Honam Mathematical J. **45** (2023), No. 3, pp. 471–490
https://doi.org/10.5831/HMJ.2023.45.3.471

SASAKIAN STATISTICAL MANIFOLDS WITH *QSM*-CONNECTION AND THEIR SUBMANIFOLDS

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Abstract. In this present paper, we study *QSM*-connection (quarter-symmetric metric connection) on Sasakian statistical manifolds. Firstly, we express the relation between the *QSM*-connection $\tilde{\nabla}$ and the torsion-free connection ∇ and obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and R of ∇. After then we obtain these relations for $\tilde{\nabla}$ and the dual connection ∇^* of ∇. Also, we give the relations between the curvature tensor $\tilde{R}$ of *QSM*-connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds. We obtain the relations between the Ricci tensor of *QSM*-connection $\tilde{\nabla}$ and the Ricci tensors of the connections ∇ and ∇^*. After these, we construct an example of a 3-dimensional Sasakian manifold admitting the *QSM*-connection in order to verify our results. Finally, we study the submanifolds with the induced connection with respect to *QSM*-connection of statistical manifolds.

1. Introduction

Statistical manifolds are Riemannian manifolds whose each point can be identified with a probability density with respect to a given measure. In 1945, the theory of these manifolds has started with a paper of Rao [26]. We know that statistical manifolds theory is information geometry. The information geometry typically deals with the study of various geometric structures on a statistical manifold, has begun as a study of the geometric structures possessed by a statistical model of probability distributions. As it is known, information geometry has many fields of study, for example; information theory, stochastic processes, dynamical systems and time series, statistical physics, quantum systems and the mathematical theory of neural networks ([8], [3]).

Amari has first introduced the notion of dual connection (or conjugate connection) in affine geometry [2]. A statistical model equipped with a Riemannian metric together with a pair of dual affine connections is called a *statistical*

Received December 2, 2022. Revised February 11, 2023. Accepted February 14, 2023.
2020 Mathematics Subject Classification. 53C40, 53D10.
Keywords and phrases. Sasakian manifolds, statistical structure, dual connection, quarter-symmetric metric connection.
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manifold. For more information about statistical manifolds and information geometry, we refer to [10], [29], [22], [5], [24], [18], [19], [21] and etc.

On the other hand, if Φ is a tensor field of type $(1,1)$, η is a 1-form and ξ is a vector field on a $(2n+1)$-dimensional differentiable manifold M, then almost contact structure (Φ,η,ξ) which is related to almost complex structures and satisfies the conditions $\Phi^2 X = -I + \eta \otimes \xi$, $\eta(\xi) = 1$ has been determined by Sasaki in 1960 [27]. In addition to this definition, different types of manifolds such as Sasakian manifold, Kenmotsu manifold, trans-Sasakian manifold and etc. have been defined and studied by many mathematicians [25], [20], [27] and etc.

Considering these notions, the differential geometry of statistical manifolds are being studied by geometers by adding different geometric structures to these manifolds. For instance, in [30] quaternionic Kähler-like statistical manifold have been studied and the authors have introduced the notion of Sasakian statistical structure and obtained the condition for a real hypersurface in a holomorphic statistical manifold to admit such a structure in [11]. In [16], the author has studied conformally-projectively flat trans-Sasakian statistical manifolds. Also in [12], the notion of a Kenmotsu statistical manifold is introduced and they have showed that, a Kenmotsu statistical manifold of constant Φ-sectional curvature is constructed from a special Kähler manifold, which is an important example of holomorphic statistical manifold. The projection of a dualistic structure has been defined on a twisted product manifold induces dualistic structures on the base and the fiber manifolds, and conversely in [8]. Also, the authors Kazan and Kazan have examined Sasakian statistical manifolds with semi-symmetric metric connection in [17].

In this paper, we aim to give the notion of *QSM*-connection $\tilde{\nabla}$ on Sasakian statistical manifolds and obtain the relations between the curvature tensor $\tilde{R}$ of *QSM*-connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds, respectively. Moreover, we give the relations between the Ricci tensor of *QSM*-connection $\tilde{\nabla}$ and the Ricci tensors of the connections ∇ and ∇^*, respectively. At the end of this, we construct an example of a 3-dimensional Sasakian manifold admitting the *QSM*-connection in order to verify our results. Finally, we study the submanifolds of this manifold.

Thus, let us give some basic notions about statistical manifolds which will be useful for us in the next sections. Throughout the paper we denote the notion of quarter-symmetric metric connection as *QSM*-connection.

We assume that M is a $(2n+1)$-dimensional manifold, g is a Riemannian metric, $\hat{\nabla}$ is the Levi-Civita connection associated with g and $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p,q) on M.

A pair (∇, g) is called a *statistical structure* on M, if ∇ is torsion-free and

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z), \ \forall X,Y,Z \in \Gamma(TM) \tag{1}$$

holds, where the equation (1) is generally called *Codazzi equation.* In this case, (M, ∇, g) is called a *statistical manifold.*

Let (∇, g) be a statistical structure on M. Then the connection ∇^* which is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2}$$

is called *conjugate or dual connection* of ∇ with respect to g. If (∇, g) is a statistical structure on M, then (∇^*, g) is a statistical structure on M, too.

For a statistical structure (∇, g), one can define the *difference tensor field* $K \in \Gamma(TM^{(1,2)})$ as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \ \forall X, Y \in \Gamma(TM), \tag{3}$$

where K satisfies

$$K(X,Y) = K(Y,X), \tag{4}$$

$$g(K(X,Y), Z) = g(Y, K(X,Z)). \tag{5}$$

Furthermore, we have

$$K = \hat{\nabla} - \nabla^* = \frac{1}{2}(\nabla - \nabla^*). \tag{6}$$

For more detailed study, we refer to [10], [11] and [32].

On the other hand in [15], Hayden introduced a metric connection with a non-zero torsion on a Riemannian manifold and this connection is called a Hayden connection. In [13], the author has introduced the quarter-symmetric linear connection in a differentiable manifold. A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection if its torsion tensor T is of the connection $\tilde{\nabla}$

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{7}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{8}$$

where $X,\ Y \in \chi(M)$ and η is a 1-form, ϕ is a (1,1)-tensor field. Also, a quarter-symmetric connection $\tilde{\nabla}$ is called a quarter-symmetric metric connection (*QSM*-connection) if it further satisfies $\tilde{\nabla} g = 0$.

2. *QSM*-Connection On Statistical Manifolds

In this section, first we define the notion of statistical manifold admitting a *QSM*-connection. After that we obtain the relation between the curvature tensor $\tilde{R}$ of the *QSM*-connection $\tilde{\nabla}$ with the curvature tensor R of the affine connection ∇.

Let M be a Riemannian manifold. A linear connection $\tilde{\nabla}$ in M is said to be quarter symmetric connection [13] if the torsion tensor T of the connection $\tilde{\nabla}$, for $\forall X, Y \in \Gamma(TM)$, satisfies

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{9}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. A linear connection $\tilde{\nabla}$ is called a metric connection with respect to a Riemannian metric g of M if and only if

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{10}$$

where $X, Y, Z \in \chi(M)$ are arbitrary vector fields on M. Then, a linear connection $\tilde{\nabla}$ satisfying (9) and (10) is called a *QSM*-connection [13].

Definition 2.1. *Let (M, ∇, g) be a statistical manifold and let U be a vector field on M. For any $X, Y \in \Gamma(TM)$, the linear connection $\tilde{\nabla}$ on M is defined by*

$$\tilde{\nabla}_X Y = \nabla_X Y - w(X)\phi Y - K(X,Y), \tag{11}$$

where $g(X,U) = w(X)$. Then, $(M, \tilde{\nabla}, g)$ is called a statistical manifold admitting a QSM-connection.

Taking (6) in (11), we get

$$\tilde{\nabla}_X Y = \nabla^*_X Y - w(X)\phi Y + K(X,Y), \tag{12}$$

for any $X, Y \in \Gamma(TM)$.

Now, we can examine the curvature tensor of a statistical manifold admitting a *QSM*-connection in the following proposition:

Proposition 2.2. *Let $(M, \tilde{\nabla}, g)$ be a statistical manifold admitting a QSM-connection. Then the relations between the curvature tensor $\tilde{R}$ of QSM-connection $\tilde{\nabla}$ with the curvature tensors R and R^* of the connections ∇ and ∇^* are given by*

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + w(X)(\nabla_Y \phi)Z - w(Y)(\nabla_X \phi)Z \\
&\quad + \phi Z \left\{ \begin{array}{c} w(X)w(\phi Y) - w(Y)w(\phi X) - g(Y, \nabla_X U - K(X,U)) \\ +g(X, \nabla_Y U - K(Y,U)) + g(w(X)Y - w(Y)X, \phi U) \end{array} \right\} \\
&\quad + K(w(Y)X - w(X)Y, \phi Z) - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) \\
&\quad + w(X)\phi(K(Y,Z)) - w(Y)\phi(K(X,Z)) + K(X, K(Y,Z)) - K(Y, K(X,Z))
\end{aligned} \tag{13}$$

and

$$\begin{aligned}\tilde{R}(X,Y)Z &= R^*(X,Y)Z + w(X)(\nabla^*_Y\phi)Z - w(Y)(\nabla^*_X\phi)Z \\ &\quad + \phi Z\left\{\begin{array}{l} w(X)w(\phi Y) - w(Y)w(\phi X) - g(Y,\nabla^*_X U - K(X,U)) \\ +g(X,\nabla^*_Y U - K(Y,U)) + g(w(X)Y - w(Y)X,\phi U)\end{array}\right\} \\ &\quad + K(w(Y)X - w(X)Y,\phi Z) - (\nabla^*_X K)(Y,Z) + (\nabla^*_Y K)(X,Z) \\ &\quad + w(X)\phi(K(Y,Z)) - w(Y)\phi(K(X,Z)) + K(X,K(Y,Z)) - K(Y,K(X,Z)) \end{aligned} \tag{14}$$

for all vector fields X, Y *and* Z *on* M.

Proof. It is known that the Riemannian curvature tensor R of M with respect to the torsion-free connection ∇ is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. Then, from (11), we can define the Riemannian curvature tensor $\tilde{R}$ of M with respect to the *QSM*-connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z, \tag{15}$$

where $\forall X,Y,Z \in \Gamma(TM)$. Here, using (11), we have

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \nabla_X\nabla_Y Z - w(X)\phi(\nabla_Y Z) - K(X,\nabla_Y Z) \\ &\quad + g(\nabla_X Y,U)\phi Z + w(X)g(\phi Y,U)\phi Z + g(K(X,Y),U)\phi Z \\ &\quad - g(Y,\nabla_X U)\phi Z + w(X)g(Y,\phi U)\phi Z + g(Y,K(X,U))\phi Z \\ &\quad - w(Y)\nabla_X\phi Z + w(Y)w(X)\phi^2 Z + w(Y)K(X,\phi Z) \\ &\quad - \nabla_X K(Y,Z) + w(X)\phi(K(Y,Z)) + K(X,K(Y,Z)). \end{aligned} \tag{16}$$

Similarly, we can find $\tilde{\nabla}_Y\tilde{\nabla}_X Z$. Also, we get

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z - w(\nabla_X Y)\phi Z + w(\nabla_Y X)\phi Z - K(\nabla_X Y,Z) + K(\nabla_Y X,Z). \tag{17}$$

Then, using (16) and (17) in (15), we obtain (13).

On the other hand, we know that R^* is the Riemannian curvature tensor of M with respect to the dual connection ∇^* which is defined by $R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z$. Thus, we obtain (14). □

Proposition 2.3. *Let* $(M,\tilde{\nabla},g)$ *be a statistical manifold admitting a QSM-connection. Then the curvature tensor* $\tilde{R}$ *associated with* $\tilde{\nabla}$ *satisfies the following conditions for* $\forall X,Y,Z \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$,

i) $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z$, *ii)* $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y =$

$$
\begin{aligned}
& w(X)(\nabla_Y\phi)Z - w(Y)(\nabla_X\phi)Z + w(Y)(\nabla_Z\phi)X \\
& - w(Z)(\nabla_Y\phi)X + w(Z)(\nabla_X\phi)Y - w(X)(\nabla_Z\phi)Y \\
& + \phi Z \left\{ \begin{array}{c} w(X)w(\phi Y) - w(Y)w(\phi X) - g(Y, \nabla_X U - K(X,U)) \\ +g(X, \nabla_Y U - K(Y,U)) + g(w(X)Y - w(Y)X, \phi U) \end{array} \right\} \\
& + \phi X \left\{ \begin{array}{c} w(Y)w(\phi Z) - w(Z)w(\phi Y) - g(Z, \nabla_Y U - K(Y,U)) \\ +g(Y, \nabla_Z U - K(Z,U)) + g(w(Y)Z - w(Z)Y, \phi U) \end{array} \right\} \\
& + \phi Y \left\{ \begin{array}{c} w(Z)w(\phi X) - w(X)w(\phi Z) - g(X, \nabla_Z U - K(Z,U)) \\ +g(Z, \nabla_X U - K(X,U)) + g(w(Z)X - w(X)Z, \phi U) \end{array} \right\} \\
& + K(w(Y)X - w(X)Y, \phi Z) + K(w(Z)Y - w(Y)Z, \phi X) \\
& + K(w(X)Z - w(Z)X, \phi Y),
\end{aligned}
$$

iii) $g(\tilde{R}(X,Y)Z, V) = g(Z, \tilde{R}(X,Y)V)$ *if and only if*

$$
\begin{aligned}
& g(Z, (\nabla_X K)(Y,V)) + g(Z, (\nabla_Y K)(X,V)) - g((\nabla_X K)(Y,Z), V) + g((\nabla_Y K)(X,Z), V) \\
& + w(X)\{g((\nabla_Y\phi)Z, V) + g((\nabla_Y\phi)V, Z) + g(\phi(K(Y,Z)), V) + g(\phi(K(Y,V)), Z)\} \\
& - w(Y)\{g((\nabla_X\phi)Z, V) + g((\nabla_X\phi)V, Z) + g(\phi(K(X,Z)), V) + g(\phi(K(X,V)), Z)\} \\
& + [g(\phi Z, V) + g(\phi V, Z)] \left\{ \begin{array}{c} w(X)w(\phi Y) - w(Y)w(\phi X) - g(Y, \nabla_X U - K(X,U)) \\ +g(X, \nabla_Y U - K(Y,U)) + g(w(X)Y - w(Y)X, \phi U) \end{array} \right\} \\
& + 3\{g(K(Y, K(X,V)), Z) - g(K(X, K(Y,V)), Z)\} \\
& + g(K(X, K(Y,Z)), V) - g(K(Y, K(X,Z)), V) \\
& = 0.
\end{aligned}
$$

Proof. Using (11) and (13) and by long calculations, one can see that (i)-(ii)-(iii) hold. □

3. *QSM*-Connection On Sasakian Statistical Manifolds

A $(2n+1)$-dimensional differentiable manifold M is said to admit an *almost contact Riemannian structure* (ϕ, η, ξ, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$
\begin{aligned}
& \phi\xi = 0, \quad \eta(\xi) = 1, \\
& g(\xi, X) = \eta(X), \quad \phi^2 X = -X + \eta(X)\xi, \\
& g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),
\end{aligned} \tag{18}
$$

for any vector fields X, Y on M . In addition, if (ϕ, η, ξ, g) satisfies the equations

$$d\eta = 0, \ \hat{\nabla}_X \xi = \phi X, \tag{19}$$

$$(\hat{\nabla}_X \phi)Y = \eta(Y)X - g(X,Y)\xi, \tag{20}$$

then M is called a *Sasakian manifold* (for detail, see [6] and [11]).

Also in [11], the authors have defined the notion of Sasakian statistical structure and have obtained the necessary and sufficient conditions for a statistical structure on an almost contact metric manifold to be a Sasakian statistical structure as follows:

Definition 3.1. *A quadruple (∇, g, ϕ, ξ) is called a Sasakian statistical structure on M if (∇, g) is a statistical structure, (g, ϕ, ξ) is a Sasakian structure on M and the formula $K(X, \phi Y) + \phi K(X, Y) = 0$ holds for any vector fields X and Y on M* [11].

Theorem 3.2. *Let (∇, g) be a statistical structure and let (g, ϕ, ξ) be an almost contact metric structure on M. Then (∇, g, ϕ, ξ) is a Sasakian statistical structure if and only if the following formulas hold* [11]*:*

$$\nabla_X \phi Y - \phi \nabla^*_X Y = g(Y, \xi)X - g(Y, X)\xi, \tag{21}$$

$$\nabla_X \xi = \phi X + g(\nabla_X \xi, \xi)\xi. \tag{22}$$

Here, we can give the following results for Sasakian statistical manifolds:

Remark 3.3. *Let $(M, \nabla, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, from* [11]*, we have the following statements:*

$$(\nabla_X \phi)Y = \eta(Y)X - g(X, Y)\xi + 2K(X, \phi Y), \tag{23}$$

$$(\nabla^*_X \phi)Y = \eta(Y)X - g(X, Y)\xi - 2K(X, \phi Y), \tag{24}$$

$$\nabla_X \xi = \phi X + \eta(\nabla_X \xi)\xi, \tag{25}$$

$$\nabla^*_X \xi = \phi X + \eta(\nabla^*_X \xi)\xi, \tag{26}$$

for any $X, Y \in \Gamma(TM)$.

Corollary 3.4. *Let $(M, \nabla, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, from Remark 3.3, we have*

$$\begin{cases} K(X, \xi) = \nabla_X \xi - \phi X & = \eta(\nabla_X \xi)\xi, \\ K(X, \xi) = \phi X - \nabla^*_X \xi & = \eta(\nabla^*_X \xi)\xi, \end{cases} \tag{27}$$

for any $X \in \Gamma(TM)$,

With reference [23], we define *QSM*-connection $\tilde{\nabla}$ for the connections ∇ and ∇^* on Sasakian statistical manifolds:

Definition 3.5. *Let $(M, \nabla, \nabla^*, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, M is called Sasakian statistical manifold with QSM-connection $\tilde{\nabla}$, if the following equation holds:*

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y - K(X, Y). \tag{28}$$

The equation (28) can also be written for dual connection, as follows:

$$\tilde{\nabla}_X Y = \nabla^*_X Y - \eta(X)\phi Y + K(X,Y), \tag{29}$$

for any $X, Y \in \Gamma(TM)$ and η is a 1-form.

Now, using (28), (29) and Proposition 2.2, we give the relations between the curvature tensor $\tilde{R}$ of *QSM*-connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds.

Theorem 3.6. *Let $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then the relations between the curvature tensor $\tilde{R}$ of QSM-connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* are*

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + 2g(X,\phi Y)\phi Z - \eta(X)g(Y,Z)\xi \\ &+ \eta(Y)g(X,Z)\xi + \{\eta(X)Y - \eta(Y)X\}\,\eta(Z) \\ &- (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X,K(Y,Z)) - K(Y,K(X,Z))\end{aligned} \tag{30}$$

and

$$\begin{aligned}\tilde{R}(X,Y)Z &= R^*(X,Y)Z + 2g(X,\phi Y)\phi Z - \eta(X)g(Y,Z)\xi \\ &+ \eta(Y)g(X,Z)\xi + \{\eta(X)Y - \eta(Y)X\}\,\eta(Z) \\ &+ (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X,K(Y,Z)) - K(Y,K(X,Z)),\end{aligned} \tag{31}$$

for all vector fields X, Y and Z on M.

Proof. Using the equations (13) and (14), the theorem is proved. □

Corollary 3.7. *Let $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then we have*

$$\tilde{R}(X,Y)\xi = R(X,Y)\xi + \eta(X)Y - \eta(Y)X - (\nabla_X K)(Y,\xi) + (\nabla_Y K)(X,\xi) \tag{32}$$

$$= R^*(X,Y)\xi + \eta(X)Y - \eta(Y)X + (\nabla^*_X K)(Y,\xi) - (\nabla^*_Y K)(X,\xi) \tag{33}$$

and

$$\begin{aligned}\tilde{R}(\xi,X)Y &= R(\xi,X)Y - g(X,Y)\xi + \eta(Y)X - (\nabla_\xi K)(X,Y) + (\nabla_X K)(\xi,Y) \\ &+ K(\xi,K(X,Y)) - K(X,K(\xi,Y))\end{aligned} \tag{34}$$

$$\begin{aligned}&= R^*(\xi,X)Y - g(X,Y)\xi + \eta(Y)X - (\nabla^*_\xi K)(X,Y) + (\nabla^*_X K)(\xi,Y) \\ &+ K(\xi,K(X,Y)) - K(X,K(\xi,Y)),\end{aligned} \tag{35}$$

for all vector fields X and Y on M.

Proof. From (27), we have $K(X,K(Y,\xi)) = \eta(\nabla_X\xi)\eta(\nabla_Y\xi)\xi$ and so, we obtain that

$$K(X,K(Y,\xi)) = K(Y,K(X,\xi)). \tag{36}$$

Then, using (25), (26) and (27) in (30) and (31), we reach the equations (32)-(35) and the proof completes. □

Now, we give the relations between the Ricci tensor $\widetilde{Ric}$ of QSM-connection $\tilde{\nabla}$ and the Ricci tensors Ric and Ric^* of the connections ∇ and ∇^*.

Theorem 3.8. *Let $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, the relations between the Ricci tensors of QSM-connection $\tilde{\nabla}$ and the connections ∇ and ∇^* are*

$$\begin{aligned} \widetilde{Ric}(X,Y) = {} & Ric(X,Y) + g(\phi X, \phi Y) - (2n)\eta(X)\eta(Y) \\ & - \sum_{i=1}^{2n+1} g((\nabla_{e_i}K)(X,Y) - (\nabla_X K)(e_i,Y) \\ & - K(e_i, K(X,Y)) + K(X, K(e_i,Y)), e_i) \end{aligned} \tag{37}$$

and

$$\begin{aligned} \widetilde{Ric}(X,Y) = {} & Ric^*(X,Y) + g(\phi X, \phi Y) - (2n)\eta(X)\eta(Y) \\ & + \sum_{i=1}^{2n+1} g((\nabla^*_{e_i}K)(X,Y) - (\nabla^*_X K)(e_i,Y) \\ & + K(e_i, K(X,Y)) - K(X, K(e_i,Y)), e_i), \end{aligned} \tag{38}$$

for all vector fields X and Y on M.

Proof. Using (18) and (30), we get (37). Similarly, using (18) and (31), we get (38). Then, the proof is completed. $\square$

Now, we give a result from Theorem 3.8:

Corollary 3.9. *Let $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. If we get the $\phi-$basis $\{e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n, \xi\}$, then we obtain*

$$\begin{aligned} \widetilde{Ric}(X,Y) = {} & Ric(X,Y) + g(\phi X, \phi Y) - (2n)\eta(X)\eta(Y) \\ & + \sum_{i=1}^{n} \left\{ \begin{array}{c} g((\nabla_X K)(e_i,Y), e_i) - g((\nabla_{e_i}K)(X,Y), e_i) + g((\nabla_X K)(\phi e_i, Y), \phi e_i) \\ +g((\nabla_{\phi e_i}K)(X,Y), \phi e_i) + g(K(e_i, K(X,Y)), e_i) - g(K(X, K(e_i,Y)), e_i) \\ +g(K(\phi e_i, K(X,Y)), \phi e_i) - g(K(X, K(\phi e_i, Y)), \phi e_i) \end{array} \right\} \\ & + g((\nabla_X K)(\xi, Y), \xi) - g((\nabla_\xi K)(X,Y), \xi) + g(K(\xi, K(X,Y)), \xi) \\ & - g(K(X, K(\xi, Y)), \xi) \end{aligned} \tag{39}$$

and

$$\begin{aligned}
\widetilde{Ric}(X,Y) &= Ric^*(X,Y) + g(\phi X, \phi Y) - (2n)\eta(X)\eta(Y) \\
&+ \sum_{i=1}^{n} \left\{ \begin{array}{c} g((\nabla^*_{e_i} K)(X,Y), e_i) - g((\nabla^*_X K)(e_i,Y), e_i) + g((\nabla^*_{\phi e_i} K)(X,Y), \phi e_i) \\ -g((\nabla^*_X K)(\phi e_i, Y), \phi e_i) + g(K(e_i, K(X,Y)), e_i) - g(K(X, K(e_i,Y)), e_i) \\ +g(K(\phi e_i, K(X,Y)), \phi e_i) - g(K(X, K(\phi e_i, Y)), \phi e_i) \end{array} \right\} \\
&- g((\nabla_X K)(\xi, Y), \xi) + g((\nabla_\xi K)(X,Y), \xi) + g(K(\xi, K(X,Y)), \xi) \\
&- g(K(X, K(\xi, Y)), \xi)
\end{aligned} \tag{40}$$

for all vector fields X and Y on M.

Now, let us construct an example of a 3-dimensional Sasakian manifold admitting the *QSM*-connection in order to verify our results.

Example 3.10. *We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$.*

We choose the vector fields $\{e_1, e_2, e_3\}$ as

$$e_1 = \frac{\partial}{\partial x},\ e_2 = -x\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) + \frac{\partial}{\partial y},\ e_3 = \frac{1}{2}\frac{\partial}{\partial z}, \tag{41}$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i,j = 1,2,3$ and $g(e_k, e_k) = 1$, $k = 1,2,3$.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = 0. \tag{42}$$

Using the linearity of ϕ and g, we have $\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$ and $g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Now, we have

$$[e_1, e_2] = -e_1 + 2e_3,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0. \tag{43}$$

The Levi-Civita connection $\hat{\nabla}$ of the metric tensor g is given by Koszul's formula which is defined as

$$\begin{aligned}
2g(\hat{\nabla}_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
&\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).
\end{aligned}$$

Taking $e_3 = \xi$ and using Koszul's formula, we get the following

$$\begin{aligned}
&\hat{\nabla}_{e_1} e_1 = e_2,\ \hat{\nabla}_{e_1} e_2 = -e_1 + e_3,\ \hat{\nabla}_{e_1} e_3 = -e_2, \\
&\hat{\nabla}_{e_2} e_1 = -e_3,\ \hat{\nabla}_{e_2} e_2 = 0,\ \hat{\nabla}_{e_2} e_3 = e_1, \\
&\hat{\nabla}_{e_3} e_1 = -e_2,\ \hat{\nabla}_{e_3} e_2 = e_1,\ \hat{\nabla}_{e_3} e_3 = 0.
\end{aligned} \tag{44}$$

By using above values, it can be easily seen that (ϕ, ξ, η, g) *is a Sasakian structure on* M. *Consequently,* (M, ϕ, ξ, η, g) *is a 3-dimensional Sasakian manifold* [7].

Now, the components of the curvature tensor, the Ricci tensor and scalar curvature with respect to the Levi-Civita connection $\hat{\nabla}$ *are obtained as*

$$\begin{aligned} &\hat{R}(e_1, e_2)e_1 = 4e_2,\ \hat{R}(e_1, e_2)e_2 = -4e_1,\ \hat{R}(e_1, e_2)e_3 = 0, \\ &\hat{R}(e_1, e_3)e_1 = -e_3,\ \hat{R}(e_1, e_3)e_2 = 0,\ \hat{R}(e_1, e_3)e_3 = e_1, \\ &\hat{R}(e_2, e_3)e_1 = 0,\ \hat{R}(e_2, e_3)e_2 = -e_3,\ \hat{R}(e_2, e_3)e_3 = e_2, \end{aligned} \tag{45}$$

$$\begin{aligned} &\widehat{Ric}(e_1, e_1) = -3,\ \widehat{Ric}(e_1, e_2) = 0,\ \widehat{Ric}(e_1, e_3) = 0, \\ &\widehat{Ric}(e_2, e_1) = 0,\ \widehat{Ric}(e_2, e_2) = -3,\ \widehat{Ric}(e_2, e_3) = 0, \\ &\widehat{Ric}(e_3, e_1) = 0,\ \widehat{Ric}(e_3, e_2) = 0,\ \widehat{Ric}(e_3, e_3) = 2 \end{aligned} \tag{46}$$

and

$$\hat{\tau} = -4, \tag{47}$$

respectively.

From (3) and (44), we have

(48)
$$\begin{aligned} &\nabla_{e_1} e_1 = e_2 + K(e_1, e_1),\ \nabla_{e_1} e_2 = -e_1 + e_3 + K(e_1, e_2),\ \nabla_{e_1} e_3 = -e_2 + K(e_1, e_3), \\ &\nabla_{e_2} e_1 = -e_3 + K(e_2, e_1),\ \nabla_{e_2} e_2 = K(e_2, e_2),\ \nabla_{e_2} e_3 = e_1 + K(e_2, e_3), \\ &\nabla_{e_3} e_1 = -e_2 + K(e_3, e_1),\ \nabla_{e_3} e_2 = e_1 + K(e_3, e_2),\ \nabla_{e_3} e_3 = K(e_3, e_3). \end{aligned}$$

So, the components of the curvature tensor, the Ricci tensor and scalar curvature with respect to the torsion-free connection ∇ *are obtained as*

(49)
$$\begin{aligned} &R(e_1, e_2)e_1 = 4e_2 + K(e_1, e_1) - 3K(e_1, e_3) - K(e_2, e_2) + \nabla_{e_1} K(e_2, e_1) - \nabla_{e_2} K(e_1, e_1), \\ &R(e_1, e_2)e_2 = -4e_1 + 2K(e_1, e_2) - 3K(e_2, e_3) + \nabla_{e_1} K(e_2, e_2) - \nabla_{e_2} K(e_1, e_2), \\ &R(e_1, e_2)e_3 = K(e_1, e_1) + K(e_1, e_3) + K(e_2, e_2) - 2K(e_3, e_3) + \nabla_{e_1} K(e_2, e_3) - \nabla_{e_2} K(e_1, e_3), \\ &R(e_1, e_3)e_1 = -e_3 - K(e_1, e_2) - K(e_2, e_3) + \nabla_{e_1} K(e_3, e_1) - \nabla_{e_3} K(e_1, e_1), \\ &R(e_1, e_3)e_2 = K(e_1, e_1) + K(e_1, e_3) - K(e_3, e_3) + \nabla_{e_1} K(e_3, e_2) - \nabla_{e_3} K(e_1, e_2), \\ &R(e_1, e_3)e_3 = e_1 + K(e_2, e_3) + \nabla_{e_1} K(e_3, e_3) - \nabla_{e_3} K(e_1, e_3), \\ &R(e_2, e_3)e_1 = -K(e_2, e_2) + K(e_3, e_3) + \nabla_{e_2} K(e_3, e_1) - \nabla_{e_3} K(e_2, e_1), \\ &R(e_2, e_3)e_2 = -e_3 + K(e_2, e_1) + \nabla_{e_2} K(e_3, e_2) - \nabla_{e_3} K(e_2, e_2), \\ &R(e_2, e_3)e_3 = e_2 - K(e_3, e_1) + \nabla_{e_2} K(e_3, e_3) - \nabla_{e_3} K(e_2, e_3), \end{aligned}$$

(50)

$$\begin{aligned}
Ric(e_1,e_1) &= -3 - g(K(e_1,e_1),e_2) + 3g(K(e_1,e_3),e_2) + g(K(e_2,e_2),e_2)\\
&\quad - g(\nabla_{e_1}K(e_2,e_1),e_2) + g(\nabla_{e_2}K(e_1,e_1),e_2) + g(K(e_1,e_2),e_3) + g(K(e_3,e_2),e_3)\\
&\quad - g(\nabla_{e_1}K(e_3,e_1),e_3) + g(\nabla_{e_3}K(e_1,e_1),e_3),\\
Ric(e_1,e_2) &= -2g(K(e_1,e_2),e_2) + 3g(K(e_2,e_3),e_2) - g(\nabla_{e_1}K(e_2,e_2),e_2)\\
&\quad + g(\nabla_{e_2}K(e_1,e_2),e_2) - g(K(e_1,e_1),e_3) + g(K(e_3,e_3),e_3) - g(K(e_3,e_1),e_3)\\
&\quad - g(\nabla_{e_1}K(e_3,e_2),e_3) - g(\nabla_{e_3}K(e_1,e_2),e_3),\\
Ric(e_1,e_3) &= -g(K(e_1,e_1),e_2) - g(K(e_2,e_2),e_2) + 2g(K(e_3,e_3),e_2) - g(K(e_1,e_3),e_2)\\
&\quad - g(\nabla_{e_1}K(e_2,e_3),e_2) + g(\nabla_{e_2}K(e_1,e_3),e_2) - g(K(e_3,e_2),e_3)\\
&\quad - g(\nabla_{e_1}K(e_3,e_3),e_3) + g(\nabla_{e_3}K(e_1,e_3),e_3),\\
Ric(e_2,e_1) &= g(K(e_1,e_1),e_1) - 3g(K(e_1,e_3),e_1) - g(K(e_2,e_2),e_1)\\
&\quad + g(\nabla_{e_1}K(e_2,e_1),e_1) - g(\nabla_{e_2}K(e_1,e_1),e_1) + g(K(e_2,e_2),e_3)\\
&\quad - g(K(e_3,e_3),e_3) - g(\nabla_{e_2}K(e_3,e_1),e_3) - g(\nabla_{e_3}K(e_2,e_1),e_3),\\
Ric(e_2,e_2) &= -3 + g(K(e_1,e_2),e_1) + g(K(e_2,e_1),e_1) - 3g(K(e_2,e_3),e_1)\\
&\quad + g(\nabla_{e_1}K(e_2,e_2),e_1) - g(\nabla_{e_2}K(e_1,e_2),e_1) - g(K(e_2,e_1),e_3)\\
&\quad - g(\nabla_{e_2}K(e_3,e_2),e_3) + g(\nabla_{e_3}K(e_2,e_2),e_3),\\
Ric(e_2,e_3) &= g(K(e_1,e_1),e_1) + g(K(e_2,e_2),e_1) - 2g(K(e_3,e_3),e_1) + g(K(e_1,e_3),e_1)\\
&\quad + g(\nabla_{e_1}K(e_2,e_3),e_1) - g(\nabla_{e_2}K(e_1,e_3),e_1) + g(K(e_3,e_1),e_3)\\
&\quad + g(\nabla_{e_2}K(e_3,e_3),e_3) - g(\nabla_{e_3}K(e_2,e_3),e_3),\\
Ric(e_3,e_1) &= -g(K(e_1,e_2),e_1) - g(K(e_3,e_2),e_1) + g(\nabla_{e_1}K(e_3,e_1),e_1) - g(\nabla_{e_3}K(e_1,e_1),e_1)\\
&\quad - g(K(e_2,e_2),e_2) + g(K(e_3,e_3),e_2) + g(\nabla_{e_2}K(e_3,e_1),e_2) - g(\nabla_{e_3}K(e_2,e_1),e_2),\\
Ric(e_3,e_2) &= g(K(e_1,e_1),e_1) - g(K(e_3,e_3),e_1) + g(K(e_3,e_1),e_1) + g(\nabla_{e_1}K(e_3,e_2),e_1)\\
&\quad - g(\nabla_{e_3}K(e_1,e_2),e_1) + g(K(e_2,e_1),e_2) + g(\nabla_{e_2}K(e_3,e_2),e_2) - g(\nabla_{e_3}K(e_2,e_2),e_2),\\
Ric(e_3,e_3) &= 2 + g(K(e_3,e_2),e_1) + g(\nabla_{e_1}K(e_3,e_3),e_1) - g(\nabla_{e_3}K(e_1,e_3),e_1) - g(K(e_3,e_1),e_2)\\
&\quad + g(\nabla_{e_2}K(e_3,e_3),e_2) - g(\nabla_{e_3}K(e_2,e_3),e_2)
\end{aligned}$$

and

(51)

$$\begin{aligned}
\tau &= -4 - g(K(e_1,e_1),e_2) + 2g(K(e_1,e_3),e_2) + g(K(e_2,e_2),e_2) + g(K(e_3,e_2),e_3)\\
&\quad + 2g(K(e_1,e_2),e_1) - 2g(K(e_3,e_2),e_1) - g(\nabla_{e_1}K(e_2,e_1),e_2) + g(\nabla_{e_2}K(e_1,e_1),e_2)\\
&\quad - g(\nabla_{e_1}K(e_3,e_1),e_3) + g(\nabla_{e_3}K(e_1,e_1),e_3) + g(\nabla_{e_1}K(e_2,e_2),e_1) - g(\nabla_{e_2}K(e_1,e_2),e_1)\\
&\quad - g(\nabla_{e_2}K(e_3,e_2),e_3) + g(\nabla_{e_3}K(e_2,e_2),e_3) + g(\nabla_{e_1}K(e_3,e_3),e_1) - g(\nabla_{e_3}K(e_1,e_3),e_1)\\
&\quad + g(\nabla_{e_2}K(e_3,e_3),e_2) - g(\nabla_{e_3}K(e_2,e_3),e_2),
\end{aligned}$$

respectively. (Similarly, the above equations can be obtained for the dual connection ∇^*.)

Finally, from (11) (for $w = \eta$ and $U = \xi$) and (44), we have

$$\begin{aligned}
&\tilde{\nabla}_{e_1} e_1 = e_2,\ \tilde{\nabla}_{e_1} e_2 = -e_1 + e_3,\ \tilde{\nabla}_{e_1} e_3 = -e_2,\\
&\tilde{\nabla}_{e_2} e_1 = -e_3,\ \tilde{\nabla}_{e_2} e_2 = 0,\ \tilde{\nabla}_{e_2} e_3 = e_1,\\
&\tilde{\nabla}_{e_3} e_1 = 0,\ \tilde{\nabla}_{e_3} e_2 = 0,\ \tilde{\nabla}_{e_3} e_3 = 0.
\end{aligned} \tag{52}$$

and the curvature tensors, Ricci tensors and scalar curvature with respect to the QSM-connection $\tilde{\nabla}$ are obtained as follows, respectively:

$$\begin{aligned}
&\tilde{R}(e_1, e_2)e_1 = 2e_2,\ \tilde{R}(e_1, e_2)e_2 = -2e_1,\ \tilde{R}(e_1, e_2)e_3 = 0,\\
&\tilde{R}(e_1, e_3)e_1 = 0,\ \tilde{R}(e_1, e_3)e_2 = 0,\ \tilde{R}(e_1, e_3)e_3 = 0,\\
&\tilde{R}(e_2, e_3)e_1 = 0,\ \tilde{R}(e_2, e_3)e_2 = 0,\ \tilde{R}(e_2, e_3)e_3 = 0,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\widetilde{Ric}(e_1, e_1) = -2,\ \widetilde{Ric}(e_1, e_2) = 0,\ \widetilde{Ric}(e_1, e_3) = 0,\\
&\widetilde{Ric}(e_2, e_1) = 0,\ \widetilde{Ric}(e_2, e_2) = -2,\ \widetilde{Ric}(e_2, e_3) = 0,\\
&\widetilde{Ric}(e_3, e_1) = 0,\ \widetilde{Ric}(e_3, e_2) = 0,\ \widetilde{Ric}(e_3, e_3) = 0
\end{aligned} \tag{54}$$

and

$$\tilde{\tau} = -4. \tag{55}$$

Hence, one can easily see that, from (18), (42), (49) and (53), the equation (30) in Theorem 3.6 is verified; from (18), (42), (50) and (54), the equation (37) in Theorem 3.8 is verified. Similarly, obtaining the above equations for dual connection ∇^*, one can easily see that, the equations (31), (38) and (40) are verified, too.

Now, by specially choosing the difference tensor field K, we obtain the equations (48)-(50) in the following Example.

Example 3.11. *If we choose the difference tensor field K as $K(X, Y) = 5\eta(X)\eta(Y)\xi$, then the curvature tensors and Ricci tensors in (49) and (50) are obtained as*

$$\begin{aligned}
&R(e_1, e_2)e_1 = 4e_2,\ R(e_1, e_2)e_2 = -4e_1,\ R(e_1, e_2)e_3 = -10e_3,\\
&R(e_1, e_3)e_1 = -e_3,\ R(e_1, e_3)e_2 = -5e_3,\ R(e_1, e_3)e_3 = e_1 - 5e_2,\\
&R(e_2, e_3)e_1 = 5e_3,\ R(e_2, e_3)e_2 = -e_3,\ R(e_2, e_3)e_3 = e_2 + 5e_1,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&Ric(e_1, e_1) = -3,\ Ric(e_1, e_2) = 5,\ Ric(e_1, e_3) = 0,\\
&Ric(e_2, e_1) = -5,\ Ric(e_2, e_2) = -3,\ Ric(e_2, e_3) = 0,\\
&Ric(e_3, e_1) = 0,\ Ric(e_3, e_2) = 0,\ Ric(e_3, e_3) = 2.
\end{aligned} \tag{57}$$

Here, one can easily see that, from (18), (42), (53)-(55) and (56)-(57), the Theorems 3.6 and 3.8 are verified.

4. Submanifolds with Induced Connection of Sasakian Statistical Manifolds

In this section, firstly we obtain the induced connection with respect to *QSM*-connection of a statistical manifold. Then we give the Gauss and Weingarten formulas for Sasakian statistical manifolds with *QSM*-connection. So, considering these formulas and taking into account Definition 3.3 in [4], we examine some results for submanifolds of Sasakian statistical manifolds with *QSM*-connection.

Let (M, ∇, ∇^*, g) be a statistical manifold and $(N, \nabla^{^N}, {\nabla^*}^{^N}, g_N)$ be a statistical submanifold of M. From [31], Gauss and Weingarten formulas for submanifold N is given by

$$\nabla_X Y = \nabla^N_X Y + h(X, Y), \tag{58}$$

$$\nabla^*_X Y = {\nabla^*_X}^{^N} Y + h^*(X, Y) \tag{59}$$

and

$$\nabla_X V = -A^*_V X + D_X V, \tag{60}$$

$$\nabla^*_X V = -A_V X + D^*_X V, \tag{61}$$

for $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$. Here, induced statistical connections ∇^N and ${\nabla^*}^{^N}$ on N are dual. Also, symmetrical and bilinear h and h^* are called embedding curvature tensors of N in M for ∇ and ∇^*, respectively. On the other hand, bilinear A_V and A^*_V are related to the embedding curvature tensors by

$$g(A_V X, Y) = g(h(X, Y), V) \qquad \text{and} \qquad g(A^*_V X, Y) = g(h^*(X, Y), V). \tag{62}$$

Now, we can give the induced connection with respect to *QSM*-connection of the statistical manifold M:

Let $(M, \tilde{\nabla}, g)$ be a statistical manifold admitting a *QSM*-connection and let N be a submanifold of M. Denote $\nabla^{'}$ and $h^{'}$ as the induced connection and the second fundamental form on the submanifold $(N, \nabla^{^N}, g_N)$ from statistical manifold M with respect to the *QSM*-connection $\tilde{\nabla}$, respectively. In this case, for $X, Y \in \Gamma(TN)$, the Gauss formula is expressed as follows:

$$\tilde{\nabla}_X Y = \nabla^{'}_X Y + h^{'}(X, Y). \tag{63}$$

We know that if the second fundamental form $h^{'}$ vanishes, then the submanifold N is called totally geodesic. Also, if $h^{'}(X, Y) = H^{'} g(X, Y)$ such that $H^{'}$ is the mean curvature vector with respect to $\tilde{\nabla}$, then N is called totally umbilical.

In following proposition, we give the relations between $\nabla^{'}$ and $\nabla^{^N}$:

Proposition 4.1. *Let $(N, \nabla^{^N}, g_N)$ be a submanifold of statistical manifold $(M, \tilde{\nabla}, g)$ such that $\tilde{\nabla}$ is a QSM-connection. Then, for $X, Y \in \Gamma(TN)$, we* *obtain*

$$\nabla^{'}_X Y = \overset{N}{\nabla}_X Y - w(X)\phi Y - K^N(X,Y), \tag{64}$$

and

$$h^{'}(X,Y) = \frac{1}{2}(h(X,Y) + h^*(X,Y)), \tag{65}$$

where $K = \frac{1}{2}(\nabla - \nabla^*)$ *and for* $U \in \Gamma(TN)$, $w(X) = g(X,U)$.

Proof. Let M be a statistical manifold with *QSM*-connection. Then, using (6) and (58) in (11), we obtain

$$\tilde{\nabla}_X Y = \overset{N}{\nabla}_X Y - w(X)\phi Y - K^N(X,Y) + \frac{1}{2}(h(X,Y) + h^*(X,Y)), \tag{66}$$

for $X, Y \in \Gamma(TN)$. Here, if we consider the tangent and normal parts of (66), then we complete the proof. □

On the other hand, we can say that the following equation also satisfies

$$\tilde{\nabla}_X Y = \overset{*N}{\nabla}_X Y - w(X)\phi Y + K^N(X,Y) + \frac{1}{2}(h(X,Y) + h^*(X,Y)), \tag{67}$$

for $X, Y \in \Gamma(TM)$.

From (66) and (67) the following result is clear:

Corollary 4.2. *Let* $(N, \overset{N}{\nabla}, g_N)$ *be a submanifold of statistical manifold* $(M, \tilde{\nabla}, g)$ *such that* $\tilde{\nabla}$ *is a QSM-connection. Then, for* $X, Y \in \Gamma(TN)$, *we get*

$$h^{'}(X,Y) = h^{'^*}(X,Y).$$

Corollary 4.3. *Let* $(N, \overset{N}{\nabla}, g_N)$ *be a submanifold of statistical manifold* $(M, \tilde{\nabla}, g)$ *such that* $\tilde{\nabla}$ *is a QSM-connection. If* N *is totally umbilical with respect to the statistical connections then* N *is totally umbilical with respect to the QSM-connection.*

Proof. Assume that N is totally umbilical with respect to the statistical connections. In this case, since we get $h(X,Y) = Hg(X,Y)$ and $h^*(X,Y) = H^*g(X,Y)$, the proof is obvious from (65). □

Now, we can give Proposition 4.1 for Sasakian statistical manifolds. For this, denote $\breve{\nabla}$ and $\breve{h}$ as the induced connection and the second fundamental form on the submanifold $(N, \overset{N}{\nabla}, g_N)$ from Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ with respect to the *QSM*-connection $\tilde{\nabla}$, respectively. Then the *QSM*-connection $\tilde{\nabla}$ is given as following for $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$

$$\tilde{\nabla}_X Y = \breve{\nabla}_X Y + \breve{h}(X,Y), \tag{68}$$

$$\tilde{\nabla}_X V = -\breve{A}_V X + \breve{D}_X V, \tag{69}$$

where $\breve{D}_X V = \breve{\nabla}^{\perp}_X V$.

Proposition 4.4. *Let $(N, \overset{N}{\nabla}, g_N)$ be a submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ such that $\tilde{\nabla}$ is a QSM-connection. Then, we have*

$$\breve{\nabla}_X Y = \overset{N}{\nabla}_X Y - \eta(X)\phi Y - K^N(X, Y) \tag{70}$$

and

$$\breve{h}(X, Y) = \frac{1}{2}(h(X, Y) + h^*(X, Y)), \tag{71}$$

for $X, Y \in \Gamma(TN)$.

Proof. Assume that M is a Sasakian statistical manifold with *QSM*-connection and let N be a submanifold of M. Then, using (6) and (28), we complete the proof. □

Proposition 4.5. *Let $(N, \overset{N}{\nabla}, g_N)$ be a submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ such that $\tilde{\nabla}$ is a* QSM*-connection. Then, we have*

$$\breve{A}_V X = \frac{1}{2}(A_V X + A^*_V X) \tag{72}$$

and

$$\breve{D}_X V = \frac{1}{2}(D_X V + D^*_X V) - \eta(X)\phi V, \tag{73}$$

for $X \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$.

Proof. Assume that M is a Sasakian statistical manifold with *QSM*-connection and let N be a submanifold of M. Then, using (6) and (60) in (28), we complete the proof. □

5. Anti-Invariant Submanifolds with Induced Connection of Sasakian Statistical Manifolds

Definition 5.1. *Let $(N, \overset{N}{\nabla}, g_N)$ be a submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ such that $\tilde{\nabla}$ is a QSM-connection. If the structure vector field ξ is tangent to N at every point of N and $\phi(TN) \subset TN^\perp$ at every point of N, then N is called an anti-invariant submanifold.*

Theorem 5.2. *Let $(N, \overset{N}{\nabla}, g_N)$ be a submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ such that ξ is tangent to N. If $g(\phi X, V) = g(\breve{h}(X, \xi), V)$, then N is an anti-invariant submanifold with respect to the induced connection from the QSM-connection $\tilde{\nabla}$, for any $X \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$.*

Proof. We know that the equation (28) is valid. Then, from (22) and (28), we have

$$\tilde{\nabla}_X \xi + K(X, \xi) = \phi X + g(\tilde{\nabla}_X \xi + K(X, \xi), \xi)\xi. \tag{74}$$

For $X, \xi \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, we get $g(\tilde{\nabla}_X \xi, V) = g(\phi X, V)$. Here, using (68), we obtain

$$g(\phi X, V) = g(\breve{h}(X, \xi), V). \tag{75}$$

Thus, for $\forall V \in \Gamma(TN^{\perp})$, we have $\breve{h}(X, \xi) = \phi X$ which implies $\phi X \in \Gamma(TN^{\perp})$. Then, the proof completes. □

Now, from the above theorem we can get the following result:

Corollary 5.3. *Let (N, ∇^{N}, g_N) be an anti-invariant submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ satisfying (75) such that ξ is tangent to N. Then N is not totally umbilical when $n \geq 1$.*

Proof. Assume that N is totally umbilical. Then $\breve{h}(X, Y) = g_N(X, Y)H$ for any $X, Y \in \Gamma(TN)$, where H is the mean curvature vector. Since $\breve{h}(\xi, \xi) = 0$, we have $g_N(\xi, \xi)H = 0$. Therefore, N is minimal and hence totally geodesic. Then from (75), for $V \in \Gamma(TN^{\perp})$, we have $\breve{h}(X, \xi) = \phi X = 0$. But this is a contradiction and the proof completes. □

Theorem 5.4. *Let (N, ∇^{N}, g_N) be an anti-invariant submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ such that ξ is tangent to N. If $K(X, \phi Y) = D_X \phi Y$, for any $X, Y \in \Gamma(TN)$, then $\phi(T_X N)$ is parallel with respect to the induced normal connection.*

Proof. Since N is an anti-invariant submanifold, we know that if $Y \in \Gamma(TN)$, then $\phi Y \in \Gamma(TN^{\perp})$. From (69), we have

$$\breve{D}_X \phi Y = \breve{\nabla}^{\perp}_X \phi Y = \tilde{\nabla}_X \phi Y + \breve{A}_{\phi Y} X.$$

Here, using (18) and (28), from (60) we obtain

$$\breve{\nabla}^{\perp}_X \phi Y = -A^{*}_{\phi Y} X + D_X \phi Y - K(X, \phi Y) + \eta(X) Y - \eta(X)\eta(Y)\xi + \breve{A}_{\phi Y} X. \tag{76}$$

For $V \in \Gamma(TN^{\perp})$ and from (76), we get

$$g(\breve{\nabla}^{\perp}_X \phi Y, V) = g(D_X \phi Y - K(X, \phi Y), V).$$

Then, using the last equation and the hypothesis and taking into account Proposition 2.12 in [28], the proof completes. □

Proposition 5.5. *Let (N, ∇^{N}, g_N) be an anti-invariant submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ satisfying (75) such that ξ is tangent to N. Then, N has parallel second fundamental form $\breve{h}$ with respect to the induced connection $\breve{\nabla}$ if and only if the following equation holds*

$$\breve{\nabla}^{\perp}_X \phi Y = \phi(\nabla^N_X Y) - \eta(X)[Y - \eta(Y)\xi - \breve{h}(Y, K^N(X, \xi))] + \phi K^N(X, Y) + \breve{h}(Y, \nabla^N_X \xi), \tag{77}$$

for any $X, Y \in \Gamma(TN)$.

Proof. For any $X, Y, Z \in \Gamma(TN)$, we have

$$(\tilde{\nabla}_X \breve{h})(Y, Z) = \breve{\nabla}^{\perp}_X h(Y, Z) - \breve{h}(\breve{\nabla}_X Y, Z) - \breve{h}(Y, \breve{\nabla}_X Z).$$

Here, taking $Z = \xi$, we find

$$(\tilde{\nabla}_X \breve{h})(Y, \xi) = \breve{\nabla}^{\perp}_X h(Y, \xi) - \breve{h}(\breve{\nabla}_X Y, \xi) - \breve{h}(Y, \breve{\nabla}_X \xi).$$

Using (70) and $\breve{h}(X, \xi) = \phi X$, we get

$$(\tilde{\nabla}_X \breve{h})(Y, \xi) = \breve{\nabla}^{\perp}_X \phi Y - \phi(\nabla^N_X Y - \eta(X)\phi Y - K^N(X, Y)) - \breve{h}(Y, \overset{N}{\nabla}_X \xi - \eta(X)\phi\xi - K^N(X, \xi)), \tag{78}$$

which completes the proof. □

Using (18) in (78), the result follows:

Corollary 5.6. *Let $(N, \overset{N}{\nabla}, g_N)$ be an anti-invariant submanifold of Sasakian statistical manifold $(M, \tilde{\nabla}, g, \phi, \eta, \xi)$ satisfying (75) such that ξ is tangent to N. Then, $\phi(T_X N)$ is parallel with respect to the induced normal connection, if N has parallel second fundamental form $\breve{h}$ with respect to the induced connection $\breve{\nabla}$ and $\phi(\nabla^N_X Y) = \eta(X)[Y - \eta(Y)\xi] - \phi K^N(X, Y) - \breve{h}(Y, K^N(X, \xi)) + \breve{h}(Y, \nabla^N_X \xi)$, for any $X, Y \in \Gamma(TN)$.*

Proof. It is clear from Theorem 5.4 and Proposition 5.5. □

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