J. Korean Soc. Ind. Appl. Math. Vol.27, No.3, 194–206, 2023 https://doi.org/10.12941/jksiam.2023.27.194

FAMILIES OF NONLINEAR TRANSFORMATIONS FOR ACCURATE EVALUATION OF WEAKLY SINGULAR INTEGRALS

BEONG IN YUN[1]

[1]DEPARTMENT OF MATHEMATICS, KUNSAN NATIONAL UNIVERSITY, GUNSAN, SOUTH KOREA
Email address: biyun@kunsan.ac.kr

ABSTRACT. We present families of nonlinear transformations useful for numerical evaluation of weakly singular integrals. First, for end-point singular integrals, we define a prototype function with some appropriate features and then suggest a family of transformations. In addition, for interior-point singular integrals, we develop a family of nonlinear transformations based on the aforementioned prototype function. We take some examples to explore the efficiency of the proposed nonlinear transformations in using the Gauss-Legendre quadrature rule. From the numerical results, we can find the superiority of the proposed transformations compared to some existing transformations, especially for the integrals with high singularity strength.

1. INTRODUCTION

In this paper, we consider evaluation of weakly singular integrals which are important for implementing numerical schemes in many areas of engineering such as elastostatics, fluid dynamics and material engineering [1, 2, 3].

There are lots of methods for numerical evaluation of weakly singular integrals, and among them, the coordinate transformation methods are known to be outstanding due to their ease of use in adaptive approaches. The literature [4, 5, 6] introduced ad hoc coordinate transformations to remove the singularity of the integrand. Telles[7] provided a self-adaptive co-ordinate transformation which is efficient for general boundary element method. Sigmoidal transformations, known to be very useful for the coordinate transformation method, are systematically classified in the literature [8, 9, 10, 11, 12]. In addition, in [13, 14], we can find somewhat elaborated transformations and their availability for the interior-point singular integrals.

Specifically, for the end-point weakly singular integral whose integrand behaves like $O\left((1-x)^{\kappa}\right)$ near $x = 1$ $(-1 < \kappa < 0)$, the following Sato-polynomial transform [6] is known to be prominent.

$$\Phi_m^{\rm Sat}(x) \,=\, 1 - \frac{(1-x)^m}{2^{m-1}}, \qquad -1 \le x \le 1, \tag{1.1}$$

Received June 20 2023; Revised September 21 2023; Accepted in revised form September 23 2023; Published online September 25 2023.

2000 *Mathematics Subject Classification.* 65D30, 26A48.

Key words and phrases. nonlinear transformation, weakly singular integral, Gauss-Legendre quadrature rule.

for an integer $m \geq 2$. On the other hand, for the interior-point weakly singular integral whose integrand behaves like $O\left((s_0 - x)^\kappa\right)$ near $x = s_0$ $(-1 < s_0 < 1)$, we recall well-known Monegato-Sloan transformation [13] Ψ_m^{MS} as follows.

$$\Psi_m^{\mathrm{MS}}(x) \;=\; s_0 \;+\; \delta_0 \cdot (x - \eta_0)^m, \qquad -1 \leq x \leq 1, \tag{1.2}$$

where

$$\delta_0 = 2^{-m}\left\{(1+s_0)^{\frac{1}{m}} + (1-s_0)^{\frac{1}{m}}\right\}^m, \quad \eta_0 = \frac{(1+s_0)^{\frac{1}{m}} - (1-s_0)^{\frac{1}{m}}}{(1+s_0)^{\frac{1}{m}} + (1-s_0)^{\frac{1}{m}}}. \tag{1.3}$$

Additionally, we recall the generalized sigmoidal transformation [14] defined as

$$\Psi_m^{\mathrm{SGM}}(x) \;=\; s_0 \;+\; 2\mathrm{Sgn}(x - \xi_0)\gamma_m\left(\frac{|x - \xi_0|}{2}\right), \tag{1.4}$$

where $\gamma_m(t)$, $0 \leq t \leq 1$, is a sigmoidal transformation of order m and $\xi_0 = 2\gamma_m^{-1}\left(\frac{1+s_0}{2}\right) - 1$. These transformations are useful for most weakly singular integrals, except when the strength of singularity is very high(that is, κ near -1). In a recent work [15], the author introduced other efficient but rather complicated rational transformations with parameters depending on the type of singularity and the location of the singular point. The transformations have the annoying problem of determining appropriate range of the parameters.

We, in this study, propose new nonlinear transformations for end-point and interior-point weakly singular integrals, respectively, with the goal of higher accuracy of the numerical integration. We also focus on the simplicity and the versatility of the transformations, regardless of the singularity strength $-1 < \kappa < 0$. The usefulness of the proposed transformation method is investigated through numerical examples for each type of the weakly singular integral.

In Section 2, we define a so-called prototype function $\phi_m(x)$, $-1 \leq x \leq 1$, of order m satisfying some reasonable features. Several basic examples of $\phi_m(x)$ are introduced. Then, for end-point weakly singular integrals, we suggest a family of nonlinear transformations $\Phi_m(x)$, associated with $\phi_m(x)$, that are bijective over the integration interval $[-1, 1]$ and weaken the singularity of the integrand for any integer $m \geq 2$. This family contains the Sato-polynomial transform, in the special case of $\phi_m(x) = x^m$. We also introduce a nonlinear transformation $G_\delta(x)$ associated with another prototype function $g(\delta; x)$ defined in (2.6).

In Section 3, for interior-point weakly singular integrals, we propose a family of nonlinear transformations $\Psi_m(x)$ based on the prototype function $\phi_m(x)$ that shift any singular point s_0 to the midpoint 0. Moreover, we derive the appropriate range of m with which the transformations become bijective over the interval $[-1, 1]$ according to the singular point s_0. A nonlinear transformation $H_\delta(x)$ based on the prototype function $g(\delta; x)$ is proposed, in addition.

In the last section, for some examples of the weakly singular integrals, we show that the proposed transformations accompanying the standard Gauss-Legendre quadrature rule are comparable to well-known existing transformations. Particularly, in the case of κ close to -1, it can be seen that proposed transformations $G_\delta(x)$ and $H_\delta(x)$ result in much better approximation errors compared to the existing transformations.

2. END-POINT SINGULAR INTEGRALS

We consider the end-point singular integral such as

$$\int_{-1}^{1} (1-\xi)^{\kappa} h(\xi)\, d\xi\,, \qquad -1 < \kappa < 0. \tag{2.1}$$

For an integer $m \geq 2$ we denote by ϕ_m a function which is defined on the interval $-1 \leq x \leq 1$ and satisfies the following properties.

(P1) $\phi_m \in C^{\infty}[0,1]$ and it has an asymptotic behavior of $O(x^m)$ near the point $x = 0$.

(P2) ϕ_m is strictly increasing over the interval $[0,1]$ with

$$\phi_m(0) = 0, \qquad \phi_m(1) = 1$$

(P3) ϕ_m is an odd or even function over the interval $[-1,1]$ according to m.

The property (P3) is required for the interior-point singular integrals considered in the next section.

Then, we define a new transformation

$$\Phi_m(x) := 1 - 2\phi_m\left(\frac{1-x}{2}\right), \qquad -1 \leq x \leq 1, \tag{2.2}$$

which has the following properties.

Lemma 2.1. *For any integer $m \geq 2$, Φ_m is strictly increasing over the interval $-1 \leq x \leq 1$ with*

$$\Phi_m(-1) = -1, \qquad \Phi_m(1) = 1$$

and its asymptotic behavior near the end-point $x = 1$ is

$$\Phi_m(x) = 1 + O\left((1-x)^m\right).$$

Proof. The proof is straightforward from the definition of Φ_m and the properties (P1) and (P2). □

If we set $\xi = \Phi_m(x)$ in (2.1), the original singularity $O\left((1-\xi)^{\kappa}\right)$ of the integrand is weakened as $O\left((1-x)^{(1+\kappa)m-1}\right)$ for every $m \geq 2$.

For example, we propose some basic types of the function ϕ_m and the related function Φ_m as follows.

(i) Polynomial type of order m:

$$\phi_m^{\mathrm{P}}(x) = x^m$$

and

$$\Phi_m^{\mathrm{P}}(x) := 1 - 2\phi_m^{\mathrm{P}}\left(\frac{1-x}{2}\right) = 1 - \frac{(1-x)^m}{2^{m-1}},$$

which is equivalent to the Sato-polynomial transformation, $\Phi_m^{\mathrm{Sat}}(x)$ in (1.1).

(ii) Exponential type of order m:

$$\phi_m^{\mathrm{E}}(x) = \left\{ \frac{e^{1+x} - e^{1-x}}{e^2 - 1} \right\}^m \tag{2.3}$$

and

$$\Phi_m^{\mathrm{E}}(x) := 1 - 2\phi_m^{\mathrm{E}}\left(\frac{1-x}{2}\right).$$

(iii) Trigonometric type of order m:

$$\phi_m^{\mathrm{T}}(x) = \tan^m\left(\frac{\pi x}{4}\right) \tag{2.4}$$

and

$$\Phi_m^{\mathrm{T}}(x) := 1 - 2\phi_m^{\mathrm{T}}\left(\frac{1-x}{2}\right). \tag{2.5}$$

On the other hand, we suggest an exponential type of infinite order,

$$g(\delta; x) = \exp\left(\frac{1}{\delta^2}\left(1 - \frac{1}{x^2}\right)\right) \tag{2.6}$$

for a parameter $\delta > 0$. This function satisfies $g(\delta; 1) = 1$ and

$$\lim_{x\to 0} \frac{d^j}{dx^j} g(\delta; x) = 0\,,$$

for all integers $j \geq 0$, instead of the property (P2). This implies the "infinite order"of $g(\delta; x)$. Then we define a function

$$G_\delta(x) := 1 - 2g\left(\delta; \frac{1-x}{2}\right). \tag{2.7}$$

We can see that $\lim_{x\to 1} \frac{d^j}{dx^j} G_\delta(x) = 0$ for all integers $j \geq 1$ with $G_\delta(-1) = -1$ and $\lim_{x\to 1} G_\delta(x) = 1$.

3. Interior-point singular integrals

For the interior-point weakly singular integral

$$\int_{-1}^{1} |\xi - s_0|^\kappa h(\xi)\, d\xi\,, \qquad -1 < \kappa < 0, \tag{3.1}$$

where $0 \leq s_0 < 1$ is assumed for simplicity, we consider a systematic coordinate transformation method without splitting the integration interval.

Using the function ϕ_m satisfying the properties (P1)–(P3), we suggest a transformation Ψ_m defined as

$$\Psi_m(x) := \begin{cases} s_0 + (1 - s_0 x)\,\phi_m(x)\,, & \text{if } \phi_m \text{ is an odd function} \\ s_0 + x\,(1 - s_0 x)\,\phi_m(x)\,, & \text{if } \phi_m \text{ is an even function} \end{cases} \tag{3.2}$$

for $-1 \le x \le 1$. It is noted that any singular point s_0 is always shifted to the midpoint 0 by the transformation Ψ_m. The following lemma shows the asymptotic behavior of $\Psi_m(x)$ near $x = 0$ and states that $\Psi_m(x)$ becomes a bijective map, from the interval $[-1, 1]$ onto itself, if m satisfies an appropriate condition relative to s_0.

Lemma 3.1. *For $0 \le s_0 < 1$ and ϕ_m used in* (3.2)*, we have the following properties.*

(1) *Ψ_m satisfies*

$$\Psi_m(-1) = -1, \qquad \Psi_m(0) = s_0, \qquad \Psi_m(1) = 1.$$

(2) *The asymptotic behavior of $\Psi_m(x)$, near $x = 0$, is*

$$\Psi_m(x) = s_0 + O\left(x^m\right)$$

when ϕ_m is an odd function and

$$\Psi_m(x) = s_0 + O\left(x^{m+1}\right)$$

when ϕ_m is an even function.

(3) *Assume that $\phi_m'(x)/\phi_m(x)$, $m \ge 2$, has a minimum over the interval $0 < x \le 1$ at $x = 1$. Then $\Psi_m(x)$ is bijective over the interval $-1 \le x \le 1$ for any m satisfying*

$$\phi_m'(1) > \frac{s_0}{1 - s_0}$$

when ϕ_m is an odd function and

$$\phi_m'(1) + 1 > \frac{s_0}{1 - s_0}$$

when ϕ_m is an even function.

Proof. Referring to the properties (P1)–(P3) of ϕ_m employed in definition of Ψ_m in (3.2), we can directly identify the equalities in the assertions (1) and (2).

For the assertion (3), first, suppose that ϕ_m is an odd function. Then the sufficient condition for $\Psi_m(x)$ to be bijective over the interval $-1 \le x \le 1$ is

$$\Psi_m'(x) = -s_0\phi_m(x) + (1 - s_0 x)\phi_m'(x) > 0$$

for all $x \ne 0$ in $[-1, 1]$. When $s_0 = 0$, $\Psi_m'(x) = \phi_m'(x) > 0$ is clear so that we assume that $0 < s_0 < 1$. For $-1 \le x < 0$, since $\phi_m(x) < 0$ and $\phi_m'(x) > 0$, we have $\Psi_m'(x) > 0$ regardless of $m \ge 2$. For $0 < x \le 1$, since $\phi_m(x) > 0$ and $\phi_m'(x) > 0$, the condition above implies that

$$\frac{1 - s_0 x}{s_0} \cdot \frac{\phi_m'(x)}{\phi_m(x)} > 1.$$

By the assumption that $\phi_m'(x)/\phi_m(x)$ has a minimum at $x = 1$, we may set

$$\frac{1 - s_0 x}{s_0} \cdot \frac{\phi_m'(x)}{\phi_m(x)} \ge \frac{1 - s_0}{s_0} \cdot \phi_m'(1) > 1.$$

Therefore, we have the following sufficient condition for Ψ_m to be bijective.

$$\phi'_m(1) \geq \frac{s_0}{1-s_0}.$$

Next, suppose that ϕ_m is an even function. The sufficient condition for $\Psi_m(x)$ to be bijective over the interval $-1 \leq x \leq 1$ is

$$\Psi'_m(x) = -s_0 x \phi_m(x) \, + \, (1-s_0 x)\left\{x\phi'_m(x) + \phi_m(x)\right\} > 0.$$

When $s_0 = 0$, $\Psi'_m(x) = x\phi'_m(x) + \phi_m(x) > 0$ is clear so that we assume that $0 < s_0 < 1$. For $-1 \leq x < 0$, since $\phi_m(x) > 0$ and $\phi'_m(x) < 0$, we have $\Psi'_m(x) > 0$ regardless of $m \geq 2$. For $0 < x \leq 1$, since $\phi_m(x) > 0$ and $\phi'_m(x) > 0$, the condition above implies that

$$\frac{1-s_0 x}{s_0} \cdot \frac{x\phi'_m(x) + \phi_m(x)}{x\phi_m(x)} > 1.$$

By the assumption that $\phi'_m(x)/\phi_m(x)$ has a minimum at $x = 1$, we set

$$\frac{1-s_0 x}{s_0} \cdot \frac{x\phi'_m(x) + \phi_m(x)}{x\phi_m(x)} \geq \frac{1-s_0}{s_0} \cdot \left\{\phi'_m(1) + 1\right\} > 1.$$

Therefore, we have the following sufficient condition for Ψ_m to be bijective.

$$\phi'_m(1) + 1 > \frac{s_0}{1-s_0}.$$

This completes the proof of the assertion (3). □

For the prototype function $\phi^{\mathrm{P}}_m(x) = x^m$, the formula (3.2) provides a simple form,

$$\Psi^{\mathrm{P}}_m(x) \; = \; s_0 \, + \, (1-s_0 x)\, x^m$$

for any odd integer $m \geq 3$. Sine

$$\frac{\phi^{\mathrm{P}\,\prime}_m(x)}{\phi^{\mathrm{P}}_m(x)} = \frac{m}{x}$$

has a minimum $\phi^{\mathrm{P}\,\prime}_m(1) = m$ over the interval $0 < x \leq 1$, from Lemma 3.1(3), we can see that the function Ψ^{P}_m becomes a bijective function for all odd integer $m \geq 3$ satisfying

$$m > \frac{s_0}{1-s_0}.$$

This condition implies that $m \geq 3$ for every $s_0 < 0.75$, $m \geq 5$ for $s_0 = 0.8$, $m \geq 11$ for $s_0 = 0.9$, for example.

For the prototype function ϕ^{E}_m defined in (2.3), from the formula (3.2), we have

$$\Psi^{\mathrm{E}}_m(x) \; = \; s_0 \, + \, (1-s_0 x)\, \phi^{\mathrm{E}}_m(x)$$

for any odd integer $m \geq 3$. Sine

$$\frac{\phi^{\mathrm{E}\,\prime}_m(x)}{\phi^{\mathrm{E}}_m(x)} = m\frac{e^{1+x} + e^{1-x}}{e^{1+x} - e^{1-x}}$$

has a minimum ${\phi_m^{\mathrm{E}}}'(1) = m\frac{e^2+1}{e^2-1}$ over the interval $0 < x \le 1$, from Lemma 3.1(3), the function Ψ_m^{E} becomes a bijective function for all odd integers m satisfying

$$m > \left(\frac{e^2-1}{e^2+1}\right)\frac{s_0}{1-s_0}. \tag{3.3}$$

The condition (3.3) implies that $m \ge 3$ for every $s_0 \le 0.75$, $m \ge 5$ for $s_0 = 0.8$ and $m \ge 7$ for $s_0 = 0.9$, for example.
In addition, using the prototype function ϕ_m^{T} defined in (2.4), we have

$$\Psi_m^{\mathrm{T}}(x) \;=\; s_0 \;+\; (1-s_0x)\,\phi_m^{\mathrm{T}}(x) \tag{3.4}$$

for every odd integer $m \ge 3$. Sine

$$\frac{{\phi_m^{\mathrm{T}}}'(x)}{\phi_m^{\mathrm{T}}(x)} = \frac{m\pi\sec^2\left(\frac{\pi}{4}x\right)}{4\tan\left(\frac{\pi}{4}x\right)}$$

has a minimum ${\phi_m^{\mathrm{T}}}'(1) = \frac{\pi m}{2}$ over the interval $0 < x \le 1$, from Lemma 2(3), we can see that Ψ_m^{T} becomes a bijective function for all odd integer $m \ge 3$ satisfying

$$m > \frac{2}{\pi}\frac{s_0}{1-s_0}.$$

This condition implies that $m \ge 3$ for every $s_0 \le 0.8$, and $m \ge 7$ for $s_0 = 0.9$, for example.

We can see that the change of variable $\xi = \Psi_m(x)$ in (3.1) weakens the singularity strength κ of the integrand to $(\kappa+1)m-1$. Moreover, Lemma2(3) shows that the minimum value of m useful for this change of variable increases as the location of the singularity s_0 approaches to the end point $x = 1$.

On the other hand, employing the even function $g(\delta;x) = \exp\left(\frac{1}{\delta^2}\left(1-\frac{1}{x^2}\right)\right)$ given in (2.6) and referring to the formula (3.2), we introduce a function

$$H_\delta(x) \;=\; s_0 \;+\; x\,(1-s_0x)\,g(\delta;x) \tag{3.5}$$

for a parameter $\delta > 0$. It can be seen that

$$\begin{aligned} H_\delta'(x) &= \frac{g(\delta;x)}{\delta^2x^2}\left(2+\delta^2x^2-2s_0x-2\delta^2s_0x^3\right) \\ &\ge \frac{g(\delta;x)}{\delta^2x^2}\left(2-2s_0-2\delta^2s_0\right) \end{aligned}$$

for all $-1 \le x \le 1$. Thus we may find a sufficient condition for $H_\delta'(x) > 0$ from $2-2s_0-2\delta^2s_0 > 0$, that is,

$$\delta < \sqrt{\frac{1-s_0}{s_0}}$$

for $0 < s_0 < 1$. This condition implies that $\delta < 1$ for every $s_0 \le 0.5$, $\delta < \frac{1}{2}$ for $s_0 = 0.8$ and $\delta < \frac{1}{3}$ for $s_0 = 0.9$ and $\delta < 0.229$ for $s_0 = 0.95$, for example.

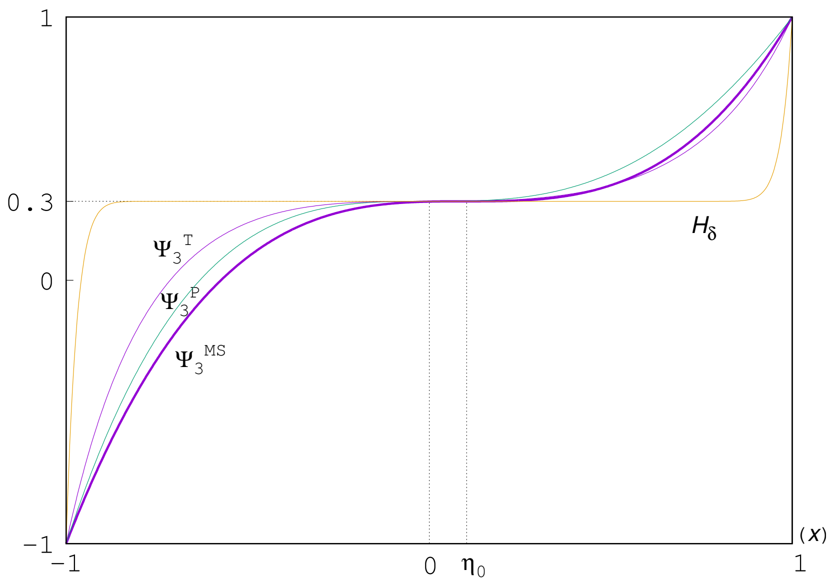


FIGURE 1. Graphs of $\Psi_3^{\mathrm{P}}(x)$, $\Psi_3^{\mathrm{T}}(x)$, $\Psi_3^{\mathrm{MS}}(x)$, and $H_\delta(x)$ with $\delta = 1/4$ for $s_0 = 0.3$.

Figure 1 shows graphs of the presented transformations $\Psi_3^{\mathrm{P}}(x)$, $\Psi_3^{\mathrm{T}}(x)$ and $H_\delta(x)$ with $\delta = 1/4$, for the case of $s_0 = 0.3$, compared with the well-known Monegato-Sloan transformation $\Psi_3^{\mathrm{MS}}(x)$ defined in (1.2).

4. NUMERICAL EXAMPLES

To explore the efficiency of the presented method, we select two typical examples in this section. First, the following example involves end-point weakly singular integrals.

Example 1. For a real $-1 < \kappa < 0$,

$$I^{[\kappa]} := \int_{-1}^{1} (1-\xi)^\kappa (1+\xi)\, d\xi$$

whose exact value is $\frac{2^{\kappa+2}}{(1+\kappa)(2+\kappa)}$.

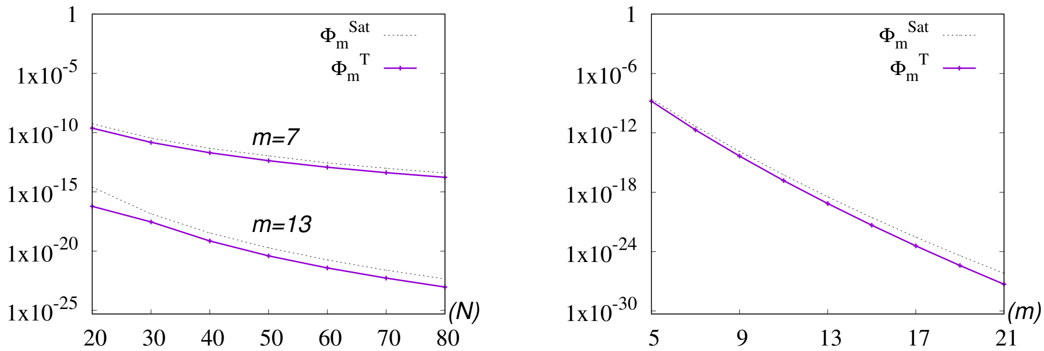
We denoted by $I_N^{[\kappa]}(\Phi_m)$ the $N-$point Gauss Legendre quadrature rule associated with the transformation Φ_m. For $\kappa = -0.5$, Figure 2 includes numerical results of the difference errors

$$E_N^{[\kappa]}(\Phi_m) := \left| I^{[\kappa]} - I_N^{[\kappa]}(\Phi_m) \right|$$

related to the presented transformation $\Phi_m = \Phi_m^{\mathrm{T}}$ in (2.5) and the Sato-transformation $\Phi_m = \Phi_m^{\mathrm{Sat}} (= \Phi_m^{\mathrm{P}})$ in (1.1). We can see that Φ_m^{T} gives slightly better errors than Φ_m^{Sat} for both the number of integration points N and the order m of the transformation. Numerical experiments show that the other presented transformation Φ_m^{E} results in the similar error tendency to that of Φ_m^{T}.

For the case of $\kappa = -0.95$, or the case where the singularity is high, Figure 3(a) shows the numerical errors $E_N^{[\kappa]}\left(\Phi_m^{\mathrm{T}}\right)$ and $E_N^{[\kappa]}\left(\Phi_m^{\mathrm{Sat}}\right)$ with a similar trend to the case of $\kappa = -0.5$. That is, the proposed transformation Φ_m^{T} gives slightly better errors than the compared transformation Φ_m^{Sat} for every number of integration points.

In addition, numerical errors related to another proposed transformation G_δ, defined in (2.7), with $\delta = \frac{1}{4}$ and $\delta = \frac{1}{8}$ are included in Fig. 3(b). This shows significantly improved errors compared with other transformations.



(a) $20 \leq N \leq 80$ ($m = 7,\ 13$) (b) $5 \leq m \leq 21$ ($N = 40$)

FIGURE 2. Graphs of the errors related to the transformations Φ_m^{T} and Φ_m^{Sat} with respect to $20 \leq N \leq 80$ in (a) and those with respect to $5 \leq m \leq 21$ in (b) for Example 1 with $\kappa = -0.5$.

Next, we consider the following example of the interior-point weakly singular integral.

Example 2. For a real $-1 < \kappa < 0$ and for the location of a singularity $-1 < s_0 < 1$,

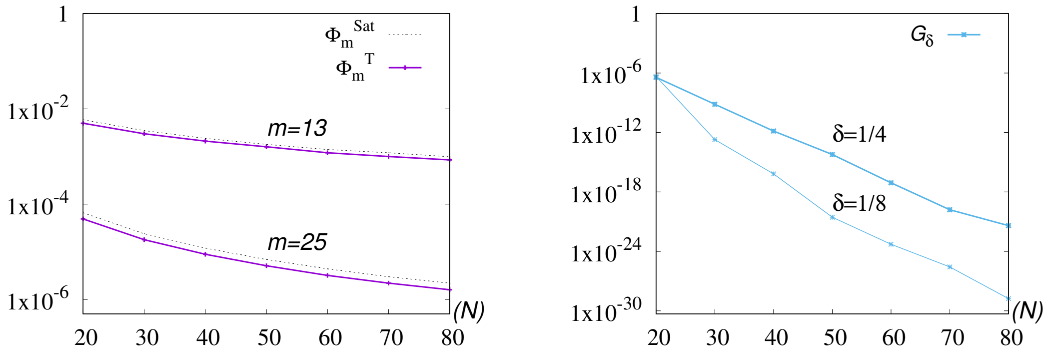
$$J^{[\kappa]} := \int_{-1}^{1} |\xi - s_0|^{\kappa}(1+\xi)\, d\xi$$

whose exact value is $\frac{1}{(1+\kappa)(2+\kappa)}\left\{(1+s_0)^{2+\kappa} + (1-s_0)^{1+\kappa}(3+2\kappa+s_0)\right\}$.

We set the difference error

$$E_N^{[\kappa]}\left(\Psi_m\right) := \left|J^{[\kappa]} - J_N^{[\kappa]}\left(\Psi_m\right)\right|$$

for $J_N^{[\kappa]}\left(\Psi_m\right)$ denoting the $N-$point Gauss Legendre quadrature rule associated with the transformation Ψ_m. Figure 4, for the case of $\kappa = -0.5$ with $s_0 = 0.1$, includes numerical errors



(a) $20 \leq N \leq 80$ $(m = 13,\ 25)$ (b) $20 \leq N \leq 80$ $(\delta = \frac{1}{4},\ \frac{1}{8})$

FIGURE 3. Graphs of the errors related to the transformations Φ_m^{T} and Φ_m^{Sat} in (a) and graphs of the errors related to the transformation G_δ in (b) for Example 1 with $\kappa = -0.95$.

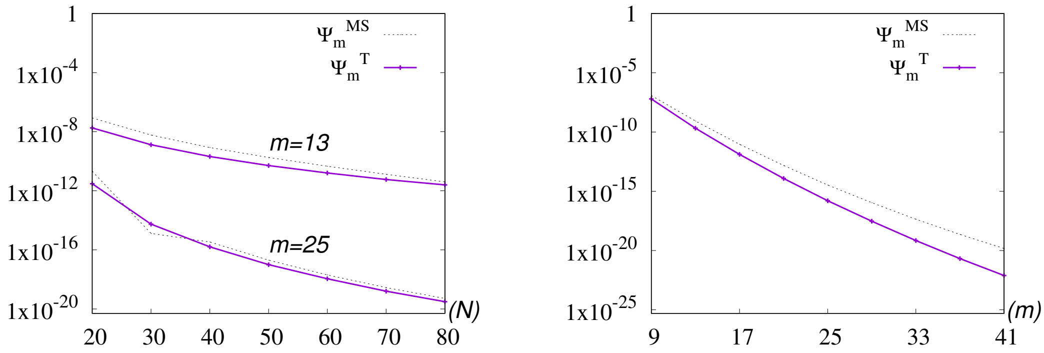
with respect to the number of integration points N in (a) and the order of the transformations m in (b). The figure shows that the proposed transformation Ψ_m^{T} gives little better errors than the Monegato-Sloan transformation Ψ_m^{MS} for both N and m. By numerical experiments, we can see that the other presented transformations Ψ_m^{P} and Ψ_m^{E} result in the similar error tendency to that of Ψ_m^{T}. Especially, it should also be noted that the presented transformations defined in (3.2) is quit simple compared to the existing transformations (1.2) – (1.4).

For a higher strength of the singularity, $\kappa = -0.95$ with $s_0 = 0.9$, Figure 5(a) shows numerical errors related to Ψ_m^{T} and Ψ_m^{MS}. It can be seen that the proposed transformation Ψ_m^{T} results in the uniformly decreasing errors, whereas Ψ_m^{MS} gives fluctuating errors as the number of integration points N increases. In addition, Fig. 5(b) includes numerical errors related to the proposed transformation H_δ, defined in (3.5), with $\delta = \frac{1}{8}$ and $\delta = \frac{1}{16}$. We can see that, like G_δ in Example 1, H_δ also gives significantly improved errors compared with other transformations.

Numerical experiments show that the error trends of the proposed transformations presented in Fig. 4 and Fig. 5 remain almost the same regardless of the strength κ and the location s_0 of the singularity.

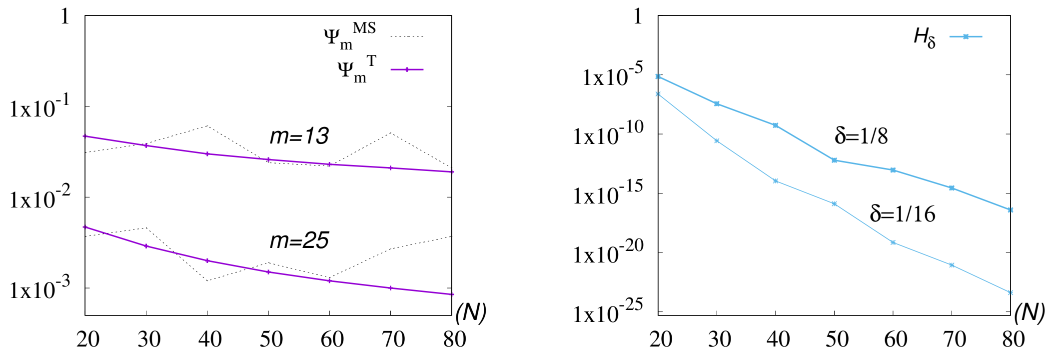
5. CONCLUSIONS

We proposed families of nonlinear transformations $\Phi_m(x)$ in (2.2) and $\Psi_m(x)$ in (3.2), based on the prototype function $\phi_m(x)$ satisfying the properties (P1)–(P3), for efficient evaluation of weakly singular integrals. Exceptionally, the transformations $G_\delta(x)$ and $H_\delta(x)$ were introduced using the function $g(\delta; x)$ in (2.6). It should be noted that the proposed transformations take simple forms compared to the existing the Monegato-Sloan transformation $\Psi_m^{\mathrm{MS}}(x)$



(a) $20 \leq N \leq 80$ $(m = 13,\ 25)$ (b) $9 \leq m \leq 41$ $(N = 40)$

FIGURE 4. Graphs of the errors related to the transformations Ψ_m^{T} and Ψ_m^{MS} with respect to $20 \leq N \leq 80$ in (a) and those with respect to $9 \leq m \leq 41$ in (b) for Example 2 with $\kappa = -0.5$ and $s_0 = 0.1$.



(a) $20 \leq N \leq 80$ $(m = 13,\ 25)$ (b) $20 \leq N \leq 80$ $(\delta = \frac{1}{8},\ \frac{1}{16})$

FIGURE 5. Graphs of the errors related to the transformations Ψ_m^{T} and Ψ_m^{MS} in (a) and graphs of the errors related to the transformation H_δ in (b) for Example 2 with $\kappa = -0.95$ and $s_0 = 0.9$.

and the generalized sigmoidal transformation $\Psi_m^{\mathrm{SGM}}(x)$ defined in (1.2) and (1.4), respectively. From the numerical results for the chosen test examples, we can summarize the conclusions as follows.

For end-point weakly singular integrals, the proposed transformations $\Phi_m^{\mathrm{T}}(x)$ and $\Phi_m^{\mathrm{E}}(x)$ provide slightly better errors than the exiting Sato-transformation $\Phi_m^{\mathrm{Sat}}(x) = \Phi_m^{\mathrm{P}}(x)$. Moreover, $G_\delta(x)$ with a parameter $\delta > 0$ gives much better errors even for the high singularity strength κ around -1.

For interior-point weakly singular integrals, the proposed transformations $\Psi_m^{\mathrm{T}}(x)$, $\Psi_m^{\mathrm{E}}(x)$ and $\Psi_m^{\mathrm{P}}(x)$ also provide slightly better errors compared to the traditional Monegato-Sloan transformation $\Psi_m^{\mathrm{MS}}(x)$. On the other hand, like $G_\delta(x)$ for the end-point weakly singular integrals, $H_\delta(x)$ also gives significantly improved errors compared with other transformations. Additionally, for κ near -1, the proposed transformations tend to uniformly reduce the error as the number of the integration points N increases, in contrast to the existing transformation $\Psi_m^{\mathrm{MS}}(x)$.

ACKNOWLEDGMENTS

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) [No. 2021R1F1A1047343].

REFERENCES

[1] M. Doblaré, *Computational aspects of the boundary element methods*,Topics in Boundary Element Research 3, Springer, Berlin, 1987.

[2] P.K. Kythe, *An introduction to boundary element method*, CRC Press, Boca Raton, 1995.

[3] M.P. Savruk, A. Kazberuk, *Method of Singular Integral Equations in Application to Problems of the Theory of Elasticity*, Stress Concentration at Notches, Springer, Cham, 2017.

[4] M. Doblaré, L. Gracia, *On non-linear transformations for integration of weakly-singular and Cauchy principal value integrals*, Int J Numer Methods Eng, **40** (1997), 3325–3358.

[5] J.M. Sanz Serna, M. Doblaré, E. Alarcon, *Remarks on methods for the computation of boundary-element integrals by co-ordinate transformation*, Commun Appl Numer Methods, **6** (1990), 121–123.

[6] M. Sato, S. Yoshiyoka, K. Tsukui, R. Yuuki, *Accurate numerical integration of singular kernels in the two-dimensional boundary element method*; Boundary Elements X Vol.1, C.A. Brebbia edt., Springer, Berlin, 1988.

[7] J.C.F. Telles, *A self-adaptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals*, Int J Numer Methods Eng, **24** (1987), 959–973.

[8] D. Elliott, *The cruciform crack problem and sigmoidal transformations*, Math Methods Appl Sci, **20** (1997), 121–132.

[9] D. Elliott, *The Euler Maclaurin formula revised*, J Austral Math Soc Ser B, **40**(E) (1998), E27–E76.

[10] P.R. Johnston, *Semi-sigmoidal transformations for evaluating weakly singular boundary element integrals*, Int J Numer Methods Eng, **47** (2000), 1709–1730.

[11] P.R. Johnston, D. Elliott, *Error estimation of quadrature rules for evaluating singular integrals in boundary element*, Int J Numer Methods Eng, **48** (2000), 949–962.

[12] P.R. Johnston, D. Elliott, *A generalization of Telles' method for evaluating weakly singular boundary element integrals*, J Comput Appl Math, **131** (2001), 223–241.

[13] G. Monegato, I.H. Sloan, *Numerical solutions of the generalized airfoil equation for an airfoil with a flap*, SIAM J Numer Anal, **34** (1997), 2288–2305.

[14] B.I. Yun, *An extended sigmoidal transformation technique for evaluating weakly singular integrals without splitting the integration interval*, SIAM J Sci Comput, **25** (2003), 284–301.

[15] B.I. Yun, *Rational transformations for evaluating singular integrals by the Gauss quadrature rule*, Mathematics, **8**, (2020), 677.