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SOME SPECTRAL AND SCATTERING PROPERTIES OF GENERALIZED EIGENPARAMETER DEPENDENT DISCRETE TRANSMISSION STURM-LIOUVILLE EQUATION

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Abstract. In this study, we set a boundary value problem (BVP) consisting of a discrete Sturm-Liouville equation with transmission condition and boundary conditions depending on generalized eigenvalue parameter. Discussing the Jost and scattering solutions of this BVP, we present scattering function and find some properties of this function. Furthermore, we obtain resolvent operator, continuous and discrete spectrum of this problem and we give an valuable asymptotic equation to get the properties of eigenvalues. Finally, we give an example to compare our results with other studies.

1. Introduction

We consider a discrete Sturm-Liouville BVP given by

$$a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n, \quad n\in\mathbb{N}\setminus\{m_0-1,m_0,m_0+1\} \tag{1}$$

with the boundary condition

$$\sum_{k=0}^{p}\left(y_1\gamma_k+y_0\beta_k\right)\lambda^k=0 \tag{2}$$

and the transmission conditions

$$\begin{aligned} y_{m_0+1}&=\zeta_1 y_{m_0-1}\\ y_{m_0+2}&=\zeta_2 y_{m_0-2}, \quad \zeta_1,\zeta_2\in\mathbb{R}, \quad \zeta_1\zeta_2\neq 0. \end{aligned} \tag{3}$$

Here $\lambda=2\cos z$ is an eigenvalue parameter, γ_k, β_k, $k=0,1,\dots,p$ are real numbers such that $\gamma_m\beta_n-\gamma_n\beta_m\neq 0$; $m,n\in\mathbb{N}\cup\mathbb{N}_0$, for $n<m\leq p$ and

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$\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences satisfying the following condition

$$\sum_{n\in\mathbb{N}} n\left(|1-a_n|+|b_n|\right)<\infty. \tag{4}$$

Throughout this work, we will assume that $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$. Because of the condition (3), BVP (1)-(3) is a kind of discontinuous boundary value problem. Discontinuous boundary value problems have great applications in various branches of natural sciences especially mathematical physics and quantum mechanics. Sturm-Liouville problems both classical and discrete cases with spectral boundary conditions are also seen in various fields of sciences. Spectral and scattering properties for these problems have been investigated by many scholars for years (see [9, 10, 7, 30, 31, 20, 29, 18, 22, 16, 13, 17, 25, 3] and the references there in). To study discontinuous boundary value problems of such problems, some conditions under titles interface conditions, point interactions, impulsive conditions, jump conditions and transmission conditions are imposed in the discontinuous point or points. Among these discontinuous, transmission (or impulsive) difference equations have been discussed in many references and a large number of authors surveyed the spectral theory of these equations also in recent years scattering properties of such equations have received a lot of attention [21, 1, 36, 6, 4, 15, 34, 32, 2, 28, 33, 5, 14, 8, 23]. Because problems with discontinuous have also been great considered in many fields of real-world problems such as earthquakes, a mass-spring-damper system with short-term perturbations, finance and pharmacotherapy [26, 35, 11, 37].

In this study, we initiate some spectral and scattering properties of the BVP (1)-(3) which consist scattering solutions, Jost solution, scattering function and its properties, resolvent operator, discrete spectrum, continuous spectrum and a useful asymptotic formula. Differently other studies in literature, the specific feature of this paper is the presence of the spectral parameter not only in the difference equation but also it is in the boundary condition at generalized polynomial form at the same time with transmission condition together. We combine impulsive condition and boundary conditions depending on generalized eigenvalue parameter with discrete Sturm-Liouville equation. This investigation is the more general form of the studies [6, 4, 34, 33]. On the other hand, [24] is a study about the spectral analysis of BVP with this kind of boundary conditions in continuous case, but it is the first which presents scattering properties of a boundary value problem consisting of boundary conditions depending on generalized spectral parameter also it is the first in discontinuity case on the aspect of spectral and scattering analysis.

The outline of this paper is as follows: In Section 2, we give some basic notations and definitions and we get the Jost solution of BVP (1)-(3). Also, we prove an auxiliary lemma and theorem to use next sections. In Section 3, we find the scattering and Jost functions of (1)-(3). In Section 4, we obtain resolvent operator, continuous spectrum and discrete spectrum of the problem.

Also, we give an important asymptotic equation in this section. Finally, we conclude our paper with a useful example.

2. Preliminaries

Let us define two semi-strips

$$D_0 := \left\{ z \in \mathbb{C} : \operatorname{Im} z > 0, -\frac{\pi}{2} \leq \operatorname{Re} z \leq \frac{3\pi}{2} \right\}, \quad D := D_0 \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right].$$

Throughout the paper, we assume that $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (1) for $z \in D$, $\lambda = 2\cos z$ and $n \in \mathbb{N} \cup \{0\}$, satisfying the initial conditions

$$P_0(z) = 0, \qquad P_1(z) = 1$$

and

$$Q_0(z) = \frac{1}{a_0}, \qquad Q_1(z) = 0,$$

respectively [12]. For each $n \geq 0$, $P_n(z)$ is a polynomial of degree $(n-1)$ and $Q_n(z)$ is a polynomial of degree $(n-2)$.

Definition 2.1. *The Wronskian of two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of (1) is defined by*

$$W[y, u] = a_n [y_n(z) u_{n+1}(z) - y_{n+1}(z) u_n(z)]$$

for $n \in \mathbb{N} \cup \{0\}$.

It is easy to see that the Wronskian is independently of n. Moreover, $P_n(z)$ and $Q_n(z)$ are linear independent solutions of (1), because $W[P, Q] = -1$ for all $z \in \mathbb{C}$ and these solutions are entire functions with respect to z. Note that, we can write the other solution $\psi_n(z)$ of (1) as a linear combination of fundamental solutions. We can express this solution as

$$\psi_n(z) = -P_n(z) \sum_{k=0}^{p} \beta_k \lambda^k + a_0 Q_n(z) \sum_{k=0}^{p} \gamma_k \lambda^k, \quad n = 0, 1, ..., m_0 - 1. \tag{5}$$

On the other hand, we show by $e(z) = \{e_n(z)\}$, $n = m_0 + 1, m_0 + 2, ...$ the bounded solution of (1) satisfying the condition

$$\lim_{n \to \infty} e^{-inz} e_n(z) = 1, \quad z \in D.$$

The solution $e(z)$ is represented by

$$e_n(z) = \rho_n e^{inz} \left(1 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right), \quad n = m_0 + 1, m_0 + 2, ...$$

in [26], where ρ_n and A_{nm} are expressed in terms of the sequences $\{a_n\}$ and $\{b_n\}$ as

$$\rho_n := \prod_{k=n}^{\infty} a_k^{-1},$$

$$A_{n1} := -\sum_{k=n+1}^{\infty} b_k,$$

$$A_{n2} := \sum_{k=n+1}^{\infty} \left\{ 1 - a_k^2 + b_k \sum_{p=k+1}^{\infty} b_p \right\},$$

$$A_{n,m+2} := A_{n+1,m} + \sum_{k=n+1}^{\infty} \left\{ \left(1 - a_k^2\right) A_{k+1,m} - b_k A_{k,m+1} \right\}$$

for $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. The function $e_n(z)$ is analytic according to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ and 2π periodic.

The equation (1) also has an unbounded solution. We will show this unbounded solution by $\breve{e}_n(z)$ for $n = m_0 + 1, m_0 + 2, ...$ satisfying $\lim_{n\to\infty} e^{inz} \breve{e}_n(z) = 1$, $z \in \overline{\mathbb{C}}_+$. It is clear from the definitions of $e_n(z)$, $\breve{e}_n(z)$ and the definition of Wronskian that

$$W\left[e_n(z), \breve{e}_n(z)\right] = -2i \sin z \tag{6}$$

for $n = m_0 + 1, m_0 + 2, ...$ and $z \in D \backslash \{0, \pi\}$.

Now, we define the following solution of BVP

$$E_n(z) := \begin{cases} \alpha(z) P_n(z) + c(z) Q_n(z), & n = 0, 1, 2, ..., m_0 - 1 \\ e_n(z), & n = m_0 + 1, m_0 + 2, ... \end{cases} \tag{7}$$

for $z \in D$. If we apply the transmission conditions (3) to this solution, we can write

$$\frac{e_{m_0+1}(z)}{\zeta_1} = \alpha(z) P_{m_0-1}(z) + c(z) Q_{m_0-1}(z),$$

$$\frac{e_{m_0+2}(z)}{\zeta_2} = \alpha(z) P_{m_0-2}(z) + c(z) Q_{m_0-2}(z). \tag{8}$$

By using (8), we obtain the coefficients $\alpha(z)$ and $c(z)$ as

$$\alpha(z) = -\frac{a_{m_0-2}}{\zeta_1 \zeta_2} \left\{ \zeta_1 e_{m_0+2}(z) Q_{m_0-1}(z) - \zeta_2 e_{m_0+1}(z) Q_{m_0-2}(z) \right\} \tag{9}$$

and

$$c(z) = \frac{a_{m_0-2}}{\zeta_1 \zeta_2} \left\{ \zeta_1 e_{m_0+2}(z) P_{m_0-1}(z) - \zeta_2 e_{m_0+1}(z) P_{m_0-2}(z) \right\} \tag{10}$$

for $z \in D$. The function $E(z) = \{E_n(z)\}$ is called the Jost solution of BVP (1)-(3).

Corollary 2.2. *Since $P_n(z) = P_n(-z)$ and $Q_n(z) = Q_n(-z)$ for $z \in D$, then we obtain that*

$$\alpha(-z) = \overline{\alpha(z)}, \quad c(-z) = \overline{c(z)}.$$

It is evident from the definitions of Wronskian and $e_n(z)$ that

$$W\left[e_n(z), e_n(-z)\right] = -2i \sin z$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. Next, we define the another solution $F(z) = \{F_n(z)\}$ of (1)-(3) by

$$F_n(z) := \begin{cases} \psi_n(z), & n = 0, 1, 2, ..., m_0 - 1 \\ d(z)e_n(z) + f(z)e_n(-z), & n = m_0 + 1, m_0 + 2, ... \end{cases} \tag{11}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. Using (3), we get that

$$d(z) = -\frac{a_{m_0+1}}{2i \sin z} \left\{\zeta_1 e_{m_0+2}(-z)\psi_{m_0-1}(z) - \zeta_2 e_{m_0+1}(-z)\psi_{m_0-2}(z)\right\} \tag{12}$$

and

$$f(z) = \frac{a_{m_0+1}}{2i \sin z} \left\{\zeta_1 e_{m_0+2}(z)\psi_{m_0-1}(z) - \zeta_2 e_{m_0+1}(z)\psi_{m_0-2}(z)\right\} \tag{13}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Corollary 2.3. *The coefficients $d(z)$ and $f(z)$ satisfy the following relationship*

$$f(z) = d(-z) = \overline{d(z)}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Theorem 2.4. *The coefficient $f(z)$ is not equal to zero for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.*

Proof. Let us suppose that there exists an element $z_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$ such that $f(z_0) = 0$. According to Corollary 2.3, we get $f(z_0) = d(z_0) = 0$. In this instance, it gives us $F_n(z_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, but this is a trivial solution of (1)-(3). It says that there is a contradiction, so $f(z) \neq 0$ for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. □

Lemma 2.5. *The Wronskian of the solutions $E_n(z)$ and $F_n(z)$ is given by*

$$W[E_n(z), F_n(z)] = \begin{cases} \dfrac{(a_{m_0-2})2i \sin z}{(a_{m_0+1})\zeta_1\zeta_2} f(z), & n = 0, 1, 2, ..., m_0 - 1 \\ \\ -2i \sin z f(z), & n = m_0 + 1, m_0 + 2, ... \end{cases}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Proof. By using the definition of Wronskian for $n = 0, 1, 2, ..., m_0 - 1$, we can write

$$\begin{aligned} W[E_n(z), F_n(z)] &= a_n \{E_n(z)F_{n+1}(z) - F_n(z)E_{n+1}(z)\} \\ &= a_{m_0+1} \{E_{m_0+1}(z)F_{m_0+2}(z) - F_{m_0+1}(z)E_{m_0+2}(z)\} \\ &= f(z)a_{m_0+1} \{e_{m_0+1}(z)e_{m_0+2}(-z) - e_{m_0+1}(-z)e_{m_0+2}(z)\} . \end{aligned}$$

Since $W[e_n(z), e_n(-z)] = -2i \sin z$, we get

$$W[E_n(z), F_n(z)] = -2i \sin z f(z).$$

Similarly, if we apply the definitions $E_n(z)$, $\alpha(z)$, $c(z)$, $F_n(z)$ and $f(z)$ given in (7), (9), (10), (11) and (13), respectively, we find

$$W[E_n(z), F_n(z)] = \frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i \sin z}{\zeta_1 \zeta_2} f(z)$$

for $n = 0, 1, 2, ..., m_0 - 1$. This completes the proof. □

3. Jost Solution and Scattering Function

In this section, we give the Jost function, scattering function and main properties of scattering function. Now, we will define the Jost function J of (1)-(3) by applying the boundary condition (2) to the Jost solution $E_n(z)$ of (1)-(3) and we write

$$\begin{aligned} J(z) &:= \sum_{k=0}^{p} (E_1 \gamma_k + E_0 \beta_k) \lambda^k \\ &= \sum_{k=0}^{p} \left(\alpha(z)\gamma_k + \frac{c(z)}{a_0}\beta_k\right) \lambda^k. \end{aligned}$$

It is obvious that

$$J(-z) = \sum_{k=0}^{p} \left(\alpha(-z)\gamma_k + \frac{c(-z)}{a_0}\beta_k\right) \lambda^k.$$

Furthermore, similarly to Sturm-Liouville equation, the function $J(z)$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$.

Lemma 3.1. *The Jost function J can be expressed as a multiple of $f(z)$*

$$J(z) = -\frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i \sin z}{a_0 \zeta_1 \zeta_2} f(z)$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Proof. It follows from (5) and (13) that

$$\begin{aligned} f(z) &= \frac{a_{m_0+1}}{2i\sin z}\frac{\zeta_1\zeta_2}{a_{m_0-2}}\left\{-c(z)\sum_{k=0}^{p}\beta_k\lambda^k - a_0\alpha(z)\sum_{k=0}^{p}\gamma_k\lambda^k\right\} \\ &= -\frac{a_{m_0+1}}{a_{m_0-2}}\frac{\zeta_1\zeta_2}{2i\sin z}\left\{\sum_{k=0}^{p}c(z)\beta_k\lambda^k + \sum_{k=0}^{p}a_0\alpha(z)\gamma_k\lambda^k\right\} \\ &= -\frac{a_{m_0+1}}{a_{m_0-2}}\frac{\zeta_1\zeta_2}{2i\sin z}a_0 J(z) \qquad (14) \end{aligned}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. It completes the proof of Lemma 3.1. □

Definition 3.2. *The function*

$$S(z) := \frac{\overline{J(z)}}{J(z)}, \qquad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$$

is called the scattering function of BVP.

It is clear from Lemma 3.1 and Definition 3.2 that scattering function can be also written in the form of the coefficient $f(z)$ as

$$S(z) = \frac{J(-z)}{J(z)} = -\frac{f(-z)}{f(z)}, \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}. \qquad (15)$$

Theorem 3.3. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, *the scattering function* $S(z)$ *satisfies*

$$S(-z) = S^{-1}(z) = \overline{S(z)}, \quad |\, S(z) \,| = 1.$$

Proof. From the Definition 3.2, we have

$$S(-z) = \frac{J(z)}{J(-z)}$$

and

$$\overline{S(z)} = \frac{\overline{J(-z)}}{\overline{J(z)}}.$$

Since $\overline{d(-z)} = d(z)$ and $d(-z) = \overline{d(z)}$, we obtain

$$S(-z) = S^{-1}(z) = \overline{S(z)}.$$

Also, since $|\, S(z) \,|^2 = \overline{S(z)} S(z)$, the equation (15) gives us

$$|\, S(z) \,| = \frac{J(z)}{J(-z)}\frac{J(-z)}{J(z)} = 1$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. It completes the proof. □

4. Resolvent Operator, Continuous and Discrete Spectrum of BVP

In this part, we try to investigate resolvent operator, continuous spectrum and discrete spectrum of related BVP. Furthermore, we present an important asymptotic equation for $f(z)$. For all $z \in D$, we will define the following solution $G(z) = \{G_n(z)\}$

$$G_n(z) := \begin{cases} \psi_n(z), & n = 0, 1, 2, ..., m_0 - 1 \\ k(z)e_n(z) + q(z)\breve{e}_n(z), & n = m_0 + 1, m_0 + 2, ... \end{cases} \tag{16}$$

By using transmission conditions (3) for $G(z)$, we find the coefficients $k(z)$ and $q(z)$

$$k(z) = -\frac{a_{m_0+1}}{2i \sin z} \{\zeta_1 \psi_{m_0-1}(z)\breve{e}_{m_0+2}(z) - \zeta_2 \psi_{m_0-2}(z)\breve{e}_{m_0+1}(z)\}$$

and

$$q(z) = \frac{a_{m_0+1}}{2i \sin z} \{\zeta_1 \psi_{m_0-1}(z)e_{m_0+2}(z) - \zeta_2 \psi_{m_0-2}(z)e_{m_0+1}(z)\}$$

for all $z \in D$. Note that, for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$

$$f(z) = q(z).$$

Also, using (7) and (16), we get

$$W[E_n(z), G_n(z)] = \begin{cases} \dfrac{(a_{m_0-2})2i \sin z}{(a_{m_0+1})\zeta_1\zeta_2} f(z), & n = 0, 1, 2, ..., m_0 - 1 \\ \\ -2i \sin z f(z), & n = m_0 + 1, m_0 + 2, ... \end{cases}$$

for $z \in D$.

Corollary 4.1. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, *it is clear that*

$$W[E(z), F(z)] = W[E(z), G(z)].$$

Theorem 4.2. *The resolvent operator of BVP is defined by*

$$R_\lambda g_n := \sum_{k=1}^{\infty} R_{nk}(z)g_k, \quad g_k \in l_2(\mathbb{N}),$$

where

$$R_{nk}(z) = \begin{cases} -\dfrac{G_k(z)E_n(z)}{W[E_k, G_k]}, & k \leq n \\ \\ -\dfrac{G_n(z)E_k(z)}{W[E_k, G_k]}, & k > n \end{cases}$$

is the Green function of (1)-(3) *for* $z \in D\backslash\{0, \pi\}$, $f(z) \neq 0$ *and* $k, n \neq m_0$.

Proof. We need to solve the following equation to find the resolvent operator of BVP

$$\nabla\left(a_n \Delta y_n\right)+h_n y_n-\lambda y_n=g(t, z), \quad g_n \in l_2(\mathbb{N}), \tag{17}$$

where $h_n = a_{n-1}+a_n+b_n$, ∇ is the backward difference operator and Δ is the forward difference operator defined by $\nabla y_n = y_n - y_{n-1}$ and $\Delta y_n = y_{n+1} - y_n$, respectively. Since $E_n(z)$ and $G_n(z)$ are fundamental solutions of (1)-(3), we can write the general solution of (17) as

$$y_n(z) = s_n E_n(z) + t_n G_n(z), \tag{18}$$

where s_n, t_n are coefficients and they are different from zero. Using the method of variation of parameters for $k \neq m_0$, we obtain s_n and t_n by

$$s_n = -\sum_{k=1}^{n} \frac{G_k g_k}{W[E,G]}, \tag{19}$$

$$t_n = -\sum_{k=n+1}^{\infty} \frac{E_k g_k}{W[E,G]}. \tag{20}$$

It follows from (18), (19) and (20) that the Green function $R_{nk}(z)$ of (1)-(3) is defined in Theorem 4.2. Also, we obtain the resolvent operator of BVP by the help of the Green function. □

Now, we can give the set of eigenvalues of (1)-(3) by using the definition of eigenvalues [27] and Theorem 4.2 as

$$\sigma_d := \{\lambda \in \mathbb{C} : \lambda = 2\cos z, z \in D_0, f(z) = 0\}.$$

Theorem 4.3. *Under the condition* (4), $f(z)$ *satisfies the following asymptotic equation for* $z \in D$

$$f(z) = e^{iz(5-p)}\left[A + o(1)\right], \quad |z| \to \infty, \quad A \neq 0,$$

where

$$A = -\frac{\zeta_1 a_{m_0+1} \beta_p \rho_{m_0+1}}{a_1 a_2 \ldots a_{m_0-3}} \left\{ \frac{a_{m_0+1}}{a_{m_0-2}} - \frac{\zeta_2}{\zeta_1} \right\}. \tag{21}$$

Proof. As mentioned earlier, $P_n(z)$ is a polynomial of degree $(n-1)$ and $Q_n(z)$ is a polynomial of degree $(n-2)$ with regard to λ. Hence, by using (5), we can obtain

$$\lim_{|z|\to\infty} \left\{ \psi_n(z) e^{iz(n-1+p)} \right\} = -\frac{B_p}{a_1 \ldots a_{n-1}} \tag{22}$$

and

$$\lim_{|z|\to\infty} \left\{ e_n(z) e^{-inz} \right\} = \rho_n \tag{23}$$

for $z \in D$ and $n = 0, 1, 2, ..., m_0 - 1$, where $\rho_n := \left(\prod_{k=n}^{\infty} a_k\right)^{-1}$. It follows from (13), (22) and (23) that

$$f(z) = \frac{a_{m_0+1}}{2i \sin z}[\zeta_1 \psi_{m_0-1}(z) e^{iz(m_0-2+p)} e_{m_0+2}(z) e^{-iz(m_0+2)z} e^{iz(4-p)}$$
$$-\zeta_2 \psi_{m_0-2}(z) e^{iz(m_0-3+p)} e_{m_0+1}(z) e^{-iz(m_0+1)z} e^{iz(4-p)}]$$

and if the last equality is written in limit form and necessary adjustments are made, we find

$$\lim_{|z|\to\infty} \left\{ f(z) e^{-iz(5-p)} \right\} = -\frac{\zeta_1 a_{m_0+1} \beta_p \rho_{m_0+1}}{a_1 a_2 ... a_{m_0-3}} \left\{ \frac{a_{m_0+1}}{a_{m_0-2}} - \frac{\zeta_2}{\zeta_1} \right\} \lim_{|z|\to\infty} \frac{1}{e^{2iz} - 1}.$$

Last equation gives that

$$\lim_{|z|\to\infty} \left\{ f(z) e^{-iz(5-p)} \right\} = A,$$

where A is given in equation (21) for all $z \in D$. It completes the proof of Theorem. □

Theorem 4.4. *Assume* (4). *Then the continuous spectrum of the operator* L *generated by* (1)-(3) *is* $[-2, 2]$, *i.e.,* $\sigma_c(L) = [-2, 2]$.

Proof. Let L_1 and L_2 denote difference operators generated in $l_2(\mathbb{N})$ by the following difference equations

$$(l_1 y) := y_{n-1} + y_{n+1}, \quad n \in \mathbb{N} \backslash \{m_0 - 1, m_0 + 1\}$$

and

$$(l_2 y) := (a_{n-1} - 1) y_{n-1} + b_n y_n + (a_n - 1) y_{n+1}, \quad n \in \mathbb{N} \backslash \{m_0 - 1, m_0, m_0 + 1\},$$

respectively, with the boundary condition (2). It is evident that $L = L_1 + L_2$ and L_2 is a compact in $l_2(\mathbb{N})$ under the assumption (4) [27]. We can also write the operator L_1 by the sum of two operators L_3 and L_4, i.e., $L_1 = L_3 + L_4$, where L_3 is a self-adjoint operator with $\sigma_c(L_3) = [-2, 2]$ and defined by the difference expression l_1 and the boundary condition (2). On the other hand, L_4 is a finite dimensional operator in $l_2(\mathbb{N})$. Since L_4 is a finite dimensional operator in $l_2(\mathbb{N})$, it is also compact operator. It gives that the sum of two compact operators $L_2 + L_4$ is a compact operator, too. It follows from that $L = L_3 + L_4 + L_2$ and by using the Weyl Theorem [19] of a compact perturbation, we get $\sigma_c(L_3) = \sigma_c(L) = [-2, 2]$. □

5. An Example

In this section, we will conclude the paper by giving a special case of (1)-(3). This special case introduces our example and it illustrates our theoretical findings.

Let us consider the following discrete transmission problem

$$y_{n-1} + y_{n+1} = 2 \cos z y_n, \quad n \in \mathbb{N} \backslash \{2, 3, 4\}$$

$$\sum_{k=0}^{p}\left(y_1\gamma_k+y_0\beta_k\right)\lambda^k=0 \tag{24}$$

$$y_4=\zeta_1 y_2$$

$$y_5=\zeta_2 y_1,$$

where ζ_1, ζ_2, γ_k, β_k, $k=0,1,...,p$ are real numbers, $\zeta_1\zeta_2\neq 0$. It can be easily seen that (24) is a special case of (1)-(3), i.e., $m_0=3$, $a(t)\equiv 1$ and $b(t)\equiv 0$. It is obvious that $e_n(z)=e^{inz}$ and the fundamental solutions $P_n(z)$ and $Q_n(z)$ of (1)-(3) have the following values for $n=0,1,2$

$$P_0(z)=0,\quad P_1(z)=1,\quad P_2(z)=\lambda$$

$$Q_0(z)=\frac{1}{a_0},\quad Q_1(z)=0,\quad Q_2(z)=-\frac{1}{a_0}.$$

Moreover, by using (7) and (13), we find $f(z)$, Jost solution and scattering function of (24) as

(25)

$$f(z)=\frac{a_4}{2i\sin z}\left\{\zeta_1\left(-\sum_{k=0}^{p}\beta_k\lambda^{k+1}-\sum_{k=0}^{p}\gamma_k\lambda^k\right)e^{5iz}+\zeta_2\left(\sum_{k=0}^{p}\beta_k\lambda^k\right)e^{4iz}\right\},$$

$$E_n(z)=\begin{cases}\alpha(z)P_n(z)+c(z)Q_n(z), & n=0,1,2\\ e^{inz}, & n=4,5,...\end{cases}$$

and

$$S(z)=e^{-8iz}\left[\frac{\zeta_1\psi_2(z)e^{-iz}-\zeta_2\psi_1(z)}{\zeta_1\psi_2(z)e^{iz}-\zeta_2\psi_1(z)}\right],$$

respectively. Also continuous spectrum of the problem (24) is $[-2,2]$ from Theorem 4.4. To get the eigenvalues of (24), it is necessary for us to find the zeros of $f(z)$ for $z\in D_0$. Because from the definition of eigenvalues, we can write

$$\sigma_d:=\{\lambda\in\mathbb{C}:\lambda=2\cos z, z\in D_0, f(z)=0\}$$

for this problem, where $f(z)$ is given by (25). If $f(z)=0$, then we write

$$\lambda e^{iz}+e^{iz}\frac{\sum_{k=0}^{p}\gamma_k\lambda^k}{\sum_{k=0}^{p}\beta_k\lambda^k}=\frac{\zeta_2}{\zeta_1} \tag{26}$$

for $z\in D_0$. We can examine the following case for the equation (26). Let assume $p=1$ and $\zeta_2=B\zeta_1$, $B\in\mathbb{R}$ in (26). By using (26), we get the same results as in [4].

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