Bull. Korean Math. Soc. **60** (2023), No. 5, pp. 1365–1374
https://doi.org/10.4134/BKMS.b220710
pISSN: 1015-8634 / eISSN: 2234-3016

# ON THE DETERMINANT OF A DUAL PERIODIC SINGULAR FIBER

Cheng Gong, Jun Lu, and Sheng-Li Tan

Abstract. Let $F$ be a periodic singular fiber of genus $g$ with dual fiber $F^*$, and let $T$ (resp. $T^*$) be the set of the components of $F$ (resp. $F^*$) by removing one component with multiplicity one. We give a formula to compute the determinant $|\det T|$ of the intersect form of $T$. As a consequence, we prove that $|\det T| = |\det T^*|$. As an application, we compute the Mordell-Weil group of a fibration $f : S \to \mathbb{P}^1$ of genus 2 with two singular fibers.

## 1. Introduction

A family of curves of genus $g$ over a smooth curve $C$ is a holomorphic map $f : S \to C$ whose general fibers $F$ are smooth curves of genus $g$, where $S$ is a smooth projective surface. We also call $f$ a fibration of genus $g$. We always assume that $f$ has a section $O$ and is relatively minimal, i.e., there is no $(-1)$-curve in any fiber. In this paper, we work over an algebraically closed field of characteristic 0.

For each singular fiber $F$ of $f$, we define $T$ to be a negative-definite sublattice spanned by the irreducible components of $F$, say $x_i$'s, disjoint with the section $O$. The determinant of $T$ is denoted to be $\det T = \det(\langle x_i, x_j\rangle)$, where $x_i$'s are the irreducible components of $F$ disjoint $O$. Then the Mordell-Weil group of $f$ is isomorphic to the quotient group of the Néron-Severi group by the subgroup generated by $O$ and all of the components in the fibers (see [13, 14]).

In order to get the structure of the Mordell-Weil group, a necessary step is to compute $\det T$ for any singular fiber $F$. Shioda [13] computed $\det T$ directly for all elliptic singular fibers $F$ based on the classification of Kodaira [9]. Oguiso-Shioda ([12]) use $\det T$ to classify of the Mordell-Weil groups of the rational elliptic surfaces. Some authors are trying to generalize this computation to the higher genus case. In [7, 8, 16], the authors computed the Mordell-Weil groups of some hyperelliptic fibrations by $\det T$. In [3], the authors even used $\det T$ to compute the Mordell-Weil groups of some non hyperelliptic fibrations. In [6],

Received October 11, 2022; Revised February 16, 2023; Accepted February 24, 2023.
2010 *Mathematics Subject Classification.* 14D06, 14C21, 14H10.
*Key words and phrases.* Modell-Weil group, singular fiber, determinant, fibrations.

the authors discussed the arithmetic properties of the Mordell-Weil groups of some examples in [7, 8, 16].

From [13], one can find the following interesting fact on an elliptic singular fiber $F$,

$$|\det T| = |\det T^*|, \tag{1}$$

where $F$ is of type II, III, IV, and the dual fiber $F^*$ is of type II$^*$, III$^*$, IV$^*$ with the lattice $T^*$, respectively. Some authors are trying to generalize the notion $F^*$ for fibers $F$ of higher genus, and to verify the equality (1) for some fibers.

The second and the third authors introduced in [10] the so-called *dual fiber* $F^*$ for any singular fiber $F$ of arbitrary genus (see Section 2), which coincides with the notion in the case for elliptic fibration. For a relatively minimal fibration $f : S \to \mathbb{P}^1$ with two singular fibers $F_1$ and $F_2$, one can find that that $F_1 = F_2^*$ and $F_2 = F_1^*$ ([4]). In this case, S. Kitagawa conjectures that the equality (1) holds for $F_i$'s and for any section $O$ of $f$.

For a periodic fiber $F$, the dual fiber $F^*$ is also periodic. In this case, the minimal normal-crossing model $\overline{F}$ (resp. $\overline{F}^*$) of $F$ (resp. $F^*$) can be written as follows:

$$\overline{F} = nC_0 + \sum_{i=1}^{b} \Gamma_i \text{ (resp. } \overline{F}^* = nC_0^* + \sum_{i=1}^{b} \Gamma_i^*), \tag{2}$$

where $\Gamma_i$'s (resp. $\Gamma_i^*$) are disjoint H-J branches with valencies $(n/\lambda_i, \lambda_i, \sigma_i)$ (resp. $(n^*/\lambda_i, \lambda_i, \sigma_i)$) and $C_0$ (resp. $C_0^*$) is a principal component meeting transversely each H-J branch at one point, respectively. Furthermore, one has $\lambda_1^* = \lambda_1, \ldots, \lambda_b^* = \lambda_b$ (see [4] and [1, Sec. 3.3]). The above terminologies can be found in Section 2 or [1, Sec. 3.3].

Let $T$ (resp. $T^*$) be the lattice generated by the irreducible components of $F$ (resp. $F^*$) except one component with multiplicity 1. Then we have the following main result.

**Theorem 1.1.** *Let $T$ follow the above definition. Then*

$$|\det T| = |\det T^*| = \frac{\prod_{j=1}^{b} \lambda_j}{n^2}.$$

*In particular, $|\det T|$ is independent of the choice of the removed component with multiplicity one.*

From Theorem 1.1, we obtain again (1) in the elliptic case found by [13].

**Corollary 1.1.** *For a periodic elliptic fiber $F$, one has*

| $F$, $F^*$ | II, II$^*$ | III, III$^*$ | IV, IV$^*$ | $I_0^*, I_0^*$ |
|---|---|---|---|---|
| $\lvert\det T\rvert$ | 1 | 2 | 3 | 4 |

As an application, we will compute the Mordell-Weil group of fibrations of genus 2 with two singular fibers.

**Theorem 1.2.** *Let $f : S \to \mathbb{P}^1$ be a relatively minimal fibration of genus $g = 2$ with two singular fibers. Assume that $S$ is a rational surface. Then the Mordell-Weil group of $f$ is as follows*:

| No. | Fibers' types in [NU] | Mordell-Weil group | $\lvert\det T\rvert$ |
|---|---|---|---|
| 1 | $\mathrm{I}^*_{0-0-0}, \mathrm{I}^*_{0-0-0}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ | 16 |
| 3 | $\mathrm{III}, \mathrm{III}$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ | 9 |
| 5 | $\mathrm{V}, \mathrm{V}^*$ | $\mathbb{Z}_3$ | 3 |
| 6 | $\mathrm{VI}, \mathrm{VI}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | 4 |
| 7 | $\mathrm{VII}, \mathrm{VII}^*$ | $\mathbb{Z}_2$ | 2 |
| 8 | $\mathrm{VIII}-1, \mathrm{VIII}-4$ | 0 | 1 |
| 9 | $\mathrm{VIII}-2, \mathrm{VIII}-3$ | 0 | 1 |
| 10 | $\mathrm{IX}-1, \mathrm{IX}-4$ | $\mathbb{Z}_5$ | 5 |
| 11 | $\mathrm{IX}-2, \mathrm{IX}-3$ | $\mathbb{Z}_5$ | 5 |

**Acknowledgements.** The authors would like to thank Professor S. Kitagawa for many useful discussions about his work on Mordell-Weil lattices. This work is supported by the Natural Science Foundation of Jiangsu Province (BK 20211305). This work is also supported by the National Key Research and Development Program of China (Grant No. 2018AAA0101001), the National Natural Science Foundation of China, the Shanghai Science and Technology Commission Foundation (Grants No. 22DZ2229014 and No. 20511100200).

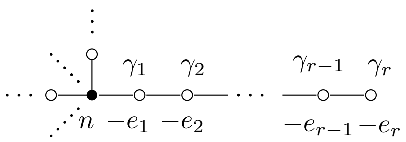
## 2. Preliminaries

Let $f : S \to C$ be a relatively minimal fibration of genus $g \geq 1$. Each singular fiber $F$ of $f$ gives a *minimal normal-crossing* model $\overline{F}$ by a partial resolution ([10, Definitions 2.1 and 2.2]), i.e., $\overline{F}$ is a normal-crossing curve in which there is no $(-1)$-curve meeting the other components at most two points.

The $n$-th root model of $F$ is a relatively minimal model of the pulling-back fiber of $F$ by a base change of degree $n$ totally ramified over $p = f(F)$ ([10, Definition 2.3]). Let $M_F$ be the least common multiplicity of the coefficients of the irreducible components of $\overline{F}$. The *dual* (resp. *semistable*) model of $F$ is the $(M_F - 1)$-th (resp. $M_F$-th) model of $F$ ([10, Definitions 2.4 and 2.5]). We say $F$ is a *periodic* fiber if the semistable model of $F$ is a smooth curve.

An irreducible component of $\overline{F}$ is said to be a *principal component* if it is either a smooth non-rational curve or a non-singular rational curve which intersects the other components transversally in 3 or more points. A *H-J branch* $\Gamma$ of $\overline{F}$ is a chain of smooth rational curves such that terminal curve intersects a principal component transversally in one point ([10, Definition 3.4]).

Let $\Gamma = \gamma_1 C_1 + \cdots + \gamma_r C_r$ be one of the H-J branches, where $C_1$ intersects a principal component $C_0$ with multiplicity $n$ and $C_k$ is a rational curve with multiplicity $\gamma_k$ and $C_k^2 = -e_k$ $(k = 1, \ldots, r)$. We call $C_r$ the end component of $\overline{F}$ in $\Gamma$.




$\Gamma$

**Lemma 2.1.** *Take* $\gamma_0 := n$, $\gamma_{r+1} := 0$, $\lambda := n/\gcd(n, \gamma_r)$ *and* $\sigma := \gamma_1/\gcd(\gamma_1, \gamma_r)$. *For* $k = 1, \dots, r$, *we have*

(1) $\gamma_{k+1} - e_k\gamma_k + \gamma_{k-1} = 0$;
(2) $n > \gamma_1 > \gamma_2 > \cdots > \gamma_r > 0$;
(3) $\gamma_r = \gcd(\gamma_k, \gamma_{k-1})$;
(4) $\frac{\gamma_{k-1}}{\gamma_k} = [e_k, \dots, e_r] := e_k - \cfrac{1}{e_{k+1} - \cfrac{1}{\ddots - \cfrac{1}{e_r}}}$.

*In particular,* $\Gamma$ *is determined by* $(n/\lambda, \lambda, \sigma)$.

*Proof.* (1) is from Zariski's Lemma. (2)-(4) are the corollaries of (1) (see also [2, Ch. III, Sec. 5]). □

**Definition 2.1.** We call $(n/\lambda, \lambda, \sigma)$ the *valency* of $\Gamma$ ([1, Sec. 3.3]).

The intersection matrix of the irreducible components in $\Gamma$ is

$$M(\Gamma) := (C_iC_j)_{1\le i,j\le r} = \begin{pmatrix} -e_1 & 1 & & & \\ 1 & -e_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -e_{r-1} & 1 \\ & & & 1 & -e_r \end{pmatrix}.$$

From a straightforward computation and Lemma 2.1, we have

$$|\det M(\Gamma)| = \lambda. \tag{3}$$

## 3. Proof of Theorem 1.1

In what follows, we assume that $f : S \to C$ has a section $O$. Let $\sigma : (\overline{S}, \overline{F}) \to (S, F)$ be a minimal partial resolution of $(S, F)$ for a periodic fiber $F$ (namely, $\overline{F}$ is the minimal normal-crossing model of $F$) and $\bar{f} := f\sigma : \overline{S} \to C$ with a section $\overline{O} = \sigma^*O$. Let $D$ be the irreducible component of $F$ intersecting the section $O$ and $\overline{D}$ be the strict transform of $D$ under $\sigma$. Let $T$ (resp. $\overline{T}$) be a negative-definite sublattice spanned by the irreducible components of $F$ (resp. $\overline{F}$) except $D$ (resp. $\overline{D}$).

We write

$$\overline{F} = n_1C_1 + \cdots + n_dC_d$$

and

$$\overline{F}_{\text{red}} = C_1 + \cdots + C_d,$$

where $C_i$ runs over all irreducible components of $\overline{F}$. We denote by $M_i$ the intersection matrix of $\overline{F}_{\text{red}} - C_i$.

We need the following key lemma.

**Lemma 3.1.** *For any $i$ and $j$, we have*

$$\frac{n_i^2}{n_j^2} = \frac{\det M_i}{\det M_j}.$$

*Proof.* Without loss of generality, we assume that $i = 1$ and $j = 2$. We have

$$\begin{pmatrix} C_1 \\ C_3 \\ \vdots \\ C_d \end{pmatrix} = P \begin{pmatrix} C_2 \\ C_3 \\ \vdots \\ C_d \end{pmatrix},$$

where

$$P = \begin{pmatrix} -\frac{n_2}{n_1} & -\frac{n_3}{n_1} & \cdots & \cdots & -\frac{n_d}{n_1} \\ & 1 & \cdots & \cdots & 0 \\ & & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}.$$

So $M_2 = PM_1P^T$. Thus we get

$$\det M_2 = (\det P)^2 \det M_1 = \left(-\frac{n_2}{n_1}\right)^2 \det M_1.$$

The proof is completed. □

Consider the expression (2) of the H-J branch $\overline{F}$. Let $\Gamma_{k1}$ be the end component of $\Gamma_k$ ($k = 1, \ldots, b$) and $\overline{M}_k$ be the intersection matrix of $\overline{F}_{\text{red}} - \Gamma_{k1}$.

**Corollary 3.1.** *Under the above assumptions and notations, we have*

$$|\det \overline{M}_k| = \frac{\prod_{i=1}^b \lambda_i}{\lambda_k^2} \quad \textit{for} \quad k = 1, 2, \ldots, b.$$

*In particular, if the section $\overline{O}$ intersecting $\Gamma_{k1}$ for some $k$, then $\lambda_k = n$ and hence*

$$|\det \overline{M}_k| = \frac{\prod_{i=1}^b \lambda_i}{n^2} = \frac{\lambda_1 \cdots \lambda_{k-1} \cdot \lambda_{k+1} \cdots \lambda_b}{n}.$$

*Proof.* Let $\overline{M}_0$ be the intersection matrix of $\overline{F}_{\text{red}} - C_0$. Since all H-J branches $\Gamma_i$ are disjoint,

$$\det \overline{M}_0 = \det M(\Gamma_1) \cdots \det M(\Gamma_b),$$

where $M(\Gamma_i)$ is the intersection matrix of the irreducible components in $\Gamma_i$ ($i = 1, \dots, b$). By (3), we have

$$\det \overline{M}_0 = \prod_{i=1}^{b} \lambda_i.$$

Note that the multiplicity of $\Gamma_{k1}$ (resp. $C_0$) in $\overline{F}$ is $\frac{n}{\lambda_k}$ (resp. $n$). From Lemma 3.1, we get

$$\frac{\det \overline{M}_k}{\det \overline{M}_0} = \frac{\left(\frac{n}{\lambda_k}\right)^2}{n^2}.$$

Thus $|\det \overline{M}_k| = \frac{\prod_{i=1}^{b} \lambda_i}{\lambda_k^2}$. □

Let $F^*$ be the dual fiber of $F$ with a minimal normal-crossing model $\overline{F}^*$ as in (2). One has $\lambda_1^* = \lambda_1, \dots, \lambda_b^* = \lambda_b$ and $n^* = n$ (see [4] and [1, Sec. 3.3]). From Corollary 3.1, we have:

**Corollary 3.2.** *Let $\overline{T}^*$ be the lattice generated by the components of $\overline{F}^*$ except one component with multiplicity one. Then*

$$|\det \overline{T}\,| = |\det \overline{T}^*\,| = \frac{\prod_{i=1}^{b} \lambda_i}{n^2}.$$

In order to complete our proof of Theorem 1.1, it's enough to claim that $|\det T| = |\det \overline{T}|$ (similarly, $|\det T^*| = |\det \overline{T}^*|$). For convenience, we can assume that the section $\overline{O}$ intersects $\Gamma_{b1}$, i.e., $\overline{D} = \Gamma_{b1}$. By Corollary 3.1, one has

$$|\det \overline{T}| = \lambda_1 \cdots \lambda_{b-1}/n. \tag{4}$$

We need to prove:

**Lemma 3.2.** $|\det T| = |\det \overline{T}|$.

*Proof.* Consider minimal partial resolution of $(S, F)$ as follows:

$$\begin{array}{ccccccccc}
(\overline{S}, \overline{F}) & \xrightarrow{\sigma_{k+1}} & (S_k, F_k) & \xrightarrow{\sigma_k} & \cdots & \longrightarrow & (S_1, F_1) & \xrightarrow{\sigma_1} & (S, F) \\
\downarrow \bar{f} & & \downarrow f_k & & & & \downarrow f_1 & & \downarrow f \\
C & = & C & = & \cdots & = & C & = & C
\end{array}$$

where $F_i$ is pulling-back fiber of $F$ in the fibration $f_i := \sigma_1 \cdots \sigma_i f : S_k \to C$ and $\sigma = \sigma_{k+1} \cdots \sigma_1$. Let $T_i$ be a negative-definite sublattice spanned by the irreducible components of $F_i$ except the strict transform of $D$ in $S_i$.

It's enough to show that $\det T_i = \det T_{i-1}$ for $i = 1, \dots, k+1$ ($T_0 := T$, $T_{k+1} := \overline{T}$). Without loss of generality, we consider only the case for $i = 1$. Let $D_0$ be the exceptional curve of $\sigma_1$, $\{D_i\}_{i=1}^{l}$'s be the basis of $T$ and $D_i'$ : $\sigma_1^* D_i - m_i D_0$ be the strict transform of $D_i$. Thus $D_0', \dots, D_l'$ are the basis of $T_1$.

Let $M = (\langle D_i, D_j\rangle)_{1\le i,j\le l}$ and $M^{'} = (\langle D_i', D_j'\rangle)_{0\le i,j\le l}$ be the intersection matrices of $T$, $T_1$. Take

$$V = \begin{pmatrix} 1 & -m_1 & -m_2 & \cdots & -m_l \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Thus one gets $\det V = 1$ and

$$\begin{pmatrix} -1 & \\ & M \end{pmatrix} = V^T M' V.$$

It implies that $|\det T| = |\det T_1|$. ☐

*Proof of Theorem 1.1.* It's from Lemma 3.2 and (4). ☐

## 4. Application

As in [15], we denote by $\mathrm{MW}(f)$ the Mordell-Weil group of $f : S \to \mathbb{P}^1$ with $s$ singular fibers $F_1, \ldots, F_s$. Let $T_i$ to be a negative-definite sublattice spanned by the irreducible components of $F_i$ which do not intersect the section $O$.

$$\mathrm{MW}(f) \cong NS(S)/\langle \oplus T_i, O, F\rangle.$$

It's well-known that the Mordell-Weil group is a finitely generated abelian group. From [15, Theorem 3], one has

$$\operatorname{rank} \mathrm{MW}(f) = \rho(S) - 2 - \sum_{i=1}^{s} (\ell(F_i) - 1)\,,$$

where $\rho(S) = \operatorname{rank} \mathrm{NS}(S)$ is the Picard number of $S$, and $\ell(F_i)$ is the number of components of $F_i$.

What can we say about a fibration $f : S \to C$ with $\operatorname{rank} \mathrm{MW}(f) = 0$? It's a very interesting problem. It's obvious that such a group is finite. By using the results in [11], one can see that $\operatorname{rank} \mathrm{MW}(f) = 0$ if $f : S \to \mathbb{P}^1$ has 2 singular fibers, or 3 singular fibers with 2 semistable ones ([4, 5]).

**Theorem 4.1** ([12], [16])**.** *Let $n$ be the order of torsion subgroup $\mathrm{MW}_{tor}$. If $|\det\ \mathrm{NS}(S)| = 1$, then*

(1) *$n^2$ divides $\prod_i \det T_i$.*
(2) *If the rank of $MW(f)$ is 0, then the order $|\mathrm{MW}(f)| = \sqrt{\prod_i \det T_i}$.*
(3) *The natural map $\mathrm{MW}_{tor} \to \oplus_i (T_i^\vee / T_i)$ is injective.*

*Remark* 4.1. The dual lattice $L^\vee$ of a lattice $L$ is defined as

$$L^\vee := \{x \in L \otimes \mathbb{Q} : (x, y) \in \mathbb{Z},\ \forall y \in L\}$$

(see [13, Sec. 6]).

*Proof of Theorem 1.2.* In [4], we have classified all fibrations $f : S \to \mathbb{P}^1$ of genus 2 with two singular fibers which are dual to each other. There are exactly 9 types of such fibrations containing at least one section. In this case, $S$ is a rational surface. We have the following computation.

By Theorem 4.1 and Theorem 1.1, we have

$$|\mathrm{MW}(f)| = |\det T_1| = |\det T_2|$$

and the following computation.

| $F$ | $\mathrm{I}^*_{0-0-0}$ | III | V | VI | VII | VIII − 1 | VIII − 2 | IX − 1 | IX − 2 |
|---|---|---|---|---|---|---|---|---|---|
| $F^*$ | $\mathrm{I}^*_{0-0-0}$ | III | V* | VI | VII* | VIII − 4 | VIII − 3 | IX − 4 | IX − 3 |
| $\lvert\det T\rvert$ | 16 | 9 | 3 | 4 | 2 | 1 | 1 | 5 | 5 |

If $|MW(f)|$ is a prime, then $MW(f)$ is a cyclic group. In what follows, we consider the cases where the orders are not a prime.

**Type** $(\mathrm{I}^*_{0-0-0}, \mathrm{I}^*_{0-0-0})$. In this case, $F_1$ is a normal-crossing divisor. The principal component is a $(-3)$-curve with multiplicity 2, there are 6 H-J branches and each branch consists of only one $(-2)$-curve with multiplicity 1. By removing one $(-2)$-curve, we get $T_1$. So the intersection matrix of $T_1$ is $-M$, where

$$M = \begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$|\det T_1| = |\det T_2| = \det M = 16$. One can compute easily that

$$M^{-1} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}$$

$2M^{-1}$ is an integral matrix. The basis of $T_i^\vee$ is a linear combination of the basis of $T_i$, and $M^{-1}$ is the presentation matrix. Thus $2T_i^\vee \subset T_i$. Because $\mathrm{MW}_{tor} \hookrightarrow \oplus_i(T_i^\vee/T_i)$, the orders of all nonzero elements in Mordell-Weil group are 2, hence $\mathrm{MW}(f) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The calculations of **Type (III, III)** and **Type (VI, VI)** are similar. ☐

## References

[1] T. Ashikaga and K. Konno, *Global and local properties of pencils of algebraic curves*, in Algebraic geometry 2000, Azumino (Hotaka), 1–49, Adv. Stud. Pure Math., 36, Math. Soc. Japan, Tokyo, 2002. https://doi.org/10.2969/aspm/03610001

[2] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 4, Springer, Berlin, 2004. https://doi.org/10.1007/978-3-642-57739-0

[3] C. Gong, S. Kitagawa, and J. Lu, *Extremal trigonal fibrations on rational surfaces*, J. Math. Soc. Japan **73** (2021), no. 2, 505–524. https://doi.org/10.2969/jmsj/82438243

[4] C. Gong, J. Lu, and S. L. Tan, *On families of complex curves over* $\mathbb{P}^1$ *with two singular fibers*, Osaka J. Math. **53** (2016), no. 1, 83–99. http://projecteuclid.org/euclid.ojm/1455892627

[5] C. Gong, X. Lu, and S. L. Tan, *Families of curves over* $\mathbb{P}^1$ *with* 3 *singular fibers*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 9-10, 375–380. https://doi.org/10.1016/j.crma.2013.05.002

[6] C. Gong and W.-Y. Xu, *On the Mordell-Weil rank of a surface fibration*, Comm. Algebra **48** (2020), no. 2, 724–732. https://doi.org/10.1080/00927872.2019.1659286

[7] S. Kitagawa, *Extremal hyperelliptic fibrations on rational surfaces*, Saitama Math. J. **30** (2013), 1–14 (2013).

[8] S. Kitagawa and K. Konno, *Fibred rational surfaces with extremal Mordell-Weil lattices*, Math. Z. **251** (2005), no. 1, 179–204. https://doi.org/10.1007/s00209-005-0797-6

[9] K. Kodaira, *On compact complex analytic surfaces. I*, Ann. of Math. (2) **71** (1960), 111–152. https://doi.org/10.2307/1969881

[10] J. Lu and S. L. Tan, *Inequalities between the Chern numbers of a singular fiber in a family of algebraic curves*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3373–3396. https://doi.org/10.1090/S0002-9947-2012-05625-X

[11] J. Lu, S. Tan, F. Yu, and K. Zuo, *A new inequality on the Hodge number* $h^{1,1}$ *of algebraic surfaces*, Math. Z. **276** (2014), no. 1-2, 543–555. https://doi.org/10.1007/s00209-013-1212-3

[12] K. Oguiso and T. Shioda, *The Mordell-Weil lattice of a rational elliptic surface*, Comment. Math. Univ. St. Paul. **40** (1991), no. 1, 83–99.

[13] T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240.

[14] T. Shioda, *Mordell-Weil lattices for higher genus fibration*, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 9, 247–250. http://projecteuclid.org/euclid.pja/1195511629

[15] T. Shioda, *Mordell-Weil lattices for higher genus fibration over a curve*, in New trends in algebraic geometry (Warwick, 1996), 359–373, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999. https://doi.org/10.1017/CBO9780511721540.015

[16] N. K. Viet, *On certain Mordell-Weil lattices of hyperelliptic type on rational surfaces*, J. Math. Sci. (New York) **102** (2000), no. 2, 3938–3977. https://doi.org/10.1007/BF02984108

Cheng Gong
Department of Mathematics
Soochow University
Shizi RD 1, Suzhou 215006, Jiangsu, P. R. China
*Email address*: cgong@suda.edu.cn

Jun Lu
Department of Mathematics
Shanghai Key Laboratory of PMMP
East China Normal University
Dongchuan RD 500, Shanghai 200241, P. R. China
*Email address*: jlu@math.ecnu.edu.cn

Sheng-Li Tan
Department of Mathematics
Shanghai Key Laboratory of PMMP
East China Normal University
Dongchuan RD 500, Shanghai 200241, P. R. China
*Email address*: sltan@math.ecnu.edu.cn