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g-METRIC SPACES AND ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENT SEQUENCES

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Abstract. In the present article, we introduce the concepts of strongly asymptotically lacunary equivalence, asymptotically statistical equivalence, and asymptotically lacunary statistical equivalence for sequences in g-metric spaces. We investigate some properties and relationships among these new concepts.

1. Introduction and Preliminaries

The notion of statistical convergence was studied at initial stage by Fast [10]. The publication of the paper affected deeply all the scientific fields. When we focus on the statistical convergence in the literature, we meet Başar [3], Belen and Mohiuddine [4], Mursaleen and Başar [20], and so many other researchers (see [2, 6, 7, 8, 9, 12, 13, 14, 15, 18, 19, 24, 25, 26, 27, 28]).

The idea of convergence of a real sequence was extended to statistical convergence as follows:

A real number sequence $u = (u_k)$ is said to be statistically convergent to the number x if for each $\varepsilon > 0$,

$$\lim_{n} \frac{1}{n} \left|\{k \leq n : |u_k - x| \geq \varepsilon\}\right| = 0,$$

where the vertical bars indicate the cardinality of the enclosed set. x is called the statistical limit of the sequence (u_k) and we write $st - \lim u_k = x$. The main idea behind statistical convergence was the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

$$\delta(A) = \lim_{n} \tfrac{1}{n} \left|\{k \in A : k \leq n\}\right|.$$

Clearly, $\delta(\mathbb{N} \setminus A) + \delta(A) = 1$ and $A \subseteq B$ implies $\delta(A) \leq \delta(B)$. It is obvious that if A is a finite set then $\delta(A) = 0$.

Pobyvanets defined asymptotically equivalent sequences and asymptotic regular matrices in 1980. Li [16] and Marouf [17] investigated the links between

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the asymptotic equivalence of two sequences in summability theory and offered several versions of asymptotic equivalence. In [17], two nonnegative sequences $u = (u_k)$ and $v = (v_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{u_k}{v_k} = 1$$

and this is denoted by $u \sim v$. Afterwards, Patterson [22] expanded these concepts by proposing an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in 2003. In Patterson's study: Let $u = (u_k)$ and $v = (v_k)$ are two nonnegative sequences. These sequences are said to be asymptotically statistical equivalent of multiple x provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{u_k}{v_k} - x \right| \geq \varepsilon \right\} \right| = 0$$

and this is denoted by $u \overset{S^x}{\sim} v$. Simply asymptotically lacunary statistical equivalent if $x = 1$. Moreover, let S^x denote the set of u and v such that $u \overset{S^x}{\sim} v$.

Lacunary statistical convergence is one of our study's additional primary themes. The concept of lacunary statistical convergence was introduced by Fridy and Orhan [11] in 1993. A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals $I_r = (k_{r-1}, k_r]$ are determined by θ and the ratio is determined $q_r = \frac{k_r}{k_{r-1}}$. Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $u = (u_k)$ is a lacunary statistically convergent (or $S_\theta-$convergent) to x if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \{ k \in I_r : |u_k - x| \geq \varepsilon \} \right| = 0.$$

In thıs case we write $S_\theta - \lim u_k = x$ and the set of all lacunary statistically convergent sequences is denoted by S_θ.

Patterson and Savaş [23] adapted the definitions in Patterson's paper [22] to lacunary sequences in 2006.

Definition 1.1. *Let θ be a lacunary sequence and the two nonnegative sequences $u = (u_k)$ and $v = (v_k)$ are said to be asymptotically lacunary statistical equivalent of multiple x provided that for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{u_k}{v_k} - x \right| \geq \varepsilon \right\} \right| = 0.$$

This situation is denoted by $u \overset{S_\theta^x}{\sim} v$ and simply asymptotically lacunary statistical equivalent if $x = 1$. Moreover, let S_θ^x denote the set of u and v such that $u \overset{S_\theta^x}{\sim} v$.

Definition 1.2. *Let θ be a lacunary sequence and the two nonnegative sequences $u = (u_k)$ and $v = (v_k)$ are said to be strong asymptotically lacunary equivalent of multiple x provided that*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{u_k}{v_k} - x \right| = 0.$$

This situation is denoted by $u \overset{N_\theta^x}{\sim} v$ and strong simply asymptotically lacunary equivalent if $x = 1$. Besides, let N_θ^x denote the set of u and v such that $u \overset{N_\theta^x}{\sim} v$.

Finally, we review generalized metric spaces. The idea of distance function may be generalized in a variety of ways. The G-metric space notion, which Mustafa and Sims [21] has investigated, is one of them and is a distinct generalization of the ordinary metric. Distances between three points are the metrics in this space. In 2018, Choi et al. [5] presented new definition, called the g-metric, for a clear generalization of the ordinary metric. In this space, g-metric of degree z determines the distance between $z + 1$ points. This definition is provided below.

Definition 1.3. *Let X be a nonempty set. A function $g : X^{z+1} \to \mathbb{R}^+$ is called a $g-$metric with order z on X if it satisfies the following conditions:*

(g_1) $g(u_0, u_1, \cdots, u_z) = 0$ if and only if $u_0 = u_1 = \cdots = u_z$,

(g_2) $g(u_0, u_1, \cdots, u_z) = g\left(u_{\rho(0)}, u_{\rho(1)}, \cdots, u_{\rho(z)}\right)$ for any permutation ρ on $\{0, 1, \cdots, z\}$,

(g_3) $g(u_0, u_1, \cdots, u_z) \leq g(v_0, v_1, \cdots, v_z)$ for all $(u_0, \cdots, u_z), (v_0, \cdots, v_z) \in X^{z+1}$ with $\{u_i : i = 1, \cdots, z\} \subset \{v_i : i = 1, \cdots, z\}$,

(g_4) For all $u_0, u_1, \cdots, u_s, v_0, v_1, \cdots, v_t, w \in X$ with $s + t + 1 = z$

$$\begin{aligned} g(u_0, u_1, \cdots, u_s, v_0, v_1, \cdots, v_t) &\leq g(u_0, u_1, \cdots, u_s, w, \cdots, w) \\ &+ g(v_0, v_1, \cdots, v_t, w, \cdots, w). \end{aligned}$$

The pair (X, g) is called a g-metric space.

In 2022, Abazari [1] extended these concepts by presenting statistical convergent and statistical Cauchy analogs of these definitions and their properties in g-metric spaces. Let $z \in \mathbb{N}, A \in \mathbb{N}^z$ and

$$A(n) = \{i_1, i_2, \cdots, i_z \leq n : (i_1, i_2, \cdots, i_z) \in A\},$$

then

$$\delta_z(A) = \lim_n \frac{z!}{n^z} |A(n)|$$

is called z-dimensional natural density of the set A.

Definition 1.4. *Let $u = (u_n)$ be a sequence in a g-metric space (X, g). Then (u_n) is statistically convergent to x, if for all $\varepsilon > 0$,*

$$\lim_n \frac{z!}{n^z} \left|\{(i_1, i_2, \cdots, i_z) \in \mathbb{N}^z : i_1, i_2, \cdots, i_z \leq n, g(x, u_{i_1}, \cdots, u_{i_z}) \geq \varepsilon\}\right| = 0,$$

and is denoted by $gS - \lim u_n = x$ or $u_n \overset{gS}{\to} x$.

2. Main Results

In this section, our aim to give notions of asymptotically statistical equivalent, lacunary statistically convergence and asymptotically lacunary statistical equivalent in $g-$metric spaces as a result of researches above. Afterwards, theorems expressing the relationship between these definitions will be presented.

Definition 2.1. *Let (X, g) be a g-metric space, two nonnegative sequences $u = (u_n)$ and $v = (v_n)$ are said to be asymptotically statistical equivalent of multiple x provided that for every $\varepsilon > 0$,*

$$\lim_n \frac{z!}{n^z} \left| \left\{ i_1, i_2, \cdots, i_z \leq n : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x \right) \geq \varepsilon \right\} \right| = 0.$$

This situation is denoted by $u \overset{gS^x}{\sim} v$ and simply asymptotically statistical equivalent if $x = 1$. Moreover, let gS^x denote the set of u and v such that $u \overset{gS^x}{\sim} v$.

Definition 2.2. *Let (X, g) be a g-metric space, $u = (u_n)$ be a sequence in this space and $\theta = (k_r)$ be a lacunary sequence. The sequence (u_n) is said to be lacunary statistically convergent to x provided that for every $\varepsilon > 0$,*

$$\lim_r \frac{z!}{h_r^z} \left| \left\{ i_1, i_2, \cdots, i_z \in I_r : g\left(x, u_{i_1}, \cdots, u_{i_z}\right) \geq \varepsilon \right\} \right| = 0.$$

This situation is denoted by $gS_\theta - \lim u_n \to x$ or $u_n \overset{gS_\theta}{\longrightarrow} x$. The set of all lacunary statistically convergent sequences is denoted by gS_θ.

Definition 2.3. *Let (X, g) be a $g-$metric space, $u = (u_n)$ be a sequence in this space and $\theta = (k_r)$ be a lacunary sequence. The sequence (u_n) is said to be $gN_\theta-$statistically convergent to x provided that*

$$\lim_r \frac{z!}{h_r^z} \sum_{i_1, i_2, \cdots, i_z \in I_r} g\left(x, u_{i_1}, \cdots, u_{i_z}\right) = 0.$$

This situation is denoted by $gN_\theta - \lim u_n \to x$ or $u_n \overset{gN_\theta}{\longrightarrow} x$. The set of all gN_θ-statistically convergent sequences is denoted by gN_θ.

Definition 2.4. *Let (X, g) be a $g-$metric space, two nonnegative sequences $u = (u_n)$ and $v = (v_n)$ in this space and θ be a lacunary sequence. (u_n) and (v_n) are said to be asymptotically lacunary statistical equivalent of multiple x provided that for every $\varepsilon > 0$,*

$$\lim_r \frac{z!}{h_r^z} \left| \left\{ i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x \right) \geq \varepsilon \right\} \right| = 0.$$

This situation is denoted by $u \overset{gS_\theta^x}{\sim} v$ and simply asymptotically lacunary statistical equivalent if $x = 1$. In addition, let gS_θ^x denote the set of u and v such that $u \overset{gS_\theta^x}{\sim} v$.

Definition 2.5. *Let (X,g) be a $g-$metric space, two nonnegative sequences $u=(u_n)$ and $v=(v_n)$ in this space and θ be a lacunary sequence. (u_n) and (v_n) are said to be strongly asymptotically lacunary equivalent of multiple x provided that*

$$\lim_r \frac{z!}{h_r^z} \sum_{i_1,i_2,\cdots,i_z \in I_r} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) = 0.$$

This situation is denoted by $u \overset{gN_\theta^x}{\sim} v$ and strongly simply asymptotically lacunary equivalent if $x=1$. Furthermore, let gN_θ^x denote the set of u and v such that $u \overset{gN_\theta^x}{\sim} v$.

Theorem 2.6. *Let (X,g) be a g-metric space, two nonnegative sequences $u=(u_n)$ and $v=(v_n)$ in this space and θ be a lacunary sequence. Then*

(i) If (u_n) and (v_n) are strongly asymptotically lacunary equivalent of multiple x, then (u_n) and (v_n) are asymptotically lacunary statistical equivalent of multiple x.

(ii) If (u_n) is bounded and (u_n) and (v_n) are asymptotically lacunary statistical equivalent of multiple x, then these sequences are strongly asymptotically lacunary equivalent of multiple x.

(iii) $gS_\theta^x \cap \ell_\infty = gN_\theta^x \cap \ell_\infty$, where ℓ_∞ is a set of bounded sequences.

Proof. (i) If $\varepsilon > 0$ and $u \overset{gN_\theta^x}{\sim} v$ then

$$\begin{aligned}
&\frac{z!}{h_r^z} \sum_{i_1,i_2,\cdots,i_z \in I_r} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \\
&\geq \frac{z!}{h_r^z} \sum_{\substack{i_1,i_2,\cdots,i_z \in I_r \\ g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon}} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \\
&\geq \varepsilon \frac{z!}{h_r^z} \left| \left\{ i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon \right\} \right|.
\end{aligned}$$

Therefore $u \overset{gS_\theta^x}{\sim} v$.

(ii) Suppose (u_n) and (v_n) are in ℓ_∞ and $u \overset{gS_\theta^x}{\sim} v$. Then we can assume that

$$g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \leq M \text{ for all } i_1, i_2, \cdots, i_z.$$

Given $\varepsilon > 0$

$$\begin{aligned}
&\frac{z!}{h_r^z} \sum_{i_1,i_2,\cdots,i_z \in I_r} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \\
&= \frac{z!}{h_r^z} \sum_{\substack{i_1,i_2,\cdots,i_z \in I_r \\ g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_m}}, x\right) \geq \varepsilon}} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \\
&+ \frac{z!}{h_r^z} \sum_{\substack{i_1,i_2,\cdots,i_z \in I_r \\ g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) < \varepsilon}} g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \\
&\leq M \frac{z!}{h_r^z} \left|\left\{i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right| + \varepsilon.
\end{aligned}$$

(iii) It is clear from (i) and (ii). □

Theorem 2.7. *Let (X, g) be a $g-$metric space, two nonnegative sequences $u = (u_n)$ and $v = (v_n)$ in this space and θ be a lacunary sequence with $\liminf q_r > 1$. If $u \overset{gS^x}{\sim} v$, then (u_n) and (v_n) are asymptotically lacunary statistical equivalent of multiple x.*

Proof. Suppose first that $\liminf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies $\frac{h_r^z}{k_r^z} \geq \left(\frac{\delta}{1+\delta}\right)^z$. If $u \overset{gS^x}{\sim} v$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}
&\frac{z!}{k_r^z} \left|\left\{i_1, i_2, \cdots, i_z \leq k_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right| \\
&\geq \frac{z!}{k_r^z} \left|\left\{i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right| \\
&\geq \left(\frac{\delta}{1+\delta}\right)^z \frac{z!}{h_r^z} \left|\left\{i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right|;
\end{aligned}$$

this completes the proof. □

Theorem 2.8. *Let (X, g) be a $g-$metric space, two nonnegative sequences $u = (u_n)$ and $v = (v_n)$ in this space and θ be a lacunary sequence with $\limsup q_n^z < \infty$. If $u \overset{gS_\theta^x}{\sim} v$, then $u = (u_n)$ and $v = (v_n)$ are asymptotically statistical equivalent of multiple x.*

Proof. If $\limsup q_n^z$, then there exists $B > 0$ such that $q_n^z < B$ for all $r \geq 1$. Let $u \overset{gS_\theta^x}{\sim} v$ and $\varepsilon > 0$. There exists $n_0 > 0$ such that for every $j \geq n_0$

$$A_j = \frac{z!}{h_j^z} \left|\left\{i_1, i_2, \cdots, i_z \in I_j : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right| < \varepsilon.$$

We can also find $M > 0$ such that $A_j < M$ for all $j = 1, 2, \cdots$. Now let n be any integer with $k_{r-1} < n < k_r$, where $r > n_0$. Then we have

$$\begin{aligned}
&\frac{z!}{n^z}\left|\left\{i_1, i_2, \cdots, i_z \le n : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&\le \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \le k_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&= \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_1 : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_2 : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \ldots + \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_{n_0} : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_{n_0+1} : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \ldots + \frac{z!}{k_{r-1}^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&= \frac{z!}{k_{r-1}^z}\frac{k_1^z}{k_1^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_1 : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \frac{z!}{k_{r-1}^z}\frac{(k_2-k_1)^z}{(k_2-k_1)^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_2 : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \ldots + \frac{z!}{k_{r-1}^z}\frac{(k_{n_0}-k_{n_0-1})^z}{(k_{n_0}-k_{n_0-1})^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_{n_0} : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \frac{z!}{k_{r-1}^z}\frac{(k_{n_0+1}-k_{n_0})^z}{(k_{n_0+1}-k_{n_0})^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_{n_0+1} : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&+ \ldots + \frac{z!}{k_{r-1}^z}\frac{(k_r-k_{r-1})^z}{(k_r-k_{r-1})^z}\left|\left\{i_1, i_2, \cdots, i_z \in I_r : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \ge \varepsilon\right\}\right| \\
&= \frac{k_1^z}{k_{r-1}^z}A_1 + \frac{(k_2-k_1)^z}{k_{r-1}^z}A_2 + \cdots + \frac{(k_{n_0}-k_{n_0-1})^z}{k_{r-1}^z}A_{n_0} \\
&+ \frac{(k_{n_0+1}-k_{n_0})^z}{k_{r-1}^z}A_{n_0+1} + \cdots + \frac{(k_r-k_{r-1})^z}{k_{r-1}^z}A_r \\
&\le \left\{\sup_{j\ge 1} A_j\right\}\frac{k_1^z + (k_2-k_1)^z + \cdots + (k_{n_0}-k_{n_0-1})^z}{k_{r-1}^z} \\
&+ \left\{\sup_{j\ge n_0} A_j\right\}\left(\frac{(k_{n_0+1}-k_{n_0})^z + \cdots + (k_r-k_{r-1})^z}{k_{r-1}^z}\right) \\
&\le M\frac{k_1^z + k_2^z - k_1^z + \cdots + k_{n_0}^z - k_{n_0-1}^z}{k_{r-1}^z} + \varepsilon\left(\frac{k_{n_0+1}^z - k_{n_0}^z + \cdots + k_r^z - k_{r-1}^z}{k_{r-1}^z}\right) \\
&= M\frac{k_{n_0}^z}{k_{r-1}^z} + \varepsilon\frac{k_r^z - k_{n_0+1}^z}{k_{r-1}^z} \le M\frac{k_{n_0}^z}{k_{r-1}^z} + \varepsilon B.
\end{aligned}$$

Since $k_{r-1}^z \to \infty$ as $n \to \infty$, it follows that

$$\frac{z!}{n^z}\left|\left\{i_1, i_2, \cdots, i_z \leq n : g\left(\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \cdots, \frac{u_{i_z}}{v_{i_z}}, x\right) \geq \varepsilon\right\}\right| \to 0.$$

This completes the proof. □

Theorem 2.9. *Let (X, g) be a $g-$metric space, $u = (u_n)$ and $v = (v_n)$ are two nonnegative sequences in this space and θ be a lacunary sequence with $1 < \inf q_r \leq \sup q_r < \infty$. Then $u \overset{gS_\theta^x}{\sim} v = u \overset{gS^x}{\sim} v$.*

Proof. The proof can be easily obtained from Theorem 2.7 and Theorem 2.8. □

3. Conclusion

The concept of g-metric was first investigated in 2018 by Choi et al. [5]. Statistical convergence in g-metric spaces was studied by Abazari [1]. Even though certain features in g-metric spaces have been examined, it is yet open to explore further properties of asymptotically equivalent sequences in g-metric spaces. So, the main results of the present paper fill up the gap in the existing literature. Some of the results presented in this article are analogous to the researches in the relevant topic, but in most situations the proofs follow a different approach. Therefore, it can be said that we have achieved more comprehensive concept.

References

[1] R. Abazari, *Statistical convergence in g-metric spaces*, Filomat **36** (2022), no. 5, 1461–1468.

[2] S. Aytar, M. Mammadov, and S. Pehlivan, *Statistical limit inferior and limit superior for sequences of fuzzy numbers*, Fuzzy Sets Syst. **157** (2006), no. 7, 976–985.

[3] F. Başar, *Summability Theory and its Applications*, 2^{nd} ed., CRC Press/Taylor & Francis Group, Boca Raton, London, New York, 2022.

[4] C. Belen and S. A. Mohiuddine, *Generalized weighted statistical convergence and application*, Appl. Math. Comput. **219** (2013), 9821–9826.

[5] H. Choi, S. Kim, and S. Yang, *Structure for g-metric spaces and related fixed point theorem*, preprint (2018); Arxiv: 1804.03651v1.

[6] A. Esi, *On D-asymptotically statistical equivalent sequences*, Appl. Math. Inf. Sci. **4** (2010), no. 2, 183–189.

[7] A. Esi, *On asymptotically lacunary statistical equivalent sequences in probabilistic normed space*, J Math Stat. **9** (2013), no. 2, 144–148.

[8] A. Esi, *On almost asymptotically lacunary statistical equivalent sequences induced probabilistic norms*, Analysis **34** (2014), 1001–1010.

[9] A. Esi, *On asymptotically double lacunary statistical equivalent sequences*, Appl. Math. Lett. **22** (2009), 1781–1785.

[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[11] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pasific J. Math. **160** (1993), no. 1, 755–762.

[12] M. Gürdal, A. Şahiner, and I. Açık, *Approximation theory in 2-Banach spaces*, Nonlinear Anal. **71** (2009), no. 5-6, 1654–1661.
[13] B. Hazarika, A. Alotaibi, and S. A. Mohiuddine, *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput. **24** (2020), 6613–6622.
[14] B. Hazarika and A. Esi, *On $\mathcal{I}$-asymptotically Wijsman generalized statistical convergence of sequences of sets*, Tatra Mt.Math.Publ. **56** (2013), 67–77.
[15] U. Kadak and S. A. Mohiuddine, *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-Gamma function and related approximation theorems*, Results Math. **73** (2018), no. 9, 1–31.
[16] J. Li, *Asymptotic equivalence of sequences and summability*, Internat J. Math. Math. Sci. **20** (1997), no. 4, 749–758.
[17] M. S. Marouf, *Asymptotic equivalence and summability*, Int. J. Math. Math. Sci. **16** (1993), no. 4, 755–762.
[18] S. A. Mohiuddine, A. Asiri, and B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst. **48** (2019), no. 5, 492–506.
[19] S. A. Mohiuddine, B. Hazarika, and A. Alotaibi, *On statistical convergence of double sequences of fuzzy valued functions.* J. Intell. Fuzzy Syst. **32** (2017), 4331–4342.
[20] M. Mursaleen and F. Başar, *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press/Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.
[21] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 289–297.
[22] R. F. Patterson, *On asymptotically statistically equivalent sequences*, Demonstr. Math. **36** (2003), no. 1, 149–153.
[23] F. Patterson and E. Savaş, *On asymptotically lacunary statistical equivalent sequences*, Thai J. Math. **4** (2006), no. 2, 267–272.
[24] S Pehlivan, A Güngan, M Mamedov, *Statistical cluster points of sequences in finite dimensional spaces*, Czechoslov. Math. J. **54** (2004), no. 1, 95–102.
[25] E. Savaş and M. Gürdal, *Generalized statistically convergent sequences of functions in fuzzy 2-normed spaces*, J. Intell. Fuzzy Syst. **27** (2014), no. 4, 2067–2075.
[26] E. Savaş and M. Gürdal, *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Scienceasia **41** (2015), no. 4, 289–294.
[27] D. Rath and B. C. Tripathy, *On statistically convergent and statistical Cauchy sequences*, Indian J. Pure Appl. Math. **25** (1994), 381–386.
[28] B. C. Tripathy and M. Sen, *On generalized statistically convergent sequences*, Indian J. Pure Appl. Math. **32** (2001), no. 11, 1689–1694.

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