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SINGLE STEP REAL-VALUED ITERATIVE METHOD FOR LINEAR SYSTEM OF EQUATIONS WITH COMPLEX SYMMETRIC MATRICES

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Abstract. For solving complex symmetric positive definite linear systems, we propose a single step real-valued (SSR) iterative method, which does not involve the complex arithmetic. The upper bound on the spectral radius of the iteration matrix of the SSR method is given and its convergence properties are analyzed. In addition, the quasi-optimal parameter which minimizes the upper bound for the spectral radius of the proposed method is computed. Finally, numerical experiments are given to demonstrate the effectiveness and robustness of the propose methods.

1. Introduction

We consider the system of linear equations of the form

$$Az = b,\ A \in \mathbb{C}^{n\times n} \ \text{ and } \ z, b \in \mathbb{C}^n. \tag{1.1}$$

Here, $A = W + iT$, and $W, T \in \mathbb{R}^{n\times n}$ are real symmetric positive definite, and $z = x+iy$, $b = p+iq$, the vectors x, y, p, q as all in $\mathbb{R}^n$. Here and in the sequel, i denotes the imaginary unit. We may encounter linear system of equations (1.1) in many problems such as diffuse optical tomography [1], an FFT-based solution of certain time-dependent PDEs [9], quantum mechanics [17], molecular scattering [15], structural dynamics [11], and lattice quantum chromodynamics [12]. The reader can refer to [8] for more examples and additional references.

Due to the universal existence and significance of the complex symmetric linear systems, several iteration methods have been presented to solve (1.1) in recent years. In the remaining part of this article, we will introduce some existing iterative methods, which involve a positive parameter α. When both W

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and T are symmetric positive semi-definite with at least one of them being positive definite, some efficient iterative methods have been proposed. Recently, based on the Hermitian and skew-Hermitian splitting (HSS) [6] method, Bai et al. developed the modified HSS (MHSS) [2] iteration method to solve the system of linear equations (1.1), which is unconditionally convergent to the exact solution. Furthermore, by taking advantage of the parameter accelerating strategy, Bai [5] presented the accelerated HSS (AHSS) iteration method. To further speed up the convergence rate, Bai [14] gave the preconditioned MHSS (PMHSS) [3] iteration method. Besides, there are also some other effective iterative methods, the Lopsided PMHSS (LPMHSS) [14] iteration method, the generalized PMHSS (GPMHSS) [10] iteration method, the efficient block splitting (PBS) [13] iteration method, the double-step scale splitting (DSS) [20] iteration method, the double-step scale splitting real-valued (DSS) [19] iteration method. The PMHSS [3], DSS [19] iteration methods can be described as follows:

The PMHSS iterative method [3]: Give any initial vector $z^{(0)} \in \mathbb{C}^n$ for $k = 0, 1, 2, \ldots$ until the iterative sequence $\{z^{(k)}\}$ converges, and calculate the following program

$$\begin{cases} (\alpha V + W) z^{(k+\frac{1}{2})} = (\alpha V - iT) z^{(k)} + b, \\ (\alpha V + T) z^{(k+1)} = (\alpha V + iW) z^{(k+\frac{1}{2})} - ib, \end{cases} \tag{1.2}$$

where $\alpha > 0$, V is a symmetric positive definite matrix and I is the identity matrix.

The iteration matrix for the PMHSS method is obtained as

$$L(V, \alpha) = (\alpha V + T)^{-1} (\alpha V + iW)(\alpha V + W)^{-1} (\alpha V - iT).$$

When $V = W$, one has

$$L(\alpha) := L(W, \alpha) = \frac{\alpha + i}{\alpha + 1} (\alpha W + T)^{-1} (\alpha W - iT).$$

The DSS iterative method [19]: Give any initial vector $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ for $k = 0, 1, 2, \ldots$ until the iterative sequence $({x^{(k)}}^T, {y^{(k)}}^T)^T$ converges, and calculate the following program

$$\begin{cases} \begin{bmatrix} \alpha T + W & 0 \\ 0 & \alpha T + W \end{bmatrix} \begin{bmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{bmatrix} = \begin{bmatrix} \alpha T & T \\ -T & \alpha T \end{bmatrix} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}, \\ \begin{bmatrix} \alpha W + T & 0 \\ 0 & \alpha W + T \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha W & -W \\ W & \alpha W \end{bmatrix} \begin{bmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{bmatrix} + \begin{bmatrix} q \\ -p \end{bmatrix}, \end{cases} \tag{1.3}$$

where $\alpha > 0$, and I is the identity matrix.

The iteration matrix for DSS method is as follows:

$$M(\alpha) = \begin{bmatrix} \alpha W + T & 0 \\ 0 & \alpha W + T \end{bmatrix}^{-1} \begin{bmatrix} \alpha W & -W \\ W & \alpha W \end{bmatrix} \begin{bmatrix} \alpha T + W & 0 \\ 0 & \alpha T + W \end{bmatrix}^{-1} \begin{bmatrix} \alpha T & T \\ -T & \alpha T \end{bmatrix}.$$

If the matrices W and T are symmetric positive semi-definite, some efficient iteration methods have been presented. Bai et al. [7] applied the HSS method to the preconditioned block linear systems and proposed the preconditioned HSS (PHSS) iteration method to improve the convergence speed of the HSS method. Besides, Wang et al. [18] developed the combination method of real part and imaginary part (CRI) iteration method. The CRI iteration method can be described as following:

The CRI iterative method [18]: Give any initial vector $z^{(0)} \in \mathbb{C}^n$ for $k = 0, 1, 2, \ldots$ until the iterative sequence $\{z^{(k)}\}$ converges, and calculate the following program

$$\begin{cases} (\alpha T + W)z^{(k+\frac{1}{2})} = (\alpha - i)Tz^{(k)} + b, \\ (\alpha W + T)z^{(k+1)} = (\alpha + i)Wz^{(k+\frac{1}{2})} - ib, \end{cases} \tag{1.4}$$

where $\alpha > 0$, and I is the identity matrix.

The iteration matrix for the CRI method is

$$\tau(\alpha) = (\alpha^2 + 1)(\alpha W + T)^{-1}W(\alpha T + W)^{-1}.$$

Moreover, Shirilord and Dehghan designed the single step [16] iteration method for solving positive semi-definite linear systems. The specific construction of the iterative method is as follows. Multiplying both sides of (1.1) by $-i\alpha$ leads to

$$\alpha Tz = i\alpha Wz - i\alpha b, \tag{1.5}$$

then, adding Wz to both sides of the above equation yields that

$$(\alpha T + W)z = (i\alpha + 1)Wz - i\alpha b. \tag{1.6}$$

Based on the above matrix splitting form, Shirilord and Dehghan [16] designed the single iterative method as follows:

$$(\alpha T + W)z^{k+1} = (i\alpha + 1)Wz^k - i\alpha b, \tag{1.7}$$

where $\alpha > 0$. In each step of the above iteration procedure, the method need solve a linear system with nonsingular symmetric positive definite coefficient matrix $\alpha T + W$ that can be determined numerically by iterative schemes or exactly by direct algorithms. The method (1.7) can be reformulated as:

$$z^{k+1} = G_{SS}(\alpha)z^k + R_{SS}(\alpha)b,$$

where

$$G_{SS}(\alpha) = (i\alpha + 1)(\alpha T + W)^{-1}W,$$

and $R_{SS}(\alpha) = -i\alpha(\alpha T + W)^{-1}$. Note that the matrix $G_{SS}(\alpha)$ is the iteration matrix of the method (1.7). Let

$$B_{SS}(\alpha) = \frac{i}{\alpha}(\alpha T + W), \qquad C_{SS}(\alpha) = \frac{i - \alpha}{\alpha}W.$$

Then $A = B_{SS}(\alpha) - C_{SS}(\alpha)$, and $G_{SS}(\alpha) = B_{SS}(\alpha)^{-1}C_{SS}(\alpha)$, therefore the matrix $B_{SS}(\alpha)$ can be considered as a preconditioner for the coefficient matrix A.

It follows from the iterative scheme (1.7) that the single iterative method involves the complex arithmetic, which may decrease its efficiency. In order to overcome this shortcoming and improve its the computational efficiency, we apply the real value acceleration technique in [4] to the single step iterative method [16] and establish the single step real-valued (SSR) iteration method to solve system of linear equations (1.1), which does not involve the complex arithmetic. Moreover, we discuss the convergence properties of the proposed method and give the quasi-optimal parameter minimizing the upper bound of the spectral radius of the iteration matrix.

The rest of this paper is organized as follows. In Section 2, we introduce the single step real-valued (SSR) iteration method for solving the system (1.1). The convergence of the SSR method is investigated and the quasi-optimal parameter minimizing the upper bound of the spectral radius of the iteration matrix is given in Section 3. In Section 4, numerical examples are reported to examine the effectiveness and robustness of the proposed method. Finally, we make brief concluding remarks in Section 5.

Throughout the paper, we use $sp(\cdot)$ and $\rho(\cdot)$ to denote the set of eigenvalue and spectral radius of the corresponding matrix, respectively. Moreover, the Euclidean norm of a vector or a matrix is represented by $\|\cdot\|_2$. Moreover the symbols I_n and I_m denote identity matrices by dimensions n and m, respectively.

2. The single step real-valued (SSR) iteration method

In this section, we will develop a single step real-valued iteration method for solving problem (1.1) by splitting the complex coefficient matrix into the real and imaginary parts and using them simultaneously.

To this end, substituting $z = x + iy$ and $b = p + iq$ into (1.6), we can get

$$(\alpha T + W)(x + iy) = (i\alpha + 1)W(x + iy) - i\alpha(p + iq).$$

After some calculations and transposition, it is easy to obtain that

$$(\alpha T + W)x + i(\alpha T + W)y = W(x - \alpha y) + iW(\alpha x + y) + \alpha q - i\alpha p. \tag{2.1}$$

Comparing the real part and imaginary part on both sides of the above equation, we can derive a single step real-valued iterative method, abbreviated as SSR iteration method.

The single step real-valued iterative method: Given any initial vector $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ for $k = 0, 1, 2, \ldots$, until the iterative sequence $({x^{(k)}}^T, {y^{(k)}}^T)^T$ converges, and calculate the following program

$$\begin{cases} (\alpha T + W)x^{(k+1)} = W(x^{(k)} - \alpha y^{(k)}) + \alpha q, \\ (\alpha T + W)y^{(k+1)} = W(\alpha x^{(k+1)} + y^{(k)}) - \alpha p, \end{cases} \tag{2.2}$$

where α is a given positive constant.

After simple calculations, we can reformulate the SSR iteration scheme into the following fixed-point equation

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = G_{SSR}(\alpha) \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + R_{SSR}(\alpha) \begin{pmatrix} p \\ q \end{pmatrix}, \tag{2.3}$$

where

$$G_{SSR}(\alpha) = \begin{pmatrix} (\alpha T + W)^{-1} W & -\alpha(\alpha T + W)^{-1} W \\ \alpha(\alpha T + W)^{-1} W (\alpha T + W)^{-1} W & (\alpha T + W)^{-1} W - \alpha^2 (\alpha T + W)^{-1} W (\alpha T + W)^{-1} W \end{pmatrix}$$

and

$$R_{SSR}(\alpha) = \begin{pmatrix} 0 & \alpha(\alpha T + W)^{-1} \\ -\alpha(\alpha T + W)^{-1} & \alpha^2 (\alpha T + W)^{-1} W (\alpha T + W)^{-1} \end{pmatrix}.$$

If let $S = (\alpha W^{-\frac{1}{2}} T W^{-\frac{1}{2}} + I_n)^{-1}$, then $G_{SSR}(\alpha)$ and $R_{SSR}(\alpha)$ can be written as

$$G_{SSR}(\alpha) = \begin{pmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} S & -\alpha S \\ \alpha S^2 & S - \alpha^2 S^2 \end{pmatrix} \begin{pmatrix} W^{\frac{1}{2}} & 0 \\ 0 & W^{\frac{1}{2}} \end{pmatrix} \tag{2.4}$$

and

$$R_{SSR}(\alpha) = \begin{pmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 0 & \alpha S \\ -\alpha S & \alpha^2 S^2 \end{pmatrix} \begin{pmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{pmatrix}.$$

Note that the matrix $G_{SSR}(\alpha)$ is an iteration matrix of the SSR iteration method.

From the iterative scheme (2.2), we can derive the following algorithmic version of the SSR iteration method.

Algorithm 2.1. The algorithm representation of the SSR iteration method:
Step 1: Calculate $t^{(k)} = W(x^{(k)} - \alpha y^{(k)}) + \alpha q$;
Step 2: Solve $(\alpha T + W) x^{(k+1)} = t^{(k)}$;
Step 3: Calculate $s^{(k)} = W(\alpha x^{(k+1)} + y^{(k)}) - \alpha p$;
Step 4: Solve $(\alpha T + W) y^{(k+1)} = s^{(k)}$;
Step 5: Set $z^{(k+1)} = x^{(k+1)} + i y^{(k+1)}$;

From the above algorithm, we can see that the action of the SSR method requires to solve two same symmetric positive definite linear subsystems with the coefficient matrix $\alpha T + W$ in every iteration step. So we can use the Cholesky factorization to solve them exactly or conjugate gradient (CG) method inexactly. Moreover, compared to the DSS and CRI methods, two linear subsystems of the SSR method are same. This makes the proposed method needs less computation, which will be verified by numerical experiments in Section 4.

3. Convergence analysis of the SSR iteration method

In this section, we analyze the convergence theory of the SSR iterative method in detail. First we give the following lemma which will be used for the convergence property of the SSR iteration method.

Lemma 3.1. *Let $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite. Let $\mu_j > 0$ and ξ_j $(j = 1,\ldots,n)$ be the eigenvalues of the matrices $\overline{S} = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ and $S = \left(\alpha W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I_n\right)^{-1}$, respectively. Then*

$$\xi_j = \frac{1}{1+\alpha\mu_j}. \tag{3.1}$$

Proof. Since the matrix $\overline{S} = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ is real symmetric positive definite, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$\overline{S} = Q\Omega Q^T,$$

where $\Omega = diag\left(\mu_1, \quad \mu_2, \quad \ldots, \quad \mu_n\right)$, and $\mu_j > 0$ $(j = 1,\ldots,n)$ are the eigenvalues of $\overline{S}$. Then

$$\begin{aligned} S &= \left(I_n + \alpha W^{-\frac{1}{2}}TW^{-\frac{1}{2}}\right)^{-1} \\ &= (I_n + \alpha\overline{S})^{-1} \\ &= \left(I_n + \alpha Q\Omega Q^T\right)^{-1} \\ &= Q(I_n + \alpha\Omega)^{-1}Q^T. \end{aligned}$$

It follows immediately that the eigenvalues of the matrix S satisfy the relation (3.1). □

In order to analyze the convergence properties of the SSR method, the following two theorems are given. First, Theorem 3.1 gives the relationship between the eigenvalues of the two matrices $G_{SSR}(\alpha)$ and $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$.

Theorem 3.1. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, and $\alpha > 0$. Let λ and μ_j $(j = 1,\ldots,n)$ be eigenvalues of the matrices $G_{SSR}(\alpha)$ and $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$, respectively. Then*

$$\lambda^2(1+\alpha\mu_j)^2 - (2(1+\alpha\mu_j) - \alpha^2)\lambda + 1 = 0. \tag{3.2}$$

Proof. From (2.4), the iteration matrix $G_{SSR}(\alpha)$ of the SSR iteration method is

$$G_{SSR}(\alpha) = \begin{pmatrix} W^{-\frac{1}{2}} & 0 \\ 0 & W^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} S & -\alpha S \\ \alpha S^2 & S - \alpha^2 S^2 \end{pmatrix} \begin{pmatrix} W^{\frac{1}{2}} & 0 \\ 0 & W^{\frac{1}{2}} \end{pmatrix}.$$

Let λ be an eigenvalue of the matrix $G_{SSR}(\alpha)$. Then

$$\begin{aligned} 0 &= \det\left(\begin{pmatrix}\lambda I_n & 0\\ 0 & \lambda I_n\end{pmatrix} - \begin{pmatrix}W^{-\frac{1}{2}} & 0\\ 0 & W^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}S & -\alpha S\\ \alpha S^2 & S-\alpha^2S^2\end{pmatrix}\begin{pmatrix}W^{\frac{1}{2}} & 0\\ 0 & W^{\frac{1}{2}}\end{pmatrix}\right)\\ &= \det\left(\begin{pmatrix}W^{-\frac{1}{2}} & 0\\ 0 & W^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\lambda I_n - S & \alpha S\\ -\alpha S^2 & \lambda I_n - S+\alpha^2S^2\end{pmatrix}\begin{pmatrix}W^{\frac{1}{2}} & 0\\ 0 & W^{\frac{1}{2}}\end{pmatrix}\right)\\ &= \det\left(\begin{pmatrix}\lambda I_n - S & \alpha S\\ -\alpha S^2 & \lambda I_n - S+\alpha^2S^2\end{pmatrix}\right). \end{aligned}$$

So the eigenvalues of the matrix $G_{SSR}(\alpha)$ could be computed from the following determinant

$$\begin{vmatrix}\lambda - \frac{1}{1+\alpha\mu_j} & \frac{\alpha}{1+\alpha\mu_j}\\ \frac{-\alpha}{(1+\alpha\mu_j)^2} & \lambda - \frac{1}{1+\alpha\mu_j} + \frac{\alpha^2}{(1+\alpha\mu_j)^2}\end{vmatrix} = 0.$$

After some calculations, it is not difficult to verify that the relationship (3.2) is valid. This completes the proof of this theorem. □

Based on Theorem 3.1, we present the following theorem giving the specific expressions of the eigenvalues of the iterative matrix $G_{SSR}(\alpha)$.

Theorem 3.2. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, and $\alpha > 0$. Let $\mu_j > 0$ $(j = 1,\dots,n)$ be eigenvalues of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ and $\mu_1 \le \mu_2 \le \cdots \le \mu_n$. Then the eigenvalues $\lambda_j^{\pm}$ $(j = 1,\dots,n)$ of the matrix $G_{SSR}(\alpha)$ can be expressed as*

(1) *If $0 < \alpha \le 2\mu_1 + 2\sqrt{1+\mu_1^2}$, then*

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm i\alpha\sqrt{4(1+\alpha\mu_j)-\alpha^2}}{2(1+\alpha\mu_j)^2}. \tag{3.3}$$

(2) *If $\alpha \ge 2\mu_n + 2\sqrt{1+\mu_n^2}$, then*

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm \alpha\sqrt{\alpha^2-4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2}. \tag{3.4}$$

(3) *If $2\mu_1 + 2\sqrt{1+\mu_1^2} < \alpha < 2\mu_n + 2\sqrt{1+\mu_n^2}$, then*

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm \alpha\sqrt{\alpha^2-4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2} \quad (j = 1,\dots,k), \tag{3.5}$$

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm i\alpha\sqrt{4(1+\alpha\mu_j)-\alpha^2}}{2(1+\alpha\mu_j)^2} \quad (j = k+1,\dots,n), \tag{3.6}$$

where k is chosen by $2\mu_k + 2\sqrt{1+\mu_k^2} < \alpha < 2\mu_{k+1} + 2\sqrt{1+\mu_{k+1}^2}$.

Proof. It follows from Theorem 3.1 that

$$\lambda^2(1+\alpha\mu_j)^2 - (2(1+\alpha\mu_j)-\alpha^2)\lambda + 1 = 0.$$

The above equation is regarded as a quadratic equation about λ, and the discriminant of the root of the equation is

$$\triangle_\lambda^j = (2(1+\alpha\mu_j) - \alpha^2)^2 - 4(1+\alpha\mu_j)^2 = \alpha^2(\alpha^2 - 4\alpha\mu_j - 4).$$

If $0 < \alpha \leq 2\mu_1 + 2\sqrt{1+\mu_1^2}$, then $0 < \alpha \leq 2\mu_j + 2\sqrt{1+\mu_j^2}$ $(j = 1, 2, \ldots, n)$. Then for all j the discriminant $\triangle_\lambda^j < 0$, and one can obtain

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm i\alpha\sqrt{4(1+\alpha\mu_j) - \alpha^2}}{2(1+\alpha\mu_j)^2}.$$

If $\alpha \geq 2\mu_n + 2\sqrt{1+\mu_n^2}$, then $\alpha \geq 2\mu_j + 2\sqrt{1+\mu_j^2}$ $(j = 1, 2, \ldots, n)$. Then for all j the discriminant $\triangle_\lambda^j \geq 0$, and we can get

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2}.$$

If $2\mu_1 + 2\sqrt{1+\mu_1^2} < \alpha < \mu_n + 2\sqrt{1+\mu_n^2}$. Without loss of generality, we assume that there exists a k such that $2\mu_1 + 2\sqrt{1+\mu_1^2} \leq \cdots \leq 2\mu_k + 2\sqrt{1+\mu_k^2} < \alpha < 2\mu_{k+1} + 2\sqrt{1+\mu_{k+1}^2} \leq \cdots \leq \mu_n + 2\sqrt{1+\mu_n^2}$. Then $\triangle_\lambda^j \geq 0$ $(j = 1, \ldots, k)$ and $\triangle_\lambda^j \leq 0$ $(j = k+1, \ldots, n)$, and we obtain that

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2} \ (j = 1, \ldots, k),$$
$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm i\alpha\sqrt{4(1+\alpha\mu_j) - \alpha^2}}{2(1+\alpha\mu_j)^2} \ (j = k+1, \ldots, n).$$

By the above analyses, the proof of this theorem is completed. □

It is well-known that the SSR iteration method is convergent if and only if $\rho(G_{SSR}(\alpha)) < 1$. However, it is difficult to obtain convergence interval of parameter by making $\rho(G_{SSR}(\alpha)) < 1$ in general. Instead, we can consider getting the upper bound of $\rho(G_{SSR}(\alpha))$, and let the upper bound be less than 1 to get the convergence interval of the parameter α involved in the SSR iteration method. To this end, on basis of Theorem 3.2 the upper bound of $\rho(G_{SSR}(\alpha))$ of the SSR iterative method is given in the below theorem.

Theorem 3.3. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, and $\alpha > 0$. Let $\mu_j > 0$ $(j = 1, \ldots, n)$ be eigenvalues of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Then*

$$\rho(G_{SSR}(\alpha)) = \max\left|\lambda_j^{\pm}\right| \leq \sigma_{SSR}(\alpha), \tag{3.7}$$

where

$$\sigma_{SSR}(\alpha) = \begin{cases} \frac{1}{1+\alpha\mu_1} & 0 < \alpha \leq 2\mu_1 + 2\sqrt{1+\mu_1^2}, \\ \frac{\alpha^2}{(1+\alpha\mu_1)2} & \alpha > 2\mu_1 + 2\sqrt{1+\mu_1^2}. \end{cases} \tag{3.8}$$

Proof. From Theorem 3.2, the eigenvalues $\lambda_j^{\pm}$ $(j = 1, \ldots, n)$ of the matrix $G_{SSR}(\alpha)$ can be expressed as

(1) If $0 < \alpha \leq 2\mu_1 + 2\sqrt{1+\mu_1^2}$, then for $j = 1, \ldots, n$

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm i\alpha\sqrt{4(1+\alpha\mu_j) - \alpha^2}}{2(1+\alpha\mu_j)^2},$$

and one has

$$\begin{aligned}
\left|\lambda_j^{\pm}\right|^2 &= \frac{\left(2(1+\alpha\mu_j) - \alpha^2\right)^2 + (\alpha\sqrt{4(1+\alpha\mu_j) - \alpha^2})^2}{4(1+\alpha\mu_j)^4} \\
&= \frac{4(1+\alpha\mu_j)^2 - 4\alpha^2(1+\alpha\mu_j) + \alpha^4 + \alpha^2(4(1+\alpha\mu_j) - \alpha^2)}{4(1+\alpha\mu_j)^4} \\
&= \frac{4(1+\alpha\mu_j)^2}{4(1+\alpha\mu_j)^4} \\
&= \frac{1}{(1+\alpha\mu_j)^2},
\end{aligned}$$

which together with $\mu_1 > 0$ results in

$$\rho(G_{SSR}(\alpha)) = \max\left|\lambda_j^{\pm}\right| = \frac{1}{1+\alpha\mu_1}.$$

(2) If $\alpha \geq 2\mu_n + 2\sqrt{1+\mu_n^2}$, then for $j = 1, \ldots, n$

$$\lambda_j^{\pm} = \frac{2(1+\alpha\mu_j) - \alpha^2 \pm \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2}.$$

It is not difficult to verify

$$2(1+\alpha\mu_j) - \alpha^2 < 0 \quad \text{and} \quad \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)} < \alpha^2 - 2(1+\alpha\mu_j),$$

then

$$\begin{aligned}
\left|\lambda_j^{+}\right| &= \frac{\alpha^2 - 2(1+\alpha\mu_j) - \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2} \\
&< \frac{\alpha^2 - 2(1+\alpha\mu_j) + \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2} = \left|\lambda_j^{-}\right|.
\end{aligned} \tag{3.9}$$

Besides, one can get

$$\frac{\alpha^2 - 2(1+\alpha\mu_j) + \alpha\sqrt{\alpha^2 - 4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2} < \frac{\alpha^2}{(1+\alpha\mu_j)^2}. \tag{3.10}$$

It follows from Inequalities (3.9) and (3.10) that

$$\rho(G_{SSR}(\alpha)) = \max\left|\lambda_j^{\pm}\right| < \frac{\alpha^2}{(1+\alpha\mu_1)^2}.$$

(3) If $2\mu_1+2\sqrt{1+\mu_1^2}<\alpha<2\mu_n+2\sqrt{1+\mu_n^2}$, from Theorem 3.2 we can get

$$\lambda_j^{\pm}=\frac{2(1+\alpha\mu_j)-\alpha^2\pm\alpha\sqrt{\alpha^2-4(1+\alpha\mu_j)}}{2(1+\alpha\mu_j)^2}\ (j=1,\dots,k),$$

$$\lambda_j^{\pm}=\frac{2(1+\alpha\mu_j)-\alpha^2\pm i\alpha\sqrt{4(1+\alpha\mu_j)-\alpha^2}}{2(1+\alpha\mu_j)^2}\ (j=k+1,\dots,m).$$

Similar to the discussions of the cases (1) and (2), we derive the following inequalities

$$\left|\lambda_j^{\pm}\right|<\frac{\alpha^2}{(1+\alpha\mu_1)^2},\ j=1,\dots,k,$$

and

$$\left|\lambda_j^{\pm}\right|<\frac{1}{1+\alpha\mu_{k+1}}\le\frac{1}{1+\alpha\mu_1},\ j=k+1,\dots,n.$$

Therefore, in this case one has

$$\rho(G_{SSR}(\alpha))=\max\left|\lambda_j^{\pm}\right|\le\max\{\frac{1}{1+\alpha\mu_1},\ \frac{\alpha^2}{(1+\alpha\mu_1)^2}\}.$$

Next, we discuss the maximum of two functions $\frac{1}{1+\alpha\mu_1}$ and $\frac{\alpha^2}{(1+\alpha\mu_1)^2}$. Let $N(\alpha)=\frac{\frac{1}{1+\alpha\mu_1}}{\frac{\alpha^2}{(1+\alpha\mu_1)^2}}=\frac{1+\alpha\mu_1}{\alpha^2}$. Then

$$\frac{dN(\alpha)}{d\alpha}=\frac{-2-\alpha\mu_1}{\alpha^3}<0\ \ (\alpha>0).$$

Thus, the function $N(\alpha)=\frac{\frac{1}{1+\alpha\mu_1}}{\frac{\alpha^2}{(1+\alpha\mu_1)^2}}=\frac{1+\alpha\mu_1}{\alpha^2}$ decreases monotonically on $2\mu_1+2\sqrt{1+\mu_1^2}<\alpha<2\mu_n+2\sqrt{1+\mu_n^2}$. Then the function $N(\alpha)=\frac{\frac{1}{1+\alpha\mu_1}}{\frac{\alpha^2}{(1+\alpha\mu_1)^2}}=\frac{1+\alpha\mu_1}{\alpha^2}$ obtains the maximum value at $\alpha=2\mu_1+2\sqrt{1+\mu_1^2}$, and the maximum value is

$$N(2\mu_1+2\sqrt{1+\mu_1^2})=\frac{1+2\mu_1^2+2\mu_1\sqrt{1+\mu_1^2}}{4+8\mu_1^2+8\mu_1\sqrt{1+\mu_1^2}}<1.$$

It follows that $\frac{1}{1+\alpha\mu_1}<\frac{\alpha^2}{(1+\alpha\mu_1)^2}$, therefore

$$\rho(G_{SSR}(\alpha))=\max\left|\lambda_j^{\pm}\right|\le\max\{\frac{1}{1+\alpha\mu_1},\ \frac{\alpha^2}{(1+\alpha\mu_1)^2}\}=\frac{\alpha^2}{(1+\alpha\mu_1)^2}.$$

Therefore, by summarizing the above conclusions, we get the upper bound of $\rho(G_{SSR}(\alpha))$ in (3.7) and (3.8). The proof of this theorem is completed. □

Based on the upper bound of spectral radius of the iteration matrix $G_{SSR}(\alpha)$ in Theorem 3.3, the following theorem gives the convergence ranges of the parameter α involved in the SSR method.

Theorem 3.4. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, and $\alpha > 0$. Let $\mu_{\min}$ and μ_{max} be smallest and largest eigenvalue of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$, respectively. Then the SSR iteration method is convergent if the parameter α satisfies one of the following conditions*

(a) $0 < \alpha \le 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$;

(b) *when* $\mu_{\min} \ge 1$, $\alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$;

(c) *when* $\frac{3}{4} < \mu_{\min} < 1$, $2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2} < \alpha < \frac{1}{1-\mu_{\min}}$.

Proof. From Theorem 3.3, we know

$$\rho(G_{SSR}(\alpha)) \le \sigma_{SSR}(\alpha) = \begin{cases} \frac{1}{1+\alpha\mu_{\min}} & 0 < \alpha \le 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}, \\ \frac{\alpha^2}{(1+\alpha\mu_{\min})^2} & \alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}. \end{cases}$$

(1) If $0 < \alpha \le 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$, then $\sigma_{SSR}(\alpha) = \frac{1}{1+\alpha\mu_{\min}}$. Due to $\mu_{\min} > 0$, it follows immediately that $\rho(G_{SSR}(\alpha)) \le \sigma_{SSR}(\alpha) < 1$.

(2) If $\alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$, in this case $\rho(G_{SSR}(\alpha)) \le \sigma_{SSR}(\alpha) = \frac{\alpha^2}{(1+\alpha\mu_{\min})^2}$. Keeping in mind that $\alpha > 0$ and $\mu_{\min} > 0$, $\frac{\alpha^2}{(1+\alpha\mu_{\min})2} < 1$ amounts to make

$$\alpha(1-\mu_{\min}) < 1. \tag{3.11}$$

(i) When $\mu_{\min} \ge 1$, Inequality (3.11) is always valid. Therefore, in this case the convergence interval of the parameter α is $\alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$.

(ii) When $\mu_{\min} < 1$, it follows from (3.11) that $\alpha < \frac{1}{1-\mu_{\min}}$. For this case of $\alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$ and $\alpha < \frac{1}{1-\mu_{\min}}$, we should make $\frac{1}{1-\mu_{\min}} > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$ holding to guarantee that there exists α such that $\rho(G_{SSR}(\alpha)) \le \sigma_{SSR}(\alpha) < 1$. If $\frac{1}{1-\mu_{\min}} > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}$, then

$$\begin{aligned} & \frac{1}{1-\mu_{\min}} > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2} \\ \Longleftrightarrow \quad & 1 - 2\mu_{\min} + 2\mu_{\min}^2 > 2(1-\mu_{\min})\sqrt{1+\mu_{\min}^2} \\ \Longleftrightarrow \quad & (1 - 2\mu_{\min} + 2\mu_{\min}^2)^2 > (2(1-\mu_{\min})\sqrt{1+\mu_{\min}^2})^2 \\ \Longleftrightarrow \quad & \frac{3}{4} > \mu_{\min}. \end{aligned}$$

Thus when $\frac{3}{4} < \mu_{\min} < 1$, the convergence interval of the parameter α is $2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2} < \alpha < \frac{1}{1-\mu_{\min}}$. Summarizing all the above cases, the convergence conditions for the SSR iteration method are given by (a)-(c) in this theorem. □

Next, we give the following theorem where the theoretical quasi-optimal parameter minimizing the upper bound of the spectral radius of the iteration matrix of the SSR iteration method is presented.

Theorem 3.5. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, and $\alpha > 0$. Let $\mu_{\min}$ and $\mu_{\max}$ be smallest and largest eigenvalue of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$, respectively. Then the quasi-optimal parameter is*

$$\alpha_* = \arg\min_{\alpha}\{\sigma_{SSR}(\alpha)\} = 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}\,. \tag{3.12}$$

And the optimal upper bound of spectral radius of the iterative matrix is

$$\sigma_{SSR}(\alpha_*) = \frac{1}{1+2\mu_{\min}^2 + 2\mu_{\min}\sqrt{1+\mu_{\min}^2}}. \tag{3.13}$$

Proof. From Theorem 3.3, the upper bound of $\rho(G_{SSR}(\alpha))$ is

$$\sigma_{SSR}(\alpha) = \begin{cases} \frac{1}{1+\alpha\mu_{\min}} & 0 < \alpha \le 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}, \\ \frac{\alpha^2}{(1+\alpha\mu_{\min})^2} & \alpha > 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}. \end{cases}$$

To find the quasi-optimal parameter α of the SSR iteration method, we minimize the upper bound $\sigma_{SSR}(\alpha)$ of $\rho(G_{SSR}(\alpha))$.

Let $h(\alpha) = \frac{1}{1+\alpha\mu_{\min}}$ and $s(\alpha) = \frac{\alpha^2}{(1+\alpha\mu_{\min})^2}$. By some calculations, we can deduce that $\frac{dh(\alpha)}{d\alpha} = -\frac{\mu_{\min}}{(1+\alpha\mu_{\min})^2} \le 0$ and $\frac{ds(\alpha)}{d\alpha} = \frac{2\alpha}{(1+\alpha\mu_{\min})^3} > 0$. We can see that the function $h(\alpha) = \frac{1}{1+\alpha\mu_{\min}}$ is decreasing for $\alpha \in (0, \mu_{\min} + 2\sqrt{1+\mu_{\min}^2}]$, and $s(\alpha) = \frac{\alpha^2}{(1+\alpha\mu_{\min})^2}$ is increasing for $\alpha \in (\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}, +\infty)$. Thus we can get

$$\alpha_* = \arg\min_{\alpha}\{\sigma_{SSR}(\alpha)\} = 2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2}\,.$$

Then

$$\begin{aligned} &\sigma_{SSR}(\alpha_*) \\ =\ &\min\{h(\alpha_*), s(\alpha_*)\} \\ =\ &\min\left\{\frac{1}{1+2\mu_{\min}^2 + 2\mu_{\min}\sqrt{1+\mu_{\min}^2}}, \frac{(2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2})^2}{(1+2\mu_{\min}^2 + 2\mu_{\min}\sqrt{1+\mu_{\min}^2})^2}\right\}. \end{aligned}$$

Due to the fact that

$$\begin{aligned} \frac{\frac{1}{1+2\mu_{\min}^2+2\mu_{\min}\sqrt{1+\mu_{\min}^2}}}{\frac{(2\mu_{\min}+2\sqrt{1+\mu_{\min}^2})^2}{(1+2\mu_{\min}^2+2\mu_{\min}\sqrt{1+\mu_{\min}^2})^2}} &= \frac{1+2\mu_{\min}^2 + 2\mu_{\min}\sqrt{1+\mu_{\min}^2}}{(2\mu_{\min} + 2\sqrt{1+\mu_{\min}^2})^2} \\ &= \frac{1+2\mu_{\min}^2 + 2\mu_{\min}\sqrt{1+\mu_{\min}^2}}{4+8\mu_{\min}^2 + 8\mu_{\min}\sqrt{1+\mu_{\min}^2}} \\ &= \frac{1}{4} < 1, \end{aligned}$$

we have the result

$$\frac{1}{1+2\mu_{\min}^2+2\mu_{\min}\sqrt{1+2\mu_{\min}^2}} < \frac{(2\mu_{\min}+2\sqrt{1+\mu_{\min}^2})^2}{(1+2\mu_{\min}^2+2\mu_{\min}\sqrt{1+2\mu_{\min}^2})^2}.$$

Therefore the optimal upper bound of spectral radius of the iterative matrix is

$$\sigma_{SSR}(\alpha_*) = \frac{1}{1+2\mu_{\min}^2+2\mu_{\min}\sqrt{1+\mu_{\min}^2}}.$$

The proof of this theorem is completed. □

4. Numerical experimental results

In this section, we use practical examples to show the effectiveness and feasibility of the proposed SSR method by compared to the existing PMHSS, CRI, DSS, and single step iteration methods in terms of iteration steps (IT), iteration time (CPU) and relative residual (RES). This numerical experiment result is run on in MATLAB 2018b on PC with Intel(R) Core(TM) i5 CPU 1.19Hz and 8GB. We adopt the zero vector $z^{(0)} = \mathbf{0}$ as the initial guess for all tested methods. When the relative residual satisfies

$$\frac{\left\|r^{(k)}\right\|_2}{\left\|r^{(0)}\right\|_2} < 10^{-10},$$

the calculation ends, where $r^{(k)} = b - Az^{(k)}$. In the following numerical experiments, we use α_* to represent the theoretical quasi-optimal parameter and α_{eps} to represent the experimental optimal parameter, and the maximum value of α is set to 1000.

Example 4.1 ([16])**.** Consider the following linear system

$$(K - \omega^2 M + i(C_H + \omega C_V))z = f,$$

where $M = I_n$, $C_H = \gamma K$ and $C_V = 10 I_N$. The matrix $K = B \otimes I_m + I_m \otimes B \in \mathbb{R}^{m\times m}$, $(n = m^2)$. In fact the matrix K is the five-point centered difference approximating the operator $-\triangle$ with homogeneous Dirichlet boundary conditions, ω is the driving circular frequency, on a uniform mesh in the unit square with $h = \frac{1}{m+1}$. Also we set $\omega = 7$, $\gamma \in \{2,\ 5\}$ and $f = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1.

Table 1. The experimentally found optimal parameters for Example 4.1 with $\gamma = 2$.

Method		m				
		64	128	256	512	1024
PMHSS [3]	α	[0.4,0.47]	[0.4,0.47]	[0.42,0.44]	[0.41,0.45]	[0.41,0.46]
Single step method [16]	α	[1.29,3.8]	[1.38,3.27]	[1.42,3.1]	[1.43,3.06]	[1.44,3.04]
SSR	α	[8.59,8.82]	[6.75,8.93]	[5.52,8.91]	[7.33,8.36]	[7.25,8.36]

Table 2. The experimentally found optimal parameters for Example 4.1 with $\gamma = 5$.

Method		m				
		64	128	256	512	1024
PMHSS [3]	α	[0.51,0.63]	[0.5,0.63]	[0.55,0.58]	[0.54,0.6]	[0.53,0.61]
CRI [18]	α	[0.9,1.11]	[0.85,1.18]	[0.89,1.13]	[0.83,1.2]	[0.87,1.15]
DSS [19]	α	[0.9,1.11]	[0.85,1.18]	[0.89,1.13]	[0.83,1.2]	[0.87,1.15]
Single step method [16]	α	[1.35,1000]	[1.38,1000]	[1.39,1000]	[1.39,1000]	[1.39,1000]
SSR	α	[12.35,46.89]	[8.65,44.55]	[18.81,19.71]	[13.66,20.57]	[10.03,11.08]

Table 3. Numerical results of the tested methods for Example 4.1 with $\gamma = 2$.

Method		m				
		64	128	256	512	1024
PMHSS [3]	α_{eps}	0.45	0.41	0.43	0.42	0.41
	IT	139	132	124	117	110
	CPU	0.2557	1.3228	6.8255	32.2788	150.8733
	RES	9.06E-11	9.12E-11	9.86E-11	9.72E-11	9.68E-11
Single step method [16]	α_{eps}	3.13	2.45	1.58	1.46	1.57
	IT	30	30	30	30	30
	CPU	0.0173	0.0925	0.4559	2.1102	9.4973
	RES	7.62E-11	7.28E-11	8.29E-11	9.56E-11	8.55E-11
SSR	α_{eps}	8.71	6.89	7.64	7.72	8.19
	IT	11	11	11	10	10
	CPU	0.0112	0.0555	0.2531	1.156	5.184
	RES	9.75E-11	9.09E-11	2.07E-11	6.74E-11	3.10E-11
	α_*	8.4901	8.4769	-	-	-
	IT	12	11	-	-	-
	CPU	0.017	0.063	-	-	-
	RES	1.38E-11	3.90E-11	-	-	-

Tables 1 and 2 show the value ranges of the parameters with the least IT of the PMHSS, SSR and Single step iterative methods processing Example 4.1 when $\gamma = 2$ and $\gamma = 5$, respectively. In addition, Table 2 also shows the value ranges of the CRI and DSS iterative methods when IT is the least. And when $\gamma = 2$, the CRI and DSS iterative methods do not converge when solving Example 4.1, so we do not list the value ranges of the parameters and the numerical results of the two methods in Table 1 and Table 3. Table 3 shows the parameter values with the least CPU and the corresponding IT, CPU and RES. It can be observed from Table 3 that the new method we established is superior to the PMHSS and Single step iterative methods in all aspects. Based on Table 2, Table 4 shows the parameter values, IT, CPU and RES of the PMHSS, CRI, DSS and SSR iterative methods when the IT is the least. It can be seen from Table 2 that the value ranges of the Single step iterative method are very wide, and it is difficult and costly to find the optimal parameters in the experiment. Therefore, in Table 4, for the Single step iterative method, we show the optimal

Table 4. Numerical results of the tested methods for Example 4.1 with $\gamma = 5$.

Method		m				
		64	128	256	512	1024
PMHSS [3]	α_{eps}	0.53	0.52	0.55	0.6	0.54
	IT	99	93	87	82	77
	CPU	0.1819	0.9656	4.7525	23.6799	114.7064
	RES	8.91E-11	9.05E-11	9.97E-11	9.96E-11	9.53E-11
CRI [18]	α_{eps}	1.07	0.9	0.92	0.89	1.15
	IT	30	29	27	26	24
	CPU	0.0389	0.173	0.8266	3.7409	15.7525
	RES	8.36E-11	6.72E-11	8.41E-11	6.63E-11	9.89E-11
DSS [19]	α_{eps}	1.07	1.07	0.89	0.97	1
	IT	30	29	27	26	24
	CPU	0.0404	0.2127	1.0312	4.8308	20.0113
	RES	8.36E-11	5.86E-11	9.58E-11	5.22E-11	7.05E-11
Single step method [16]	α_{eps}	446	254	663	336	270
	IT	16	16	16	16	16
	CPU	0.0112	0.0602	0.2892	1.3333	6.0798
	RES	2.79E-11	3.09E-11	3.22E-11	3.24E-11	3.24E-11
SSR	α_{eps}	30.3	26.43	19.43	19.47	11.01
	IT	8	8	7	7	7
	CPU	0.0084	0.0459	0.1946	0.9325	4.2535
	RES	3.15E-11	1.70E-11	9.74E-11	4.26E-11	9.47E-11
	α_*	20.2350	20.2074	-	-	-
	IT	8	8	-	-	-
	CPU	0.0147	0.0726	-	-	-
	RES	3.19E-11	1.15E-11	-	-	-

Table 5. The experimentally found optimal parameters for Example 4.2.

Method		n				
		60^2	70^2	80^2	90^2	100^2
PMHSS [3]	α	[1.53,4.4]	[1.53,4.4]	[1.53,4.4]	[1.53,4.4]	[1.53,4.4]
CRI [18]	α	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]
DSS [19]	α	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]	[0.81,1.23]
Single step method [16]	α	[2.04,3.54]	[2.04,3.54]	[2.04,3.54]	[2.04,3.54]	[2.04,3.54]
SSR	α	[10.29,10.31]	[10.29,10.31]	[10.29,10.31]	[10.29,10.31]	[10.29,10.31]

value when α is taken as an integer, the corresponding IT, CPU and RES. It can be observed from Table 4 that the IT, CPU of the SSR iterative method are also better than those of other ones. On the other hand, Tables 3 and 4 also show the results of the SSR method with α selected as the theoretical quasi-optimal parameter. However, the quasi-optimal parameter given in this paper is related to the smallest eigenvalue $\mu_{\min}$ of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$. When the order of the matrix is large, the amount of eigenvalue calculation exceeds the memory of computer, so there is no result. For this reason, we do not list

TABLE 6. Numerical results of the tested methods for Example 4.2.

Method		n				
		60^2	70^2	80^2	90^2	100^2
PMHSS [3]	α_{eps}	1.78	1.59	1.56	1.84	1.53
	IT	48	48	48	48	48
	CPU	3.164	5.9379	10.5617	17.2928	27.7211
	RES	8.80E-11	9.57S-11	9.76E-11	8.67E-11	9.97E-11
CRI [18]	α_{eps}	0.96	0.89	0.83	0.83	0.82
	IT	26	26	26	26	26
	CPU	2.0775	3.7188	6.7332	11.1903	18.7065
	RES	7.20E-11	7.87E-11	9.20E-11	9.20E-11	9.52E-11
DSS [19]	α_{eps}	0.95	0.87	0.81	0.82	0.81
	IT	26	26	26	26	26
	CPU	3.6652	6.84	11.9586	19.3768	30.3267
	RES	7.25E-11	8.21E-11	9.87E-11	9.52E-11	9.87E-11
Single step method [16]	α_{eps}	2.97	2.74	2.89	3.44	2.06
	IT	23	23	23	23	23
	CPU	1.0219	1.9205	3.4505	5.6164	8.9782
	RES	9.31E-11	9.16E-11	9.25E-11	9.85E-11	9.92E-11
SSR	α_{eps}	10.31	10.31	10.3	10.3	10.3
	IT	8	8	8	8	8
	CPU	0.5493	1.0852	1.9676	3.3663	5.4611
	RES	9.69E-11	9.69E-11	8.94E-11	8.94E-11	8.94E-11
	α_*	10.8187	10.8187	10.8187	10.8187	10.8187
	IT	10	10	10	10	10
	CPU	0.6882	1.2042	2.2634	3.8145	6.3598
	RES	3.86E-12	3.86E-12	3.86E-12	3.86E-12	3.86E-12

the results of the SSR iteration method with quasi-optimal parameters when $m \geq 256$ in Tables 3 and 4.

Example 4.2 ([16])**.** Consider the complex linear system $Az = f$, where A has a quasi-tridiagonal form

$$\begin{pmatrix} 1+4i & \frac{1}{8} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{1}{8} & 1+4i & \frac{1}{8} & 0 & \vdots & 0 \\ 0 & \frac{1}{8} & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \frac{1}{8} & 0 \\ 0 & \vdots & 0 & \frac{1}{8} & 1+4i & \frac{1}{8} \\ \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{8} & 1+4i \end{pmatrix}_{n\times n},$$

and f can be determined when exact solution is $z = \begin{pmatrix}1, & \frac{1}{2,} & \cdots & \frac{1}{n-1} & \frac{1}{n}\end{pmatrix}^T$.

Table 5 shows the ranges of the optimal parameters involved in the tested methods used to solve Example 4.2. The parameters in Table 6 are selected the

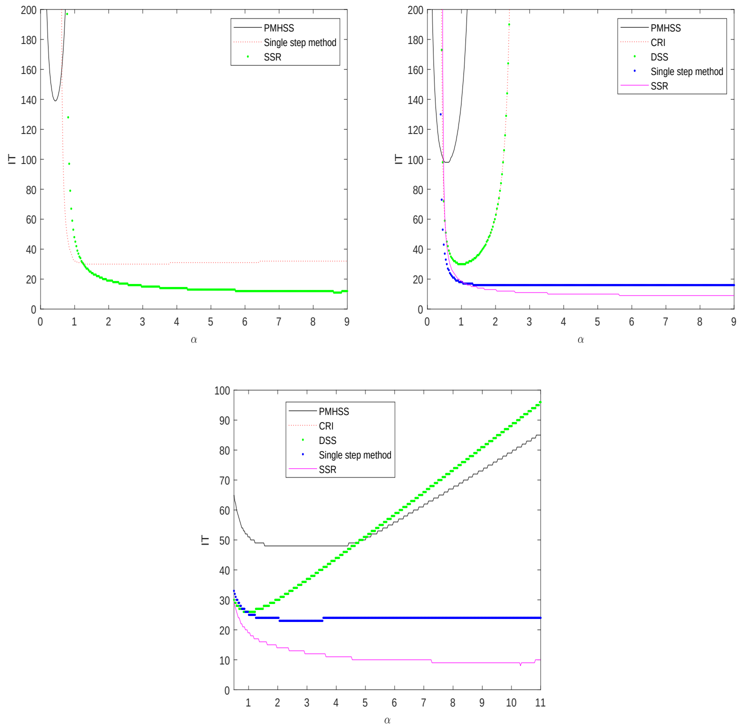

FIGURE 1. Iterations (IT) versus α for the PMHSS, CRI, DSS, SSR and Single step method. (left Example 4.1 for $m = 64$ $\gamma = 2$, middle Example 4.1 for $m = 64$ $\gamma = 5$ and right Example 4.2 for $n = 60^2$).

ones with the least IT from Table 5. Similar to Tables 3 and 4, and Table 6 also shows the IT, CPU and RES of each tested method. In addition, numerical results of the SSR method with the theoretical quasi-optimal parameter α_* are listed in Table 6. From the results shown in Table 6, we can see that the SSR iterative method returns better numerical results than the other tested ones in terms of IT and CPU times.

Figure 1 shows the number of iteration steps for the PMHSS, CRI, DSS, SSR and Single step iteration methods versus parameter α.

From Figure 1, it can be seen that the IT of the SSR method are always less than that of the other tested ones for the two examples. In addition, it can be seen that the IT of the SSR method are relatively stable with the increases of the parameter α, which means that the proposed method is not sensitive to the selection of the parameter α.

5. Concluding remarks

In the paper, for solving the system of linear equations (1.1), we present a new single step real-valued (SSR) iterative method, which avoids involving complex arithmetic compared to the single step iteration method one derived in [16]. And compared to the DSS [20] and CRI [18] methods, two linear subsystems of the SSR method are the same. Besides, we establish the convergence theories for the proposed iteration method and give the quasi-optimal parameter minimizing the upper bound of the spectral radius of the iteration matrix of the SSR iteration method. Finally, numerical experiments demonstrate that our new method can return better computing efficiency than the PMHSS, CRI, DSS and single step iteration methods.

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