversely, if $k_1 = 0$ and $r(s) = as + b$, where $a, b \in \mathbb{R}$, $a \neq \mp 1$, from (30) we get $K = 0$ and this completes the proof. □

Theorem 2.7. *The canal hypersurfaces* $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$*, given by* (13) *in* E_1^4*, are minimal if and only if* $k_1 = 0$ *and the radius* $r(s)$ *satisfies*

$$\int \frac{dr}{\sqrt{\varepsilon_1\lambda + \left(\frac{c_1}{r}\right)^{\frac{4}{3}}}} = \pm s + c_2,$$

where $c_1, c_2 \in \mathbb{R}$*.*

Proof. Firstly, let us suppose that the canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$ in E_1^4 are minimal. Then from (31) we get

$$\begin{aligned} &-3r^2k_1^2\left(\varepsilon_1\lambda - r'^2\right)f_j^2 - \varepsilon_2\lambda k_1 r\sqrt{r'^2 - \lambda\varepsilon_1}\left(5\varepsilon_1\lambda - 5r'^2 - 6rr''\right)f_j \\ &+ 2 + 2r'^4 - 5\varepsilon_1\lambda rr'' + 3r^2r''^2 + r'^2\left(-4\varepsilon_1\lambda + 5rr''\right) = 0. \end{aligned} \tag{53}$$

Since $\{f_j, f_j^2, 1\}$ are linearly independent, we have

$$\begin{cases} -3r^2k_1^2\left(\varepsilon_1\lambda - r'^2\right) = 0, \\ \varepsilon_2\lambda k_1 r\sqrt{r'^2 - \lambda\varepsilon_1}\left(5\varepsilon_1\lambda - 5r'^2 - 6rr''\right) = 0, \\ 2 + 2r'^4 - 5\varepsilon_1\lambda rr'' + 3r^2r''^2 + r'^2\left(-4\varepsilon_1\lambda + 5rr''\right) = 0. \end{cases} \tag{54}$$

From (54), we get $k_1 = 0$ and $2 + 2r'^4 - 5\varepsilon_1\lambda rr'' + 3r^2r''^2 + r'^2\left(-4\varepsilon_1\lambda + 5rr''\right) = 0$; i.e.,

$$\left(2\varepsilon_1\lambda - 2r'^2 - 3rr''\right)\left(\varepsilon_1\lambda - r'^2 - rr''\right) = 0. \tag{55}$$

When $k_1 = 0$, from (31) we have $r'^2 - \lambda\varepsilon_1 + rr'' \neq 0$ and so it must be

$$-2(r'^2 - \varepsilon_1\lambda) - 3rr'' = 0. \tag{56}$$

Now, let us solve the equation (56).

If we take $r'(s) = h(s)$, we get

$$r'' = h' = \frac{dh}{dr}\frac{dr}{ds} = \frac{dh}{dr}h. \tag{57}$$

Using (57) in (56), we have

$$3r\frac{dh}{dr}h + 2h^2 - 2\varepsilon_1\lambda = 0. \tag{58}$$

From (56), $r'(s) = h(s) \neq 0$ and so we reach that

$$\frac{3h}{2(\varepsilon_1\lambda - h^2)}dh = \frac{dr}{r}. \tag{59}$$

By integrating (59), we have

$$h = \pm\sqrt{\varepsilon_1\lambda + \left(\frac{c_1}{r}\right)^{\frac{4}{3}}}, \tag{60}$$

where c_1 is constant. Since $r' = \frac{dr}{ds} = h$, from (60) we get

$$\int \frac{dr}{\sqrt{\varepsilon_1\lambda + \left(\frac{c_1}{r}\right)^{\frac{4}{3}}}} = \pm\int ds. \tag{61}$$

Conversely, if $k_1 = 0$ and $r(s)$ satisfies $\int \frac{dr}{\sqrt{\varepsilon_1\lambda + \left(\frac{c_1}{r}\right)^{\frac{4}{3}}}} = \pm s + c_2$, where $c_1, c_2 \in \mathbb{R}$, then we have $H = 0$ and this completes the proof. □

Now, if

$$H_sK_t - H_tK_s = 0,\ H_sK_w - H_wK_s = 0,\ H_tK_w - H_wK_t = 0 \tag{62}$$

hold on a hypersurface, then we call the hypersurface as $(H,K)_{st}$-Weingarten, $(H,K)_{sw}$-Weingarten, $(H,K)_{tw}$-Weingarten hypersurface, respectively, where $H_s = \frac{\partial H}{\partial s}$ and so on. So, from (30) and (31) we have:

Theorem 2.8. *The canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$, given by* (13) *in E_1^4, are $(H,K)_{tw}$-Weingarten.*

Proof. From (30) and (31), we get $H_t^{\{j;\lambda\}}K_w^{\{j;\lambda\}} - H_w^{\{j;\lambda\}}K_t^{\{j;\lambda\}} = 0$ and this completes the proof. □

Theorem 2.9. *The canal hypersurfaces $\mathfrak{C}^{\{j;\lambda\}}(s,t,w)$, given by* (13) *in E_1^4, are $(H,K)_{sw}$-Weingarten if and only if $k_1 = 0$ or $r(s) = constant$.*

Proof. From (30) and (31), we have

$$H_s^{\{j;\lambda\}}K_w^{\{j;\lambda\}} - H_w^{\{j;\lambda\}}K_s^{\{j;\lambda\}} \tag{63}$$

$$= -\frac{\begin{pmatrix} 2\varepsilon_3^2\varepsilon_4^2\lambda^{2j}k_1r'\left(r'^2-\lambda\varepsilon_1\right)^2 \\ \left(\left(\varepsilon_2^2\lambda^2-1\right)f_j^2k_1^2r^2+r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j+rr''\right) \\ \begin{pmatrix} \varepsilon_2\lambda\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+rr''\right) \\ +k_1r\left(\left(2-\varepsilon_2^2\lambda^2\right)\left(r'^2-\lambda\varepsilon_1\right)-2\left(\varepsilon_2^2\lambda^2-1\right)rr''\right)f_j \end{pmatrix} f_{j_w} \end{pmatrix}}{3r^4\left(r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j+rr''\right)^5}.$$

If $H_s^{\{j;\lambda\}}K_w^{\{j;\lambda\}}-H_w^{\{j;\lambda\}}K_s^{\{j;\lambda\}}=0$, then from (63) we have

$$2k_1r'\left(r'^2-\lambda\varepsilon_1\right)^2\begin{pmatrix} r'^2-\lambda\varepsilon_1+rr'' \\ +\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j \end{pmatrix} \begin{pmatrix} \varepsilon_2\lambda\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+rr''\right) \\ +k_1r\left(r'^2-\lambda\varepsilon_1\right)f_j \end{pmatrix} f_{j_w}=0 \tag{64}$$

and so,

$$k_1r'\begin{pmatrix} r'^2-\lambda\varepsilon_1+rr'' \\ +\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j \end{pmatrix}\begin{pmatrix} \varepsilon_2\lambda\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+rr''\right) \\ +k_1r\left(r'^2-\lambda\varepsilon_1\right)f_j \end{pmatrix}=0. \tag{65}$$

From (63), the second and third component of (65) cannot be zero and so it must be

$$k_1r'=0. \tag{66}$$

This completes the proof. □

Theorem 2.10. *The canal hypersurfaces $\mathfrak{C}^{\{4;\lambda\}}(s,t,w)$, given by* (13) *in E_1^4, are $(H,K)_{st}$-Weingarten. Also, the canal hypersurfaces $\mathfrak{C}^{\{1;\lambda\}}(s,t,w)$, $\mathfrak{C}^{\{2;\lambda\}}(s,t,w)$, $\mathfrak{C}^{\{3;\lambda\}}(s,t,w)$, given by* (13) *in E_1^4, are $(H,K)_{st}$-Weingarten if and only if $k_1=0$ or $r(s)=constant$.*

Proof. From (30) and (31), we have

$$H_s^{\{j;\lambda\}}K_t^{\{j;\lambda\}}-H_t^{\{j;\lambda\}}K_s^{\{j;\lambda\}} \tag{67}$$

$$= -\frac{\begin{pmatrix} 2\varepsilon_3^2\varepsilon_4^2\lambda^{2j}k_1r'\left(r'^2-\lambda\varepsilon_1\right)^2 \\ \left(\left(\varepsilon_2^2\lambda^2-1\right)k_1^2r^2f_j^2+r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j+rr''\right) \\ \begin{pmatrix} \varepsilon_2\lambda\sqrt{r'^2-\lambda\varepsilon_1}\left(r'^2-\lambda\varepsilon_1+rr''\right) \\ +k_1r\left(\left(2-\varepsilon_2^2\lambda^2\right)\left(r'^2-\lambda\varepsilon_1\right)-2\left(\varepsilon_2^2\lambda^2-1\right)rr''\right)f_j \end{pmatrix} f_{j_t} \end{pmatrix}}{3r^4\left(r'^2-\lambda\varepsilon_1+\varepsilon_2\lambda k_1r\sqrt{r'^2-\lambda\varepsilon_1}f_j+rr''\right)^5}.$$

Using (33), for $j=1,2,3$ the proof can be seen with similar method in Theorem 2.9. Also, for $j=4$ we have $f_{4_t}=0$ and so $H_s^{\{4;\lambda\}}K_t^{\{4;\lambda\}}-H_t^{\{4;\lambda\}}K_s^{\{4;\lambda\}}=0$ and this completes the proof. □

3. Tubular hypersurfaces generated by non-null curves

In this section, we give the above results, which are given for the canal hypersurfaces in E_1^4, for tubular hypersurfaces.

Theorem 3.1. *The tubular hypersurfaces* $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$ *that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a non-null curve with non-null Frenet vectors in* E_1^4 *can be parametrized by*

$$(68)\quad \begin{cases} \mathfrak{T}^{\{1;1\}}(s,t,w) = \beta(s) \mp r(s)\left(\cos t \cos wF_2(s) + \sin t \cos wF_3 + \sin wF_4\right), \\ \mathfrak{T}^{\{2;1\}}(s,t,w) = \beta(s) \mp r(s)\left(\cosh t \sinh wF_2(s) + \cosh wF_3 + \sinh t \sinh wF_4\right), \\ \mathfrak{T}^{\{2;-1\}}(s,t,w) = \beta(s) \mp r(s)\left(\cosh t \cosh wF_2(s) + \sinh wF_3 + \sinh t \cosh wF_4\right), \\ \mathfrak{T}^{\{3;1\}}(s,t,w) = \beta(s) \mp r(s)\left(\sinh t \sinh wF_2(s) + \cosh t \sinh wF_3 + \cosh wF_4\right), \\ \mathfrak{T}^{\{3;-1\}}(s,t,w) = \beta(s) \mp r(s)\left(\sinh t \cosh wF_2(s) + \cosh t \cosh wF_3 + \sinh wF_4\right), \\ \mathfrak{T}^{\{4;1\}}(s,t,w) = \beta(s) \mp r(s)\left(\cosh wF_2(s) + \sinh t \sinh wF_3 + \cosh t \sinh wF_4\right), \\ \mathfrak{T}^{\{4;-1\}}(s,t,w) = \beta(s) \mp r(s)\left(\sinh wF_2(s) + \sinh t \cosh wF_3 + \cosh t \cosh wF_4\right). \end{cases}$$

Also, there is no tubular hypersurface $\mathfrak{T}^{\{1;-1\}}(s,t,w)$ *that is formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a timelike curve in* E_1^4*.*

Furthermore the tubular hypersurfaces $\mathfrak{T}^{\{j;0\}}(s,t,w)$ *that are formed as the envelope of a family of null hypercones whose centers lie on a non-null curve in* E_1^4 *can be parametrized by* (27)*.*

Proof. Let the center curve $\beta : I \subseteq \mathbb{R} \to E_1^4$ be an arc-length parametrized timelike or spacelike curve with non-zero curvature. Then, the parametrization of the envelope of pseudo hyperspheres (resp. pseudo hyperbolic hyperspheres or null hypercones) defining the tubular hypersurfaces $\mathfrak{T}^{\{j;\lambda\}}$ in E_1^4 can be given by

$$(69)\quad \mathfrak{T}^{\{j;\lambda\}}(s,t,w) - \beta(s) = a_1(s,t,w)F_1(s) + a_2(s,t,w)F_2(s) + a_3(s,t,w)F_3(s) + a_4(s,t,w)F_4(s).$$

Furthermore, since $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$ lies on the pseudo hyperspheres (resp. pseudo hyperbolic hyperspheres or null hypercones), we have

$$(70)\quad g(\mathfrak{T}^{\{j;\lambda\}}(s,t,w) - \beta(s), \mathfrak{T}^{\{j;\lambda\}}(s,t,w) - \beta(s)) = \lambda r^2$$

which leads to from (69) that

$$(71)\quad \varepsilon_1 a_1^2 + \varepsilon_2 a_2^2 + \varepsilon_3 a_3^2 + \varepsilon_4 a_4^2 = \lambda r^2$$

and

$$(72)\quad \varepsilon_1 a_1 a_{1_s} + \varepsilon_2 a_2 a_{2_s} + \varepsilon_3 a_3 a_{3_s} + \varepsilon_4 a_4 a_{4_s} = 0,$$

where $\lambda = 1$, $\lambda = -1$ and $\lambda = 0$ if the tubular hypersurfaces are obtained by pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones, respectively; r is constant radius $a_i = a_i(s,t,w)$, $a_{i_s} = \frac{\partial a_i(s,t,w)}{\partial s}$, and so on.

With similar procedure in the proof of Theorem 2.1, we reach that

$$(73)\quad a = 0.$$

Hence, using (73) in (71), we have

$$(74)\quad \varepsilon_2 a_2^2 + \varepsilon_3 a_3^2 + \varepsilon_4 a_4^2 = \lambda r^2.$$

From (69) and (74), we obtain the parametric expressions of the tubular hypersurfaces $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$ as (68) and (27). □

Now, let us give the Gaussian and mean curvatures of tubular hypersurfaces $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$ given by (68) with similar method used for canal hypersurfaces.

Theorem 3.2. *The Gaussian and mean curvatures of tubular hypersurfaces $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$, given by (68) in E_1^4, are*

$$(75)\quad \begin{cases} K^{\{1;1\}} = \frac{k_1 \cos t \cos w}{r^2(1+rk_1 \cos t \cos w)},\ H^{\{1;1\}} = \frac{2+3rk_1 \cos t \cos w}{3r(1+rk_1 \cos t \cos w)}; \\ K^{\{2;1\}} = \frac{k_1 \cosh t \sinh w}{r^2(1+rk_1 \cosh t \sinh w)},\ H^{\{2;1\}} = \frac{2+3rk_1 \cosh t \sinh w}{3r(1+rk_1 \cosh t \sinh w)}; \\ K^{\{2;-1\}} = -\frac{k_1 \cosh t \cosh w}{r^2(1+rk_1 \cosh t \cosh w)},\ H^{\{2;-1\}} = -\frac{2+3rk_1 \cosh t \cosh w}{3r(1+rk_1 \cosh t \cosh w)}; \\ K^{\{3;1\}} = \frac{k_1 \sinh t \sinh w}{r^2(1-rk_1 \sinh t \sinh w)},\ H^{\{3;1\}} = \frac{2-3rk_1 \sinh t \sinh w}{3r(-1+rk_1 \sinh t \sinh w)}; \\ K^{\{3;-1\}} = -\frac{k_1 \sinh t \cosh w}{r^2(-1+rk_1 \sinh t \cosh w)},\ H^{\{3;-1\}} = -\frac{2-3rk_1 \sinh t \cosh w}{3r(1-rk_1 \sinh t \cosh w)}; \\ K^{\{4;1\}} = \frac{k_1 \cosh w}{r^2(1-rk_1 \cosh w)},\ H^{\{4;1\}} = \frac{2-3rk_1 \cosh w}{3r(-1+rk_1 \cosh w)}; \\ K^{\{4;-1\}} = -\frac{k_1 \sinh w}{r^2(1-rk_1 \sinh w)},\ H^{\{4;-1\}} = -\frac{2-3rk_1 \sinh w}{3r(-1+rk_1 \sinh w)}. \end{cases}$$

From (75), we can state the following theorem:

Theorem 3.3. *The tubular hypersurfaces $\mathfrak{T}^{\{j;\lambda\}}(s,t,w)$, given by (68) in E_1^4, are $(H,K)_{tw}$-Weingarten, $(H,K)_{ts}$-Weingarten and $(H,K)_{sw}$-Weingarten hypersurfaces.*

4. Visualization

In this section, we construct examples for canal hypersurfaces $\mathfrak{C}^{\{1;\lambda\}}(s,t,w)$ and $\mathfrak{C}^{\{3;\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose center lie on a timelike curve and a spacelike curve with timelike binormal vector in E_1^4, separately.

Firstly, let us take the unit speed timelike curve

$$(76)\qquad \beta_1(s) = \left(2\sinh s, 2\cosh s, \sqrt{3}\cos s, \sqrt{3}\sin s\right)$$

in E_1^4. The Frenet vectors and curvatures of the curve (76) are

$$(77)\quad \begin{cases} F_1^{\beta_1} = \left(2\cosh s, 2\sinh s, -\sqrt{3}\sin s, \sqrt{3}\cos s\right), \\ F_2^{\beta_1} = \left(\frac{2}{\sqrt{7}}\sinh s, \frac{2}{\sqrt{7}}\cosh s, -\sqrt{\frac{3}{7}}\cos s, -\sqrt{\frac{3}{7}}\sin s\right), \\ F_3^{\beta_1} = \left(-\sqrt{3}\cosh s, -\sqrt{3}\sinh s, 2\sin s, -2\cos s\right), \\ F_4^{\beta_1} = \left(\sqrt{\frac{3}{7}}\sinh s, \sqrt{\frac{3}{7}}\cosh s, \frac{2}{\sqrt{7}}\cos s, \frac{2}{\sqrt{7}}\sin s\right), \\ k_1^{\beta_1} = \sqrt{7},\ k_2^{\beta_1} = 4\sqrt{\frac{3}{7}},\ k_3^{\beta_1} = \frac{1}{\sqrt{7}}. \end{cases}$$

From (26), the canal hypersurfaces $\mathfrak{C}^{\{1;\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on the curve (76) in E_1^4 are

$$\mathfrak{C}^{\{1;1\}}(s,t,w) = \begin{pmatrix} -2s\left(-4+\sqrt{15}\cos w\sin t\right)\cosh s+\frac{2}{7}\left(7+\sqrt{35}s\left(2\cos t\cos w+\sqrt{3}\sin w\right)\right)\sinh s, \\ -2s\left(-4+\sqrt{15}\cos w\sin t\right)\sinh s+\frac{2}{7}\left(7+\sqrt{35}s\left(2\cos t\cos w+\sqrt{3}\sin w\right)\right)\cosh s, \\ -4s\left(\sqrt{3}-\sqrt{5}\cos w\sin t\right)\sin s+\left(\sqrt{3}-2\sqrt{\frac{5}{7}}s\left(\sqrt{3}\cos t\cos w-2\sin w\right)\right)\cos s, \\ \left(\sqrt{3}-2\sqrt{\frac{5}{7}}s\left(\sqrt{3}\cos t\cos w-2\sin w\right)\right)\sin s+4s\left(\sqrt{3}-\sqrt{5}\cos w\sin t\right)\cos s \end{pmatrix} \tag{78}$$

and

$$\mathfrak{C}^{\{1;-1\}}(s,t,w) = \begin{pmatrix} -2s\left(4+3\cos w\sin t\right)\cosh s+\frac{2}{7}\left(7+2\sqrt{21}s\cos t\cos w+3\sqrt{7}s\sin w\right)\sinh s, \\ -2s\left(4+3\cos w\sin t\right)\sinh s+\frac{2}{7}\left(7+\sqrt{21}s\left(2\cos t\cos w+\sqrt{3}\sin w\right)\right)\cosh s, \\ 4\sqrt{3}s\left(1+\cos w\sin t\right)\sin s+\left(\sqrt{3}-\frac{2}{\sqrt{7}}s\left(3\cos t\cos w-2\sqrt{3}\sin w\right)\right)\cos s, \\ \left(\sqrt{3}-\frac{2}{\sqrt{7}}s\left(3\cos t\cos w-2\sqrt{3}\sin w\right)\right)\sin s-4\sqrt{3}s\left(1+\cos w\sin t\right)\cos s \end{pmatrix}, \tag{79}$$

where the radius function has been taken as $r(s) = 2s$. From (30), (31) and (32), the Gaussian, mean and principal curvatures of the canal hypersurfaces $\mathfrak{C}^{\{1;1\}}$ and $\mathfrak{C}^{\{1;-1\}}$ are obtained as

$$\begin{cases} K^{\{1;1\}} = \frac{5\left(\sqrt{35}+14s\cos t\cos w\right)\cos t\cos w}{4s^2\left(5+2\sqrt{35}s\cos t\cos w\right)^2}, \\ H^{\{1;1\}} = \frac{1}{3}\left(\frac{1}{s}+\frac{5\left(\sqrt{35}+14s\cos t\cos w\right)\cos t\cos w}{\left(5+2\sqrt{35}s\cos t\cos w\right)^2}\right), \\ \mu_1^{\{1;1\}} = \mu_2^{\{1;1\}} = \frac{1}{2s},\ \mu_3^{\{1;1\}} = \frac{5\left(\sqrt{35}+14s\cos t\cos w\right)\cos t\cos w}{\left(5+2\sqrt{35}s\cos t\cos w\right)^2} \end{cases} \tag{80}$$

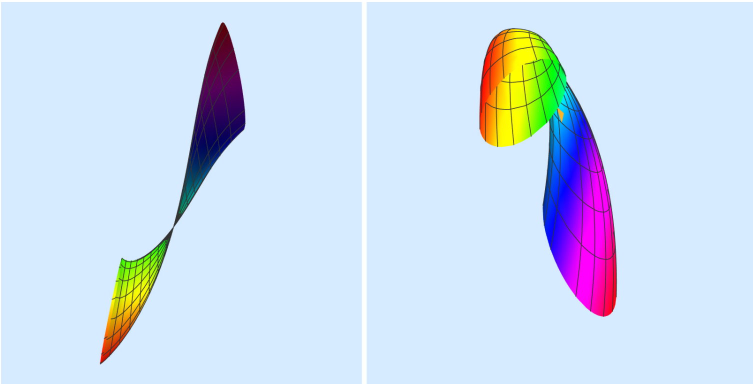
and

$$\begin{cases} K^{\{1;-1\}} = -\frac{3\left(\sqrt{21}-14s\cos t\cos w\right)\cos t\cos w}{4s^2\left(3-2\sqrt{21}s\cos t\cos w\right)^2}, \\ H^{\{1;-1\}} = -\frac{-3+5\sqrt{21}s\cos t\cos w-42s^2\cos^2 t\cos^2 w}{s\left(3-2\sqrt{21}s\cos t\cos w\right)^2}, \\ \mu_1^{\{1;-1\}} = \mu_2^{\{1;-1\}} = \frac{-1}{2s},\ \mu_3^{\{1;-1\}} = \frac{3\left(\sqrt{21}-14s\cos t\cos w\right)\cos t\cos w}{\left(3-2\sqrt{21}s\cos t\cos w\right)^2}, \end{cases} \tag{81}$$

respectively. In the following figures, one can see the projections of the canal hypersurfaces (78) and (79) for $w = 2$ and $r(s) = 2s$ into $x_2x_3x_4$-spaces in (A) and (B) in Figure 1, respectively.

Secondly, let us take the unit speed spacelike curve with timelike binormal vector

$$\beta_2(s) = \left(\sqrt{3}\sinh s, \sqrt{3}\cosh s, 2\cos s, 2\sin s\right) \tag{82}$$



(A) (B)

FIGURE 1.

in E_1^4. The Frenet vectors and curvatures of the curve (82) are

$$\left\{\begin{array}{l} F_1^{\beta_2} = \left(\sqrt{3}\cosh s, \sqrt{3}\sinh s, -2\sin s, 2\cos s\right), \\ F_2^{\beta_2} = \left(\sqrt{\frac{3}{7}}\sinh s, \sqrt{\frac{3}{7}}\cosh s, -\frac{2}{\sqrt{7}}\cos s, -\frac{2}{\sqrt{7}}\sin s\right), \\ F_3^{\beta_2} = \left(2\cosh s, 2\sinh s, -\sqrt{3}\sin s, \sqrt{3}\cos s\right), \\ F_4^{\beta_2} = \left(\frac{2}{\sqrt{7}}\sinh s, \frac{2}{\sqrt{7}}\cosh s, \sqrt{\frac{3}{7}}\cos s, \sqrt{\frac{3}{7}}\sin s\right), \\ k_1^{\beta_2} = \sqrt{7},\ k_2^{\beta_2} = 4\sqrt{\frac{3}{7}},\ k_3^{\beta_2} = \frac{1}{\sqrt{7}}. \end{array}\right. \tag{83}$$

From (26), the canal hypersurfaces $\mathfrak{C}^{\{3;\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on the curve (82) in E_1^4 are

$$\begin{aligned} &\mathfrak{C}^{\{3;1\}}(s,t,w) \\ &= \begin{pmatrix} \frac{\sqrt{3}}{7}\begin{pmatrix} 28s\left(-1+\cosh t\cosh w\right)\cosh s \\ +\left(7+2\sqrt{21}s\cosh w\sinh t+4\sqrt{7}s\sinh w\right)\sinh s \end{pmatrix}, \\ \frac{\sqrt{3}}{7}\begin{pmatrix} 28s\left(-1+\cosh t\cosh w\right)\sinh s \\ +\left(7+2\sqrt{21}s\cosh w\sinh t+4\sqrt{7}s\sinh w\right)\cosh s \end{pmatrix}, \\ -6s\sin s\cosh t\cosh w+2\left(\cos s+4s\sin s\right) \\ +\frac{2s}{\sqrt{7}}\left(-2\sqrt{3}\cosh w\sinh t+3\sinh w\right)\cos s, \\ 2s\left(-4+3\cosh t\cosh w\right)\cos s \\ +\frac{2}{7}\left(7-2\sqrt{21}s\cosh w\sinh t+3\sqrt{7}s\sinh w\right)\sin s \end{pmatrix} \end{aligned} \tag{84}$$

and

$$\mathfrak{C}^{\{3;-1\}}(s,t,w) \tag{85}$$

$$= \begin{pmatrix} 4s\left(\sqrt{3}+\sqrt{5}\cosh t\cosh w\right)\cosh s \\ +\left(\sqrt{3}+2\sqrt{\frac{5}{7}}s\left(\sqrt{3}\cosh w\sinh t+2\sinh w\right)\right)\sinh s, \\ 4s\left(\sqrt{3}+\sqrt{5}\cosh t\cosh w\right)\sinh s \\ +\left(\sqrt{3}+2\sqrt{\frac{5}{7}}s\left(\sqrt{3}\cosh w\sinh t+2\sinh w\right)\right)\cosh s, \\ -2s\left(4+\sqrt{15}\cosh t\cosh w\right)\sin s \\ +\frac{2}{7}\left(7+\sqrt{35}s\left(-2\cosh w\sinh t+\sqrt{3}\sinh w\right)\right)\cos s, \\ 2s\left(4+\sqrt{15}\cosh t\cosh w\right)\cos s \\ +\frac{2}{7}\left(7+\sqrt{35}s\left(-2\cosh w\sinh t+\sqrt{3}\sinh w\right)\right)\sin s \end{pmatrix},$$

where the radius function has been taken as $r(s) = 2s$. From (30), (31) and (32), the Gaussian, mean and principal curvatures of the canal hypersurfaces $\mathfrak{C}^{\{3;1\}}$ and $\mathfrak{C}^{\{3;-1\}}$ are obtained as

$$(86)\quad \begin{cases} K^{\{3;1\}} = -\frac{\left(\sqrt{21}+14s\cosh w\sinh t\right)\cosh w\sinh t}{4s^2\left(\sqrt{3}+2\sqrt{7}s\cosh w\sinh t\right)^2}, \\ H^{\{3;1\}} = -\frac{3+5\sqrt{21}s\cosh w\sinh t+42s^2\cosh^2 w\sinh^2 t}{s\left(3+2\sqrt{21}s\cosh w\sinh t\right)^2}, \\ \mu_1^{\{3;1\}} = \mu_2^{\{3;1\}} = \frac{-1}{2s},\ \mu_3^{\{3;1\}} = -\frac{3\left(\sqrt{21}+14s\cosh w\sinh t\right)\cosh w\sinh t}{\left(3+2\sqrt{21}s\cosh w\sinh t\right)^2} \end{cases}$$

and

$$(87)\quad \begin{cases} K^{\{3;-1\}} = -\frac{5\left(-\sqrt{35}+14s\cosh w\sinh t\right)\cosh w\sinh t}{4s^2\left(5-2\sqrt{35}s\cosh w\sinh t\right)^2}, \\ H^{\{3;-1\}} = -\frac{1}{3}\left(\frac{1}{s}+\frac{5\left(-\sqrt{35}+14s\cosh w\sinh t\right)\cosh w\sinh t}{\left(5-2\sqrt{35}s\cosh w\sinh t\right)^2}\right), \\ \mu_1^{\{3;-1\}} = \mu_2^{\{3;-1\}} = \frac{1}{2s},\ \mu_3^{\{3;-1\}} = \frac{5\left(-\sqrt{35}+14s\cosh w\sinh t\right)\cosh w\sinh t}{\left(5-2\sqrt{35}s\cosh w\sinh t\right)^2}, \end{cases}$$

respectively. In the following figures, one can see the projections of the canal hypersurfaces (84) and (85) for $w = 2$ and $r(s) = 2s$ into $x_2x_3x_4$-spaces in (A) and (B) in Figure 2, respectively.

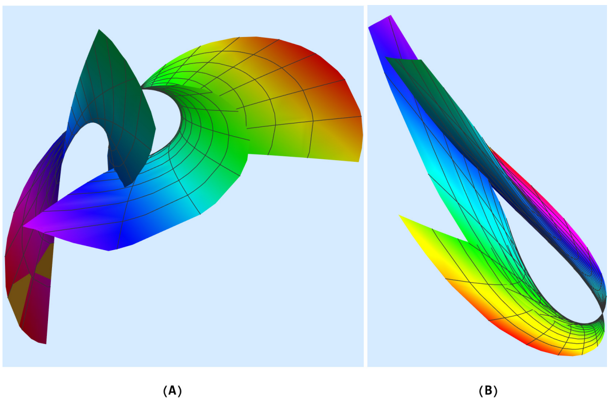


FIGURE 2.

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