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COMPLEX SYMMETRIC WEIGHTED COMPOSITION-DIFFERENTIATION OPERATORS ON H^2

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Abstract. In this paper, we study the complex symmetric weighted composition-differentiation operator $D_{\psi,\phi}$ with respect to the conjugation $JW_{\xi,\tau}$ on the Hardy space H^2. As an application, we characterize the necessary and sufficient conditions for such an operator to be normal under some mild conditions. Finally, the spectrum of $D_{\psi,\phi}$ is also investigated.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. The Hardy space H^2 is the space of all $f \in H(\mathbb{D})$ with square summable power series coefficients; that is, $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

where $\{a_n\}$ is the sequence of Maclaurin coefficients for f. The space $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space. In other words, for any $w \in \mathbb{D}$, there exists a unique function $K_w \in H^2$ such that

$$f(w) = \langle f, K_w \rangle$$

for any $f \in H^2$. It is well known that $K_w(z) = \frac{1}{1-\bar{w}z}$.

Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The operator $C_\phi : H^2 \to H^2$ given by $C_\phi f = f(\phi)$ is called a composition operator. The weighted composition operator $W_{\psi,\phi}$ on H^2 is defined as $W_{\psi,\phi} f = \psi \cdot (f \circ \phi)$. For $f \in H(\mathbb{D})$, the composition-differentiation operator D_ϕ (see [3]) is defined by

$$D_\phi f(z) = f'(\phi(z)), \ \ z \in \mathbb{D},$$

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and the weighted composition-differentiation operator $D_{\psi,\phi}$ (see [4]) is defined by

$$D_{\psi,\phi} f(z) = \psi(z) f'(\phi(z)), \ \ z \in \mathbb{D}.$$

Let H and $\mathcal{L}(H)$ represent a separable complex Hilbert space and the class of continuous linear operators on H, respectively. An operator $C : H \to H$ is called a conjugation on H if it is

(i) conjugate-linear or anti-linear: $C(\alpha x + \beta y) = \bar{\alpha} C(x) + \bar{\beta} C(y)$ for any $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$;
(ii) isometric: $\|Cx\| = \|x\|$ for any $x \in H$;
(iii) involutive: $C^2 = I$, where I is the identity operator.

It is easy to check that

$$(Jf)(z) = \overline{f(\bar{z})}$$

is a conjugation on the space H^2.

An operator $T \in \mathcal{L}(H)$ is called complex symmetric if there exists a conjugation C on H such that

$$T = CT^*C.$$

In this situation, we say that T is complex symmetric with respect to C or that T is C-symmetric. The class of complex symmetric operators includes all normal operators, binormal operators, Hankel operators, compressed Toeplitz operators and Volterra integration operator. Complex symmetric operators can be regarded as a generalization of complex symmetric matrices. The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen in [6–9].

Recently, the study of complex symmetric composition operators, weighted composition operator and weighted composition-differentiation operators on some analytic function spaces has attracted the interest of many researchers. Fatehi and Hammond in [4] studied some properties of weighted composition-differentiation operators. Han and Wang studied complex symmetric weighted composition-differentiation operators on the Hardy space and the Bergman space in [11] and [12], respectively. Liu et al. studied complex symmetric weighted composition-differentiation operators on the weighted Bergman space A^2_α and the derivative Hardy space in [16]. See [2–20] for more results and applications pertaining to complex symmetric operators.

In this paper, we investigate complex symmetric weighted composition-differentiation operators $D_{\psi,\phi}$ on the Hardy space H^2. The paper is organized as follows. Section 2 provides conditions on ϕ and ψ relating to when the operator $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$. As an application, we obtain necessary and sufficient conditions for the operator $D_{\psi,\phi}$ to be normal under some mild conditions. In Section 3, we characterize the spectrum of the operator $D_{\psi,\phi}$.

2. Main results

For any positive integer n, let

$$K_w^{(n)}(z) = \frac{n! z^n}{(1-\bar{w}z)^{1+n}}, \ \ z \in \mathbb{D}.$$

From Lemma 2.16 in [1], we see that $K_w^{(n)}$ acts as the reproducing kernel for point evaluation of the n-th derivative:

$$f^{(n)}(w) = \langle f, K_w^{(n)} \rangle$$

for each $f \in H^2$. The following lemma can be found in [12].

Lemma 2.1. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. If $D_{\psi,\phi}$ is bounded on H^2, then*

$$D^*_{\psi,\phi} K_w = \overline{\psi(w)} K^{(1)}_{\phi(w)}$$

and

$$D^*_{\psi,\phi} K^{(1)}_w = \overline{\psi'(w)} K^{(1)}_{\phi(w)} + \overline{\psi(w)\phi'(w)} K^{(2)}_{\phi(w)}$$

for every $w \in \mathbb{D}$.

Let

$$\xi(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \ \text{ and } \ \tau(z) = \frac{\lambda(a-z)}{1-\bar{a}z},$$

where $a \in \mathbb{D}$ and $|\lambda| = 1$. From [13, Lemma 2], we see that $JW_{\xi,\tau}$ is a conjugation on H^2 if and only if $\lambda a = \bar{a}$. Obviously, for any $f \in H(\mathbb{D})$,

$$JW_{\xi,\tau} f(z) = \frac{\sqrt{1-|a|^2}}{1-az} \overline{f\left(\frac{\lambda(a-\overline{z})}{1-\overline{a z}}\right)}, \ z \in \mathbb{D}.$$

Proposition 2.2. *Let $\xi(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$ and $\tau(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, where $a \in \mathbb{D}$ and $|\lambda| = 1$ such that $\lambda a = \bar{a}$. Let ϕ be a nonconstant analytic self-map of $\mathbb{D}$ and $\psi \in H^\infty$. If $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$, then the following statements hold*:

(a) *Either $\psi \equiv 0$ or ψ has a single vanishing point $\bar{a}$.*

(b) *If ψ is not identically zero, then ϕ is univalent and*

$$\psi = \frac{JW_{\xi,\tau} \overline{\psi(\beta)} K^{(1)}_{\phi(\beta)}}{\overline{\xi(\bar{\beta})\tau(\bar{\beta})} K^2_{\tau(\bar{\beta})}(\phi)}$$

for any $\beta \in \mathbb{D} \backslash \bar{a}$.

Proof. (a) Assume that $\psi(\beta) = 0$ for some $\beta \in \mathbb{D}$. Lemma 2.1 gives that

$$D^*_{\psi,\phi} K_\beta = \overline{\psi(\beta)} K^{(1)}_{\phi(\beta)} = 0.$$

Since $D_{\psi,\phi} JW_{\xi,\tau} = JW_{\xi,\tau} D^*_{\psi,\phi}$ and $JW_{\xi,\tau}$ is an isometry, it follows that

$$\|D_{\psi,\phi} JW_{\xi,\tau} K_\beta\| = \|D^*_{\psi,\phi} K_\beta\| = 0.$$

Therefore,

$$\begin{aligned} D_{\psi,\phi}JW_{\xi,\tau}K_\beta(z) &= \psi(z)\left(JW_{\xi,\tau}K_\beta(z)\right)' \circ \phi(z) \\ &= \psi(z)\frac{\sqrt{1-|a|^2}(a-\beta\bar{\lambda})}{[1-\beta a-(a-\beta\bar{\lambda})\phi(z)]^2} = 0 \end{aligned}$$

for all $z \in \mathbb{D}$. Thus $\psi \equiv 0$ or $\beta = \bar{a}$.

(b) Assume that ϕ is not univalent. Then there are two distinct points $w_1, w_2 \in \mathbb{D}$ such that $\phi(w_1) = \phi(w_2)$. Since ψ is not identically zero, it follows from (a) that at least one of $\psi(w_1)$ and $\psi(w_2)$ is not 0. If neither $\psi(w_1)$ or $\psi(w_2)$ is 0, let

$$f = \frac{K_{w_1}}{\overline{\psi(w_1)}} - \frac{K_{w_2}}{\overline{\psi(w_2)}}$$

and observe that f is a nonconstant function in H^2. For any $g \in H^2$, we have that

$$\begin{aligned} \langle g, D^*_{\psi,\phi}f\rangle &= \langle D_{\psi,\phi}g, f\rangle \\ &= \langle D_{\psi,\phi}g, \frac{K_{w_1}}{\overline{\psi(w_1)}} - \frac{K_{w_2}}{\overline{\psi(w_2)}}\rangle \\ &= \frac{1}{\psi(w_1)}\langle D_{\psi,\phi}g, K_{w_1}\rangle - \frac{1}{\psi(w_2)}\langle D_{\psi,\phi}g, K_{w_2}\rangle \\ &= g'(\phi(w_1)) - g'(\phi(w_2)) = 0. \end{aligned}$$

Since $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$, we have

$$\|D_{\psi,\phi}JW_{\xi,\tau}f\| = \|JW_{\xi,\tau}D^*_{\psi,\phi}f\| = \|D^*_{\psi,\phi}f\| = 0.$$

Therefore,

$$D_{\psi,\phi}JW_{\xi,\tau}f(z) = \psi(z)\left(JW_{\xi,\tau}f(z)\right)' \circ \phi(z) = 0$$

for any $z \in \mathbb{D}$. Since ψ is not identically zero, we have $(JW_{\xi,\tau}f(z))' \circ \phi(z) = 0$ for any $z \in \mathbb{D}$. Since ϕ is a nonconstant analytic self-map of $\mathbb{D}$, we obtain that $JW_{\xi,\tau}f$ is a constant function; that is, f is a constant function, which is a contradiction. Thus ϕ is univalent. If either $\psi(w_1)$ or $\psi(w_2)$ is 0, without loss of generality, assume that $\psi(w_1) = 0$; that is $w_1 = \bar{a}$. Let

$$f = \overline{\psi(w_2)}K_{w_1}.$$

Then

$$D^*_{\psi,\phi}\overline{\psi(w_2)}K_{w_1} = \overline{\psi(w_2)\psi(w_1)}K^{(1)}_{w_1} = 0,$$

which means that f is a constant. Indeed, the kernel of $D_{\psi,\phi}$ and $D^*_{\psi,\phi}$ consists of the constant functions in $H(\mathbb{D})$. Hence $\psi(w_2) = 0$; that is $w_2 = \bar{a}$, which is a contradiction.

Let $\beta \in \mathbb{D}$ be given. Since $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$, using Lemma 2.1, we get that

$$JW_{\xi,\tau}\overline{\psi(\beta)}K^{(1)}_{\phi(\beta)}(z) = JW_{\xi,\tau}D^*_{\psi,\phi}K_\beta(z)$$

$$\begin{aligned}
&= D_{\psi,\phi} JW_{\xi,\tau} K_\beta(z) \\
&= \psi(z)\left(JW_{\xi,\tau} K_\beta(z)\right)' \circ \phi(z) \\
&= \psi(z) \frac{\sqrt{1-|a|^2}(a-\beta\bar{\lambda})}{[1-a\beta+(\beta\bar{\lambda}-a)\phi(z)]^2} \\
&= \psi(z)\overline{\xi(\bar{\beta})\tau(\bar{\beta})} K^2_{\tau(\bar{\beta})}(\phi(z))
\end{aligned}$$

for any $z \in \mathbb{D}$. Since $\overline{\xi(\bar{\beta})\tau(\bar{\beta})} K^2_{\tau(\bar{\beta})}(\phi(z))$ does not vanish on $\mathbb{D}$, it follows that

$$\psi(z) = \frac{JW_{\xi,\tau}\overline{\psi(\beta)}K^{(1)}_{\phi(\beta)}(z)}{\overline{\xi(\bar{\beta})\tau(\bar{\beta})} K^2_{\tau(\bar{\beta})}(\phi(z))}.$$

The proof is complete. □

It is clear that $D_{\psi,\phi}$ must be identically zero if $\psi \equiv 0$. This case is trivial. From now on, we always assume that ψ is not identically zero. From Proposition 2.2, when we study the $JW_{\xi,\tau}$-symmetric operator $D_{\psi,\phi}$ on the Hardy space H^2, the only nontrivial case is that ψ has a single vanishing point $\bar{a}$. In the following theorem, we give a detailed characterization for the operator $D_{\psi,\phi}$ to be $JW_{\xi,\tau}$-symmetric on the Hardy space H^2.

Theorem 2.3. *Let $\xi(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$, $\tau(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, where $a \in \mathbb{D}$ and $|\lambda| = 1$ such that $\lambda a = \bar{a}$. Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$ be not identically zero such that $D_{\psi,\phi}$ is bounded on H^2. Then $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ if and only if*

$$\psi(z) = \frac{(1-|a|^2)^2\psi'(\bar{a})(z-\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}$$

and

$$\phi(z) = \phi(\bar{a}) + \frac{(1-|a|^2)\phi'(\bar{a})(z-\bar{a})}{1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z}.$$

Proof. First, suppose that $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$. Then

$$JW_{\xi,\tau} D^*_{\psi,\phi} K_w(z) = D_{\psi,\phi} JW_{\xi,\tau} K_w(z) \tag{1}$$

for any $z, w \in \mathbb{D}$. It follows from Lemma 2.1 that

$$\begin{aligned}
JW_{\xi,\tau} D^*_{\psi,\phi} K_w(z) &= JW_{\xi,\tau}\overline{\psi(w)}K^{(1)}_{\phi(w)}(z) \\
&= J\xi(z)\overline{\psi(w)}K^{(1)}_{\phi(w)}(\tau(z)) \\
&= \overline{\xi(\bar{z})}\psi(w)\overline{K^{(1)}_{\phi(w)}(\tau(\bar{z}))} \\
&= \frac{\sqrt{1-|a|^2}}{1-az} \cdot \frac{\psi(w)\overline{\tau(\bar{z})}}{[1-\phi(w)\overline{\tau(\bar{z})}]^2}
\end{aligned}$$

$$= \frac{\sqrt{1-|a|^2}\psi(w)\bar{\lambda}(\bar{a}-z)}{[1-az-\bar{\lambda}(\bar{a}-z)\phi(w)]^2}$$

and

$$\begin{aligned} D_{\psi,\phi}JW_{\xi,\tau}K_w(z) &= D_{\psi,\phi}J\xi(z)K_w(\tau(z)) \\ &= \psi(z)\left(\overline{\xi(\bar{z})K_w(\tau(\bar{z}))}\right)' \circ \phi(z) \\ &= \psi(z)\left(\frac{\sqrt{1-|a|^2}}{1-w\overline{\lambda a}+(w\bar{\lambda}-a)z}\right)' \circ \phi(z) \\ &= \left(\frac{\sqrt{1-|a|^2}(a-w\bar{\lambda})\psi(z)}{[1-w\overline{\lambda a}+(w\bar{\lambda}-a)z]^2}\right)' \circ \phi(z) \\ &= \frac{\sqrt{1-|a|^2}(a-w\bar{\lambda})\psi(z)}{[1-w\overline{\lambda a}+(w\bar{\lambda}-a)\phi(z)]^2}. \end{aligned}$$

Therefore, we obtain that

$$\frac{\sqrt{1-|a|^2}(a-w\bar{\lambda})\psi(z)}{[1-w\overline{\lambda a}+(w\bar{\lambda}-a)\phi(z)]^2} = \frac{\sqrt{1-|a|^2}\psi(w)\bar{\lambda}(\bar{a}-z)}{[1-az-\bar{\lambda}(\bar{a}-z)\phi(w)]^2} \tag{2}$$

for any $z, w \in \mathbb{D}$. From Proposition 2.2, we see that $\psi(\bar{a}) = 0$. Set $\psi(z) = (z-\bar{a})^k g(z)$, where g is analytic on $\mathbb{D}$ with $g(\bar{a}) \neq 0$ and k is a positive integer. Then (2) becomes

$$\frac{(z-\bar{a})^{k-1}g(z)}{[1-w\overline{\lambda a}+(w\bar{\lambda}-a)\phi(z)]^2} = \frac{(w-\bar{a})^{k-1}g(w)}{[1-az-\bar{\lambda}(\bar{a}-z)\phi(w)]^2} \tag{3}$$

for any $z, w \in \mathbb{D}$. If $k > 1$, let $w = \bar{a}$ in (3). We obtain that $g \equiv 0$. This contradicts the assumption that $g(\bar{a}) \neq 0$. Therefore, $k = 1$ and

$$\frac{g(z)}{[1-w\overline{\lambda a}+(w\bar{\lambda}-a)\phi(z)]^2} = \frac{g(w)}{[1-az-\bar{\lambda}(\bar{a}-z)\phi(w)]^2} \tag{4}$$

for any $z, w \in \mathbb{D}$. Let $w = \bar{a}$ in (4). Noting that $\lambda a = \bar{a}$, we get

$$g(z) = \frac{(1-|a|^2)^2 g(\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}.$$

Since $\psi(z) = (z-\bar{a})g(z)$, we obtain that

$$\psi(z) = \frac{(1-|a|^2)^2\psi'(\bar{a})(z-\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}, \tag{5}$$

where $\psi'(\bar{a}) = g(\bar{a}) \neq 0$. (2) and (5) give that

$$\begin{aligned} &\frac{az-|a|^2-\bar{\lambda}wz+aw}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2[1-aw-\bar{\lambda}(\bar{a}-w)\phi(z)]^2} \\ &= \frac{az-|a|^2-\bar{\lambda}wz+aw}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)w]^2[1-az-\bar{\lambda}(\bar{a}-z)\phi(w)]^2} \end{aligned}$$

for any $z, w \in \mathbb{D}$, which means that

$$\begin{aligned}&[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z]^2[1 - aw - \bar\lambda(\bar a - w)\phi(z)]^2\\ = &[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)w]^2[1 - az - \bar\lambda(\bar a - z)\phi(w)]^2\end{aligned}$$

for any $z, w \in \mathbb{D}$. By taking the derivative with respect to z, we have that

$$\begin{aligned}&[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z](\bar\lambda\phi(\bar a) - a)[1 - aw - \bar\lambda(\bar a - w)\phi(z)]^2\\ &- [1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z]^2[1 - aw - \bar\lambda(\bar a - w)\phi(z)]\bar\lambda(\bar a - w)\phi'(z)\\ = &[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)w]^2[1 - az - \bar\lambda(\bar a - z)\phi(w)](\bar\lambda\phi(w) - a).\end{aligned}$$

Letting $z = \bar a$, we obtain that

$$\begin{aligned}&(1 - |a|^2)(\bar\lambda\phi(\bar a) - a)[1 - aw - a\phi(\bar a) + \bar\lambda w\phi(\bar a)]^2\\ &- (1 - |a|^2)^2[1 - aw - a\phi(\bar a) + \bar\lambda w\phi(\bar a)]\bar\lambda(\bar a - w)\phi'(\bar a)\\ = &[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)w]^2(1 - |a|^2)(\bar\lambda\phi(w) - a).\end{aligned}$$

Therefore,

$$\phi(z) = \phi(\bar a) + \frac{(1 - |a|^2)\phi'(\bar a)(z - \bar a)}{1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z}.$$

Conversely, assume that

$$\psi(z) = \frac{(1 - |a|^2)^2\psi'(\bar a)(z - \bar a)}{[1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z]^2}$$

and

$$\phi(z) = \phi(\bar a) + \frac{(1 - |a|^2)\phi'(\bar a)(z - \bar a)}{1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z}.$$

By our hypothesis, $D_{\psi,\phi}$ is bounded on H^2. Hence, it is enough to verify that (1) holds for any $z, w \in \mathbb{D}$. Indeed, we have

$$\begin{aligned}&JW_{\xi,\tau}D^*_{\psi,\phi}K_w(z)\\ = &\frac{\sqrt{1 - |a|^2}\psi(w)\bar\lambda(\bar a - z)}{[1 - az - \bar\lambda(\bar a - z)\phi(w)]^2}\\ = &\frac{(1 - |a|^2)^{5/2}\bar\lambda(\bar a - z)\psi'(\bar a)(w - \bar a)}{\{[1 - az - \bar\lambda(\bar a - z)\phi(\bar a)][1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)w] - \bar\lambda(\bar a - z)(1 - |a|^2)\phi'(\bar a)(w - \bar a)\}^2}\end{aligned}$$

and

$$\begin{aligned}&D_{\psi,\phi}JW_{\xi,\tau}K_w(z)\\ = &\frac{\sqrt{1 - |a|^2}(a - w\bar\lambda)\psi(z)}{[1 - w\overline{\lambda a} + (w\bar\lambda - a)\phi(z)]^2}\\ = &\frac{(1 - |a|^2)^{5/2}\bar\lambda(\bar a - z)\psi'(\bar a)(w - \bar a)}{\{[1 - aw - \bar\lambda(\bar a - w)\phi(\bar a)][1 - a\phi(\bar a) + (\bar\lambda\phi(\bar a) - a)z] - \bar\lambda(\bar a - w)(1 - |a|^2)\phi'(\bar a)(z - \bar a)\}^2}.\end{aligned}$$

Since the numerators of $JW_{\xi,\tau}D^*_{\psi,\phi}K_w(z)$ and $D_{\psi,\phi}JW_{\xi,\tau}K_w(z)$ are the same, it is enough to verify their denominators are also the same. Write

$$A=[1-az-\bar{\lambda}(\bar{a}-z)\phi(\bar{a})][1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)w]-\bar{\lambda}(\bar{a}-z)(1-|a|^2)\phi'(\bar{a})(w-\bar{a})$$

and

$$B=[1-aw-\bar{\lambda}(\bar{a}-w)\phi(\bar{a})][1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]-\bar{\lambda}(\bar{a}-w)(1-|a|^2)\phi'(\bar{a})(z-\bar{a}).$$

By a simple calculation, we obtain that

$$\begin{aligned}A &= [1-az-\bar{\lambda}(\bar{a}-z)\phi(\bar{a})][1-aw-(a-\bar{\lambda}w)\phi(\bar{a})]-(a-\bar{\lambda}z)(1-|a|^2)\phi'(\bar{a})(w-\bar{a})\\ &= (1-az)(1-aw)-(1-az)(a-\bar{\lambda}w)\phi(\bar{a})-(1-aw)(a-\bar{\lambda}z)\phi(\bar{a})\\ &\quad +(a-\bar{\lambda}z)(a-\bar{\lambda}w)\phi(\bar{a})+(1-|a|^2)\phi'(\bar{a})\bar{\lambda}(\bar{a}-w)(\bar{a}-z)=B.\end{aligned}$$

The proof is complete. ☐

Next, we give a sufficient and necessary condition for $JW_{\xi,\tau}$-symmetric weighted composition-differentiation operator $D_{\psi,\phi}$ to be normal under the assumption that $\phi(\bar{a})=\bar{a}$.

Theorem 2.4. *Let $\xi(z)=\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$, $\tau(z)=\frac{\lambda(a-z)}{1-\bar{a}z}$, where $a\in\mathbb{D}$ and $|\lambda|=1$ such that $\lambda a=\bar{a}$. Let $\psi\in H(\mathbb{D})$ be not identically zero and ϕ be an analytic self-map of $\mathbb{D}$ with $\phi(\bar{a})=\bar{a}$. If $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ on H^2, then $D_{\psi,\phi}$ is normal if and only if $\phi(z)=\phi'(0)z$ and $\psi(z)=\phi'(0)z$.*

Proof. Since $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ and $\phi(\bar{a})=\bar{a}$, Theorem 2.3 yields that

$$\phi(z)=\bar{a}+\phi'(\bar{a})(z-\bar{a}) \text{ and } \psi(z)=\psi'(\bar{a})(z-\bar{a}). \tag{6}$$

For each $z,w\in\mathbb{D}$, we obtain that

$$\begin{aligned}D^*_{\psi,\phi}D_{\psi,\phi}K_w(z) &= \bar{w}D^*_{\psi,\phi}\frac{\psi(z)}{(1-\bar{w}\phi(z))^2}\\ &= \bar{w}D^*_{\psi,\phi}\frac{\psi'(\bar{a})(z-\bar{a})}{[1-\bar{w}(\bar{a}+\phi'(\bar{a})(z-\bar{a}))]^2}\\ &= \bar{w}\psi'(\bar{a})D^*_{\psi,\phi}\frac{(z-\bar{a})}{[1-\overline{wa}-\bar{w}\phi'(\bar{a})(z-\bar{a})]^2}\\ &= \frac{\bar{w}\psi'(\bar{a})}{(1-\overline{wa})^2}D^*_{\psi,\phi}K^{(1)}_{\sigma(w)}(z-\bar{a}),\end{aligned}$$

where $\sigma(w)=\frac{w\overline{\phi'(\bar{a})}}{1-wa}$. Lemma 2.1 gives that

$$\begin{aligned}&D^*_{\psi,\phi}D_{\psi,\phi}K_w(z)\\ &= \frac{\bar{w}\psi'(\bar{a})}{(1-\overline{wa})^2}\left(\overline{\psi'(\sigma(w))}K^{(1)}_{\phi(\sigma(w))}(z-\bar{a})+\overline{\psi(\sigma(w))\phi'(\sigma(w))}K^{(2)}_{\phi(\sigma(w))}(z-\bar{a})\right)\end{aligned}$$

$$= \frac{\bar{w}|\psi'(\bar{a})|^2}{(1-\overline{wa})^2}\cdot\left(\frac{z-\bar{a}}{[1-\overline{\phi(\sigma(w))}(z-\bar{a})]^2}+\frac{2\overline{(\sigma(w)-\bar{a})\phi'(\bar{a})}(z-\bar{a})^2}{[1-\overline{\phi(\sigma(w))}(z-\bar{a})]^3}\right)$$
$$= \frac{\bar{w}|\psi'(\bar{a})|^2}{(1-\overline{wa})^2}\cdot\frac{[1-\overline{\phi(\sigma(w))}(z-\bar{a})](z-\bar{a})+2\overline{(\sigma(w)-\bar{a})\phi'(\bar{a})}(z-\bar{a})^2}{[1-\overline{\phi(\sigma(w))}(z-\bar{a})]^3}$$
$$= \bar{w}|\psi'(\bar{a})|^2\cdot\frac{(1-\overline{wa})(z-\bar{a})+[\bar{w}|\phi'(\bar{a})|^2+(|a|^2\bar{w}-a)\overline{\phi'(\bar{a})}+|a|^2\bar{w}-a](z-\bar{a})^2}{[1-\overline{wa}-[(a-a\overline{\phi'(\bar{a})})(1-\overline{wa})+\bar{w}|\phi'(\bar{a}|^2)](z-\bar{a})]^3},$$

and

$$\begin{aligned}
&D_{\psi,\phi}D^*_{\psi,\phi}K_w(z)\\
&= D_{\psi,\phi}\overline{\psi(w)}K^{(1)}_{\phi(w)}(z)\\
&= D_{\psi,\phi}\overline{\psi(w)}\frac{z}{(1-\overline{\phi(w)}z)^2}\\
&= \psi(z)\overline{\psi(w)}\left(\frac{1}{(1-\overline{\phi(w)}\phi(z))^2}+\frac{2\overline{\phi(w)}\phi(z)}{(1-\overline{\phi(w)}\phi(z))^3}\right)\\
&= \psi(z)\overline{\psi(w)}\cdot\frac{1+\overline{\phi(w)}\phi(z)}{(1-\overline{\phi(w)}\phi(z))^3}\\
&= |\psi'(\bar{a})|^2(z-\bar{a})(\bar{w}-a)\\
&\quad\times\frac{1+|a|^2+\overline{a\phi'(\bar{a})}(\bar{w}-a)+[a\phi'(\bar{a})+|\phi'(\bar{a})|^2(\bar{w}-a)](z-\bar{a})}{\{1-|a|^2-\overline{a\phi'(\bar{a})}(\bar{w}-a)-[a\phi'(\bar{a})+|\phi'(\bar{a})|^2(\bar{w}-a)](z-\bar{a})\}^3}.
\end{aligned}$$

Suppose that $D_{\psi,\phi}$ is normal, then for any $z,w\in\mathbb{D}$, we have that

$$D^*_{\psi,\phi}D_{\psi,\phi}K_w(z)=D_{\psi,\phi}D^*_{\psi,\phi}K_w(z),$$

which implies that

$$\begin{aligned}
&\bar{w}\cdot\frac{1-\overline{wa}+[\bar{w}|\phi'(\bar{a})|^2+(|a|^2\bar{w}-a)\overline{\phi'(\bar{a})}+|a|^2\bar{w}-a](z-\bar{a})}{[1-\overline{wa}-((a-a\overline{\phi'(\bar{a})})(1-\overline{wa})+\bar{w}|\phi'(\bar{a}|^2))(z-\bar{a})]^3}\\
&= (\bar{w}-a)\frac{1+|a|^2+\overline{a\phi'(\bar{a})}(\bar{w}-a)+[a\phi'(\bar{a})+|\phi'(\bar{a})|^2(\bar{w}-a)](z-\bar{a})}{\{1-|a|^2-\overline{a\phi'(\bar{a})}(\bar{w}-a)-[a\phi'(\bar{a})+|\phi'(\bar{a})|^2(\bar{w}-a)](z-\bar{a})\}^3}
\end{aligned}$$

for any $z,w\in\mathbb{D}$. Thus the constant term must be 0; that is

$$\bar{w}(1-\overline{wa})[1-|a|^2-\overline{a\phi'(\bar{a})}(\bar{w}-a)]^3=(1-\overline{wa})^3(\bar{w}-a)[1+|a|^2+\overline{a\phi'(\bar{a})}(\bar{w}-a)]$$

for any $w\in\mathbb{D}$. Let $w=\bar{a}$. We obtain that $a(1-|a|^2)^4=0$. Hence, $a=0$, $\phi(z)=\phi'(0)z$ and $\psi(z)=\psi'(0)z$.

Conversely, suppose that $\phi(z)=z$ and $\psi(z)=\phi'(0)z$. We see that $D_{\psi,\phi}$ is normal from Proposition 2.7 in [11]. □

The next theorem gives a sufficient and necessary condition for $JW_{\xi,\tau}$-symmetric weighted composition-differentiation operator $D_{\psi,\phi}$ to be normal if $\phi'(\bar{a})=0$.

Theorem 2.5. *Let* $\xi(z)=\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$, $\tau(z)=\frac{\lambda(a-z)}{1-\bar{a}z}$, *where* $a\in\mathbb{D}$ *and* $|\lambda|=1$ *such that* $\lambda a=\bar{a}$. *Let* $\psi\in H(\mathbb{D})$ *be not identically zero and* ϕ *be an analytic self-map of* $\mathbb{D}$ *with* $\phi'(\bar{a})=0$. *If* $D_{\psi,\phi}$ *is complex symmetric with respect to the conjugation* $JW_{\xi,\tau}$, *then* $D_{\psi,\phi}$ *is normal if and only if* $a=0$ *and* $-\overline{\phi(0)}=\bar{\lambda}\phi(0)$.

Proof. Since $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ and $\phi'(\bar{a})=0$, Theorem 2.3 yields that

$$\phi(z)=\phi(\bar{a}) \text{ and } \psi(z)=\frac{(1-|a|^2)^2\psi'(\bar{a})(z-\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}. \tag{7}$$

For each $z,w\in\mathbb{D}$, we obtain that

$$\begin{aligned}
D^*_{\psi,\phi}D_{\psi,\phi}K_w(z) &= \bar{w}D^*_{\psi,\phi}\frac{\psi(z)}{(1-\bar{w}\phi(z))^2}\\
&= \frac{\bar{w}}{(1-\bar{w}\phi(\bar{a}))^2}D^*_{\psi,\phi}\frac{(1-|a|^2)^2\psi'(\bar{a})(z-\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}\\
&= \frac{\bar{w}(1-|a|^2)^2\psi'(\bar{a})}{(1-\bar{w}\phi(\bar{a}))^2}D^*_{\psi,\phi}\frac{z-\bar{a}}{[1-|a|^2+(\bar{\lambda}\phi(\bar{a})-a)(z-\bar{a})]^2}\\
&= \frac{\bar{w}\psi'(\bar{a})}{(1-\bar{w}\phi(\bar{a}))^2}D^*_{\psi,\phi}K^{(1)}_{\xi}(z-\bar{a})\\
&= \frac{\bar{w}\psi'(\bar{a})}{(1-\bar{w}\phi(\bar{a}))^2}\cdot\overline{\psi'(\xi)}K^{(1)}_{\phi(\bar{a})}(z-\bar{a}),
\end{aligned}$$

where

$$\xi=\frac{\bar{a}-\lambda\overline{\phi(\bar{a})}}{1-|a|^2},\quad K^{(1)}_{\phi(\bar{a})}(z-\bar{a})=\frac{z-\bar{a}}{[1+\overline{a\phi(\bar{a})}-\overline{\phi(\bar{a})z}]^2}$$

and

$$\psi'(\xi)=\frac{(1-|a|^2)^2\psi'(\bar{a})[1+a\phi(\bar{a})-2|a|^2-(\bar{\lambda}\phi(\bar{a})-a)\xi]}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)\xi]^3}.$$

Therefore,

$$\begin{aligned}
&D^*_{\psi,\phi}D_{\psi,\phi}K_w(z)\\
&=\frac{(1-|a|^2)^4|\psi'(\bar{a})|^2\bar{w}(z-\bar{a})[1-2|a|^2+2|a|^4-|a|^2a\phi(\bar{a})+|\phi(\bar{a})|^2-\overline{a\phi(\bar{a})}]}{(1-\bar{w}\phi(\bar{a}))^2[1+\overline{a\phi(\bar{a})}-\overline{\phi(\bar{a})z}]^2[1-2|a|^2+\overline{a\phi(\bar{a})}-|\phi(\bar{a})|^2+a|a|^2\phi(\bar{a})]^3}.
\end{aligned}$$

For each $z,w\in\mathbb{D}$, we have that

$$\begin{aligned}
&D_{\psi,\phi}D^*_{\psi,\phi}K_w(z)\\
&= D_{\psi,\phi}\overline{\psi(w)}K^{(1)}_{\phi(\bar{a})}(z)\\
&= D_{\psi,\phi}\overline{\psi(w)}\frac{z}{(1-\overline{\phi(\bar{a})}z)^2}\\
&= \psi(z)\overline{\psi(w)}\left(\frac{1}{(1-\overline{\phi(\bar{a})}\phi(z))^2}+\frac{2\overline{\phi(\bar{a})}\phi(z)}{(1-\overline{\phi(\bar{a})}\phi(z))^3}\right)
\end{aligned}$$

$$= \psi(z)\overline{\psi(w)} \cdot \frac{1+|\phi(\bar{a})|^2}{(1-|\phi(\bar{a})|^2)^3}$$
$$= \frac{(1-|a|^2)^4|\psi'(\bar{a})|^2(\bar{w}-a)(z-\bar{a})(1+|\phi(\bar{a})|^2)}{[1-\overline{a\phi(\bar{a})}+(\lambda\overline{\phi(\bar{a})}-\bar{a})\bar{w}]^2[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2(1-|\phi(\bar{a})|^2)^3}.$$

Suppose that $D_{\psi,\phi}$ is normal. Then for any $z, w \in \mathbb{D}$, we have that

$$D^*_{\psi,\phi}D_{\psi,\phi}K_w(z) = D_{\psi,\phi}D^*_{\psi,\phi}K_w(z),$$

which implies that

$$\frac{\bar{w}(z-\bar{a})[1-2|a|^2+2|a|^4-|a|^2a\phi(\bar{a})+|\phi(\bar{a})|^2-\overline{a\phi(\bar{a})}]}{(1-\bar{w}\phi(\bar{a}))^2[1+\overline{a\phi(\bar{a})}-\overline{\phi(\bar{a})z}]^2[1-2|a|^2+\overline{a\phi(\bar{a})}-|\phi(\bar{a})|^2+a|a|^2\phi(\bar{a})]^3} \tag{8}$$
$$= \frac{(\bar{w}-a)(z-\bar{a})(1+|\phi(\bar{a})|^2)}{[1-\overline{a\phi(\bar{a})}+(\lambda\overline{\phi(\bar{a})}-\bar{a})\bar{w}]^2[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2(1-|\phi(\bar{a})|^2)^3}$$

for any $z, w \in \mathbb{D}$. Letting $w = 0$, we obtain that $a(z-\bar{a})(1+|\phi(\bar{a})|^2) = 0$ for any $z \in \mathbb{D}$. Thus, $a = 0$. Then by (8) we get

$$\frac{\bar{w}z(1+|\phi(\bar{a})|^2)}{(1-\phi(0)\bar{w})^2(1-\overline{\phi(0)}z)^2(1-|\phi(0)|^2)^3}$$
$$= \frac{\bar{w}z(1+|\phi(\bar{a})|^2)}{(1+\lambda\overline{\phi(0)}\bar{w})^2(1+\bar{\lambda}\phi(0)z)^2(1-|\phi(0)|^2)^3}$$

for any $z, w \in \mathbb{D}$. This implies that

$$(1-\phi(0)\bar{w})^2(1-\overline{\phi(0)}z)^2 = (1+\lambda\overline{\phi(0)}\bar{w})^2(1+\bar{\lambda}\phi(0)z)^2 \tag{9}$$

for any $z, w \in \mathbb{D}$. Letting $w = 0$ in (9), we see that

$$(1-\overline{\phi(0)}z)^2 = (1+\bar{\lambda}\phi(0)z)^2$$

for any $z \in \mathbb{D}$. By taking the derivative with respect to z, we have

$$2(1-\overline{\phi(0)}z)(-\overline{\phi(0)}) = 2\bar{\lambda}\phi(0)(1+\bar{\lambda}\phi(0)z) \tag{10}$$

for any $z \in \mathbb{D}$. Considering $z = 0$, we get that $-\overline{\phi(0)} = \bar{\lambda}\phi(0)$.

Conversely, assume that $a = 0$ and $-\overline{\phi(0)} = \bar{\lambda}\phi(0)$. The desired result follows from a direct computation. □

3. Spectral properties

In this section, we completely characterize the spectral properties of the compact $JW_{\xi,\tau}$-symmetric weighted composition-differentiation operator $D_{\psi,\phi}$ satisfying the condition that $\phi(\bar{a}) = 0$ in Theorem 2.4 and $\phi'(\bar{a}) = 0$ in Theorem 2.5, respectively. We use $\sigma(T)$ to denote the spectrum of the operator $T \in \mathcal{B}(H)$.

Theorem 3.1. *Let* $\xi(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$, $\tau(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, *where* $a \in \mathbb{D}$ *and* $|\lambda| = 1$ *such that* $\lambda a = \bar{a}$. *Let* $n \in \mathbb{N}^+$, $\psi \in H(\mathbb{D})$ *be not identically zero and* ϕ *be a nonconstant analytic self-map of* $\mathbb{D}$ *with* $\phi(\bar{a}) = \bar{a}$ *such that* $D_{\psi,\phi}$ *is compact on* H^2. *If* $D_{\psi,\phi}$ *is complex symmetric with respect to the conjugation* $JW_{\xi,\tau}$, *then*

$$\sigma(D_{\psi,\phi}) = \{0\} \cup \{n\psi'(\bar{a})\phi'^{(n-1)}(\bar{a}), n \in \mathbb{N}^+\}.$$

Moreover, $f(z) = (z - \bar{a})^n$ *is an eigenvector of* $D_{\psi,\phi}$ *with respect to the eigenvalue* $n\psi'(\bar{a})\phi'^{(n-1)}(\bar{a})$ *for each* $n \in \mathbb{N}^+$.

Proof. Assume that $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ and $\phi(\bar{a}) = \bar{a}$. Theorem 2.3 gives that

$$\phi(z) = \bar{a} + \phi'(\bar{a})(z - \bar{a}) \text{ and } \psi(z) = \psi'(\bar{a})(z - \bar{a}). \tag{11}$$

For each nonzero $\gamma \in \sigma(D_{\psi,\phi})$, since $D_{\psi,\phi}$ is compact on H^2, there is a nonzero element $f \in H^2$ such that $D_{\psi,\phi}f = \gamma f$; that is,

$$\psi(z)f'(\phi(z)) = \gamma f(z), \;\; z \in \mathbb{D}.$$

Write $f(z) = \sum_{n=0}^{\infty} a_n(z - \bar{a})^n$. Then for any $z \in \mathbb{D}$,

$$\sum_{n=0}^{\infty} \psi'(\bar{a})na_n\phi'^{(n-1)}(\bar{a})(z - \bar{a})^n = \sum_{n=0}^{\infty} \gamma a_n(z - \bar{a})^n.$$

Therefore,

$$\psi'(\bar{a})na_n\phi'^{(n-1)}(\bar{a}) = \gamma a_n.$$

Since $f \not\equiv 0$, there is $n_0 \in \mathbb{N}^+$ such that $a_{n_0} \neq 0$. Hence, $\gamma = n_0\psi'(\bar{a})\phi'^{(n_0-1)}(\bar{a})$. If there is another $n_1 \in \mathbb{N}^+$ such that $n_1 \neq 0$, then $\gamma = n_1\psi'(\bar{a})\phi'^{(n_1-1)}(\bar{a})$. Since ϕ is nonconstant, we obtain that $\phi'(\bar{a}) \neq 0$. Write $\phi'(\bar{a}) = re^{i\theta}$, where $\theta \in [0, 2\pi], r > 0$. Then

$$e^{i(n_0-n_1)\theta} = \frac{n_1 r^{n_1-n_0}}{n_0},$$

which implies that $n_0 = n_1$. Thus, $f(z) = a_{n_0}(z - \bar{a})^{n_0}$.

Conversely, if there is $n_0 \in \mathbb{N}^+$ such that $\gamma = n_0\psi'(\bar{a})\phi'^{(n_0-1)}(\bar{a})$. Set $f(z) = (z - \bar{a})^{n_0}$. By a simple calculation, we obtain that $D_{\psi,\phi}f = \gamma f$. □

Theorem 3.2. *Let* $\xi(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$, $\tau(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, *where* $a \in \mathbb{D}$ *and* $|\lambda| = 1$ *such that* $\lambda a = \bar{a}$. *Let* $n \in \mathbb{N}^+$, $\psi \in H(\mathbb{D})$ *be not identically zero and* ϕ *be a nonconstant analytic self-map of* $\mathbb{D}$ *with* $\phi'(\bar{a}) = 0$ *such that* $D_{\psi,\phi}$ *is compact on* H^2. *If* $D_{\psi,\phi}$ *is complex symmetric with respect to the conjugation* $JW_{\xi,\tau}$, *then*

$$\sigma(D_{\psi,\phi}) = \begin{cases} \{0\} \cup \{\psi'(\phi(\bar{a}))\}, & \textit{when } \psi'(\phi(\bar{a})) \neq 0, \\ \{0\}, & \textit{when } \psi'(\phi(\bar{a})) = 0. \end{cases} \tag{12}$$

Moreover, if $\psi'(\phi(\bar{a})) \neq 0$, *then* ψ *is an eigenvector of* $D_{\psi,\phi}$ *with respect to the eigenvalue* $\psi'(\phi(\bar{a}))$.

Proof. Assume that $D_{\psi,\phi}$ is complex symmetric with respect to the conjugation $JW_{\xi,\tau}$ and $\phi'(\bar{a}) = 0$. Theorem 2.3 yields that

$$\phi(z) = \phi(\bar{a}) \text{ and } \psi(z) = \frac{(1-|a|^2)^2\psi'(\bar{a})(z-\bar{a})}{[1-a\phi(\bar{a})+(\bar{\lambda}\phi(\bar{a})-a)z]^2}. \tag{13}$$

For each nonzero $\gamma \in D_{\psi,\phi}$, since $D_{\psi,\phi}$ is compact on H^2, there is a nonzero element $f \in H^2$ such that $D_{\psi,\phi}f = \gamma f$; that is,

$$\psi(z)f'(\phi(z)) = \gamma f(z), \quad z \in \mathbb{D}.$$

Since $\phi(z) = \phi(\bar{a})$, we have $\psi(z)f'(\phi(\bar{a})) = \gamma f(z)$ for any $z \in \mathbb{D}$. This means that $f'(\phi(\bar{a})) \neq 0$ and $\psi'(z)f'(\phi(\bar{a})) = \gamma f'(z)$ for any $z \in \mathbb{D}$. Considering $z = \phi(\bar{a})$, we have that $\gamma = \psi'(\phi(\bar{a}))$. When $\psi'(\phi(\bar{a})) = 0$, this contradicts $\gamma \neq 0$. In this case, we obtain that $\sigma(D_{\psi,\phi}) = \{0\}$. When $\psi'(\phi(\bar{a})) \neq 0$, it follows that for any $z \in \mathbb{D}$,

$$D_{\psi,\phi}\psi(z) = \psi(z)\psi'(\phi(z)) = \psi(z)\psi'(\phi(\bar{a})).$$

In this case, $\sigma(D_{\psi,\phi}) = \{0\} \cup \psi'(\phi(\bar{a}))$. □

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