Honam Mathematical J. **45** (2023), No. 3, pp. 380–396
https://doi.org/10.5831/HMJ.2023.45.3.380

ON ADJOINT CURVES OF FRAMED CURVES AND SOME RULED SURFACES

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Abstract. In this study, we introduce the adjoint curves of framed curves. We examine some characterizations of adjoint curve of a framed curve. In addition, we give the conditions for framed curves and adjoint curves to be Bertrand and Mannheim curves. Then, we introduce adjoint curves of Frenet-type framed curves and give ruled surfaces related to adjoint curves. Finally, we create normal and binormal surfaces of the framed adjoint curves and obtain some characterizations of these surfaces and we support by the results with figures.

1. Introduction

Creating a new curve with Frenet vectors of a curve is an interesting topic in differential geometry. These curves, called a special curve, have a wide place in the literature. Some interesting ones of these curves are involute-evolute mates, Bertrand mates, Mannheim pairs, and integral curves [1, 2, 5, 11, 13, 15]. Adjoint curves are integral curves created with the binormal vector of the curve. An adjoint curve (or conjugate mate) is defined by the integral of binormal vector with any parameter s of a curve [16].

On the other hand, framed curves are a type of curve used to study a curve that can have singular points. Framed curves have an important place in smooth theory both because of their contributions to singularity theory and because of their interesting results. A framed curve is a smooth curve that can be defined by a moving frame with singular points [6]. In [3], the existence and uniqueness conditions are given for framed curves. In [19]-[21], [23], framed rectifying curves and framed normal curves are defined. Also, Bertrand and Mannheim curves of framed curves which are generalizations of regular Bertrand and Mannheim curves are given [9]. On the other hand, many ruled surfaces have been examined so that the base curve is a framed

Received September 12, 2021. Revised March 14, 2023. Accepted March 21, 2023.
2020 Mathematics Subject Classification. 53A04, 58K05, 58K30
Key words and phrases. framed curves, framed adjoint curves, Bertrand curves, Mannheim curves, ruled surfaces.
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curve [4, 7, 8, 10, 14, 22]. In [18], framed curves in Lie groups are introduced and some physical applications are given.

In this paper, we introduce the adjoint curves of framed curves. We examine some characterizations of adjoint curve of a framed curve. In addition, we give the conditions for framed curves and adjoint curves to be Bertrand and Mannheim curves with general frame. Then, we introduce adjoint curves of Frenet-type framed curves and give ruled surfaces related to adjoint curves. Finally, we create normal and binormal surfaces of the framed adjoint curves and obtain some characterizations of these surfaces. We examine the types of singularity and cross-cap conditions of normal and binormal surfaces of adjoint curves.

2. Framed Curves

A framed curve is defined as a smooth curve with singular points in Euclidean space [6]. Now consider any $\gamma : I \to \mathbb{R}^3$ curve that can have a singular point. The set

$$\Delta_\mu = \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, i, j = 1, 2\}$$

has three-dimensional a smooth manifold structure. This structure is defined as the "Stiefel manifold" in manifold theory. Let $\mu = (\mu_1, \mu_2) \in \Delta_\mu$. A unit vector is defined by $\nu = \mu_1 \times \mu_2$.

The definition of the framed curve can be given:

Definition 2.1. *If $\langle \gamma^{'}(s), \mu_i(s) \rangle = 0$ for each $s \in I$ and $i = 1, 2$, the structure $(\gamma, \mu) : I \to \mathbb{R}^3 \times \Delta_\mu$ is called a framed curve. Moreover, $\gamma : I \to \mathbb{R}^3$ is called a framed base curve if there exists $\mu : I \to \Delta_\mu$ such that (γ, μ) is a framed curve* [6].

Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\nu = \mu_1 \times \mu_2$. The derivative formula is given by

$$\begin{pmatrix} \nu^{'}(s) \\ \mu_1^{'}(s) \\ \mu_2^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \\ \mu_2(s) \end{pmatrix},$$

where $l(s) = \langle \mu_1^{'}(s), \mu_2(s) \rangle$, $m(s) = \langle \nu(s), \mu_1^{'}(s) \rangle$ and $n(s) = \langle \nu(s), \mu_2^{'}(s) \rangle$. In addition, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that:

$$\gamma^{'}(s) = \alpha(s)\nu(s).$$

If s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$. Hence, the α function is important to characterize singular points. Consequently, $(l(s), m(s), n(s), \alpha(s))$ are called the framed curvature of γ.

In general, the moving frame of a framed curve does not have geometric meaning. However, we can consider a moving frame with geometric meaning under a certain condition.

Definition 2.2. *$\gamma : I \to \mathbb{R}^3$ is called a Frenet-type framed base curve if there exist a regular spherical curve $\mathcal{T} : I \to S^2$ and a function $\alpha : I \to \mathbb{R}$ such that $\gamma^{'}(s) = \alpha(s)\mathcal{T}(s)$ for all $s \in I$. Then we call $\mathcal{T}(s)$ a unit tangent vector and $\alpha(s)$ a speed function of $\gamma(s)$, where $\mathcal{N}(s) = \dfrac{\mathcal{T}^{'}(s)}{\|\mathcal{T}'(s)\|}, \mathcal{B}(s) = \mathcal{T}(s) \times \mathcal{N}(s)$* [7].

Then we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \mathcal{T}^{'}(s) \\ \mathcal{N}^{'}(s) \\ \mathcal{B}^{'}(s)) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}(s) \\ \mathcal{N}(s) \\ \mathcal{B}(s) \end{pmatrix},$$

where

$$\kappa(s) = \|\mathcal{T}^{'}(s)\|, \quad \tau(s) = \frac{det(\mathcal{T}(s), \mathcal{T}^{'}(s), \mathcal{T}^{''}(s))}{\|\mathcal{T}'(s)\|^2}.$$

We can easily check that $(\gamma, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta_2$ is a framed curve, so that we can apply the theory of framed curves.

3. Adjoint Curves of Framed Curves

Definition 3.1. *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ be a framed curve. The adjoint curve $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ of the (γ, μ_1, μ_2) framed curve is defined by*

$$\overline{\gamma}(s) = \int_{s_0}^{s} \beta(s)\mu_2(s)ds, \tag{3.1}$$

where $\beta(s)$ is a smooth function.

Proposition 3.2. *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ be a framed curve. If $\langle \mu_2(s), \overline{\mu}_1(s) \rangle = 0$ and $\langle \mu_2(s), \overline{\mu}_2(s) \rangle = 0$ for each $s \in I$, the $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ adjoint curve is also the framed curve.*

Proof. Definition 2.1 and equation (3.1) are combined to show the desired. □

Assume that $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ and $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ are framed curves. Let $\{\nu, \mu_1, \mu_2\}$ and (l, m, n, α) be the moving frame and framed curvatures of $\gamma(s)$, respectively. In addition, let $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ and $(\bar{l}, \overline{m}, \overline{n}, \overline{\alpha})$ be the moving frame and framed curvatures of $\overline{\gamma}(s)$, respectively:

Theorem 3.3. *Let* $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ *be a framed curve and* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_\mu$ *adjoint curve. Then, we have*

$$\begin{aligned}
\overline{\nu}(s) &= \mu_2(s), \\
\overline{\mu}_1(s) &= \cos\varphi(s)\nu(s) - \sin\varphi(s)\mu_1(s), \\
\overline{\mu}_2(s) &= \sin\varphi(s)\nu(s) + \cos\varphi(s)\mu_1(s), \\
\bar{l}(s) &= -\varphi^{'}(s) - m(s), \\
\overline{m}(s) &= -n(s)\cos\varphi(s) - l(s)\sin\varphi(s), \\
\overline{n}(s) &= -n(s)\sin\varphi(s) + l(s)\cos\varphi(s), \\
\overline{\alpha}(s) &= \beta(s).
\end{aligned}$$

Proof. By differentiating equation (3.1), we get $\overline{\gamma}^{'}(s) = \mu_2(s)$. By taking the norm of both sides and using $\overline{\nu}$ and μ_2 as unit vectors, we can write $\overline{\alpha}(s) = \beta(s)$. Then, we get

$$\overline{\nu}(s) = \mu_2(s). \tag{3.2}$$

From equation (3.2), we get

$$\begin{pmatrix} \overline{\mu}_1(s) \\ \overline{\mu}_2(s) \end{pmatrix} = \begin{pmatrix} \cos\varphi(s) & -sin\varphi(s) \\ sin\varphi(s) & \cos\varphi(s) \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \end{pmatrix}, \tag{3.3}$$

where $\varphi(s) : I \to \mathbb{R}$ is a smooth function. By differentiating of equation (3.3), we get

$$\begin{aligned}
\overline{\mu}_1^{'}(s) &= (-\varphi^{'}(s)\sin\varphi(s) - m(s)\sin\varphi(s))\nu(s) \\
&+ (-m(s)\cos\varphi(s) - \varphi^{'}(s)\cos\varphi(s))\mu_1(s) \\
&+ (-n(s)\cos\varphi(s) - l(s)\sin\varphi(s))\mu_2(s),
\end{aligned}$$

$$\begin{aligned}
\overline{\mu}_2^{'}(s) &= (\varphi^{'}(s)\cos\varphi(s) + m(s)\cos\varphi(s))\nu(s) \\
&+ (-m(s)\sin\varphi(s) - \varphi^{'}(s)\sin\varphi(s))\mu_1(s) \\
&+ (-n(s)\sin\varphi(s) + l(s)\cos\varphi(s))\mu_2(s).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\bar{l}(s) &= \langle \overline{\mu}_1^{'}(s), \overline{\mu}_2(s) \rangle, \\
&= (-\varphi^{'}(s)\sin\varphi(s) - m(s)\sin\varphi(s))\sin\varphi(s) \\
&+ (-m(s)\cos\varphi(s) - \varphi^{'}(s)\cos\varphi(s))\cos\varphi(s), \\
&= -\varphi^{'}(s) - m(s),
\end{aligned}$$

$$\begin{aligned}
\overline{m}(s) &= \langle \overline{\mu}_1^{'}(s), \overline{\nu}(s) \rangle = \langle \overline{\mu}_1^{'}(s), \mu_2(s) \rangle, \\
&= -n(s)\cos\varphi(s) - l(s)\sin\varphi(s),
\end{aligned}$$

$$\begin{aligned}
\overline{n}(s) &= \langle \overline{\mu}_2^{'}(s), \overline{\nu}(s) \rangle = \langle \overline{\mu}_2^{'}(s), \mu_2(s) \rangle, \\
&= -n(s)\sin\varphi(s) + l(s)\cos\varphi(s).
\end{aligned}$$

□

Theorem 3.4. *Let $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ be an adjoint curve of framed curve (γ, μ_1, μ_2). Then (γ, μ_1, μ_2) and $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ are Bertrand curves if and only if*

$$\varphi = \left(\frac{2k+1}{2}\right)\pi, \quad \alpha(s) + \lambda m(s) = 0, \quad l(s) = \frac{\beta(s)}{\lambda},$$

where k is odd number and λ is a constant.

Proof. Let us assume that γ and $\overline{\gamma}$ are pairs of Bertrand curves. Therefore, we have

$$\overline{\gamma}(s) = \gamma(s) + \lambda\mu_1(s), \qquad \mu_1(s) = \overline{\mu}_1(s). \tag{3.4}$$

Consequently, according to equation (3.4), we get $\varphi(s) = \left(\frac{2k+1}{2}\right)\pi$, where k is odd number. Also, by differentiating equation (3.4), we get

$$\overline{\gamma}^{'}(s) = \alpha(s)\nu(s) + \lambda^{'}(s)\mu_1(s) + \lambda(l(s)\mu_2(s) + m(s)\nu(s).$$

Since $\overline{\gamma}^{'}(s) = \beta(s)\mu_2(s)$ from definition adjoint curve, we have

$$\beta(s)\mu_2(s) = (\alpha(s) + \lambda m(s))\nu(s) + \lambda^{'}(s)\mu_1(s) + \lambda l(s)\mu_2(s).$$

Therefore, we get

$$(\alpha(s) + \lambda m(s)) = 0, \quad \lambda = constant, \quad l(s) = \frac{\beta(s)}{\lambda}.$$

Conversely, if there exists a nonzero constant λ such that $\alpha(s) + \lambda m(s) = 0$ for all $s \in I$, then (γ, μ_1, μ_2) is a Bertrand curve [9]. □

Theorem 3.5. *Let $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ be an adjoint curve of framed curve (γ, μ_1, μ_2). Then (γ, μ_1, μ_2) and $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ are Mannheim curves if and only if*

$$\varphi = k\pi, \quad \alpha(s) + \lambda m(s) = 0, \quad l(s) = \frac{\beta(s)}{\lambda},$$

where k is a even number and λ is a constant.

Proof. Suppose that γ and $\overline{\gamma}$ are pairs of Mannheim curves. Therefore, we have

$$\overline{\gamma}(s) = \gamma(s) + \lambda\mu_1(s), \qquad \mu_1(s) = \overline{\mu}_2(s). \tag{3.5}$$

Consequently, according to equation (3.5), we get $\varphi(s) = 0$. By taking the derivative of (3.5), results similar to Theorem 3.4 are obtained. Conversely, if there exists a nonzero constant λ such that $\alpha(s) + \lambda m(s) = 0$ for all $s \in I$, then (γ, μ_1, μ_2) is a Mannheim curve [9]. □

4. Adjoint Curves of Frenet-type Framed Curves and Ruled Surfaces

In this section, firstly, we introduce the adjoint curves of framed curves relative to the Frenet-type frame. Then, we create ruled surfaces with Frenet-type framed curve and adjoint curve. Finally, we introduce normal and binormal surfaces of framed adjoint curves.

4.1. Adjoint Curves of Frenet-type Framed Curves

Definition 4.1. *Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a framed curve. The adjoint curve $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ of the $(\gamma, \mathcal{N}, \mathcal{B})$ Frenet-type framed curve is defined by*

$$\overline{\gamma}(s) = \int_{s_0}^{s} \sigma(s)\mathcal{B}(s)ds, \tag{4.1}$$

where σ is a smooth function.

Let $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ is a moving frame and (κ, τ, α) are framed curvatures of $\gamma(s)$ and $\{\overline{\mathcal{T}}, \overline{\mathcal{N}}, \overline{\mathcal{B}}\}$ is a moving frame and $(\overline{\kappa}, \overline{\tau}, \overline{\alpha})$ are framed curvatures of $\overline{\gamma}(s)$:

Theorem 4.2. *Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a framed curve and $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ adjoint curve. Then the following relations hold where $\tau(s) > 0$:*

$$\begin{array}{rcl} \overline{\mathcal{T}}(s) & = & \mathcal{B}(s), \\ \overline{\mathcal{N}}(s) & = & -\mathcal{N}(s), \\ \overline{\mathcal{B}}(s) & = & \mathcal{T}(s), \\ \overline{\kappa}(s) & = & \tau(s), \\ \overline{\tau}(s) & = & \kappa(s), \\ \overline{\alpha}(s) & = & \sigma(s). \end{array}$$

Proof. By differentiating of equation (4.1), we get $\overline{\gamma}'(s) = \sigma(s)\mathcal{B}(s)$. By taking the norm of both sides and using $\overline{\mathcal{T}}$ and $\mathcal{B}$ as unit vectors, we can write $\overline{\alpha}(s) = \sigma(s)$. Then, we get

$$\overline{\mathcal{T}}(s) = \mathcal{B}(s).$$

Since $\tau(s) > 0$, $\overline{\mathcal{T}}(s)$ is regular spherical curve. Consequently, by using derivative formulas of Frenet-type framed curves, we get

$$\begin{array}{rcl} \overline{\mathcal{N}}(s) & = & \dfrac{\overline{\mathcal{T}}'(s)}{\|\overline{\mathcal{T}}'(s)\|} = \dfrac{\mathcal{B}'(s)}{\|\mathcal{B}'(s)\|} = \dfrac{-\tau(s)\mathcal{N}(s)}{\tau(s)} = -\mathcal{N}(s), \\[2ex] \overline{\mathcal{B}}(s) & = & \overline{\mathcal{T}}(s) \wedge \overline{\mathcal{N}}(s) = -\mathcal{B}(s) \wedge \mathcal{N}(s) = \mathcal{T}(s), \\[2ex] \overline{\kappa}(s) & = & \|\overline{\mathcal{T}}'(s)\| = \|\mathcal{B}'(s)\| = \tau(s), \\[2ex] \overline{\tau}(s) & = & \dfrac{\langle \overline{\mathcal{T}}(s) \wedge \overline{\mathcal{T}}'(s), \overline{\mathcal{T}}''(s) \rangle}{\|\overline{\mathcal{T}}'(s)\|^2} = \dfrac{\langle \mathcal{B}(s) \wedge \mathcal{B}'(s), \mathcal{B}''(s) \rangle}{\|\mathcal{B}'(s)\|^2} = \kappa(s). \end{array}$$

□

Corollary 4.3. *The Frenet-type framed curve $(\gamma, \mathcal{N}, \mathcal{B})$ is a framed helix if and only if the Frenet-type framed adjoint curve $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ is a framed helix.*

4.2. Ruled surfaces with adjoint curves of Frenet-type framed curves

In this section, we examine the structure of ruled surfaces, whose base curve is a Frenet-type framed curve with singular points, and whose direction is a regular adjoint curve (i. e $\sigma = 1$):

Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a framed curve and $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ be an adjoint curve. The ruled surface with base curve γ and director curve $\overline{\gamma}$ is defined by

$$\psi(s,u) = \gamma(s) + u\overline{\gamma}(s).$$

Proposition 4.4. *Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a framed curve and $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ be an adjoint curve. The following characterizations can be given for the ruled surface $\psi(s,u)$:*

i. $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ is developable if and only if then $\langle \overline{\gamma}, \alpha\mathcal{N} \rangle = 0$.

ii. $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface is not a conical surface.

iii. $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface is not a cylindrical surface.

iv. If (s_0, u_0) is a singular point of $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface, then

$$\alpha(s_0)(\overline{\gamma}(s_0) \wedge \mathcal{T}(s_0)) + u_0(\overline{\gamma}(s_0) \wedge \mathcal{B}(s_0)) = 0. \tag{4.2}$$

Proof. i. For the distribution parameter of the $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface, we get

$$\begin{aligned} \frac{det(\gamma^{'}, \overline{\gamma}, \overline{\gamma}^{'})}{\|\overline{\gamma}^{'}\|^2} &\Leftrightarrow \frac{det(\alpha\mathcal{T}, \overline{\gamma}, \mathcal{B})}{\|\mathcal{B}\|^2} \\ &\Leftrightarrow -det(\overline{\gamma}, \alpha\mathcal{T}, \mathcal{B}) = -\langle \overline{\gamma}, \alpha\mathcal{T} \wedge \mathcal{B} \rangle = \langle \overline{\gamma}, \alpha\mathcal{N} \rangle. \end{aligned}$$

Consequently, the ruled surface is developable if and only if $\langle \overline{\gamma}, \alpha\mathcal{N} \rangle = 0$.

ii. We consider the striction curve $\rho(s)$ defined by

$$\rho(s) = \gamma(s) - \frac{\langle \gamma^{'}(s), \overline{\gamma}^{'}(s) \rangle}{\langle \overline{\gamma}^{'}(s), \overline{\gamma}^{'}(s) \rangle}\overline{\gamma}(s) = \gamma(s) - \frac{\langle \alpha(s)\mathcal{T}(s), \mathcal{B}(s) \rangle}{\langle \mathcal{B}(s), \mathcal{B}(s) \rangle}\overline{\gamma}(s) = \gamma(s).$$

Since $\rho^{'}(s) = \gamma^{'}(s) = \alpha(s)\mathcal{T}(s) \neq 0$ for all $s \in I$, we get (ii).

iii. The direction vector of $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface is non-constant. Therefore, $\psi(s,u)$ is not a cylindrical surface.

iv. By a direct calculation, we get

$$\frac{\partial\psi(s,u)}{\partial s} \wedge \frac{\partial\psi(s,u)}{\partial u} = \alpha(\overline{\gamma} \wedge \mathcal{T}) + u(\overline{\gamma} \wedge \mathcal{B}).$$

If (s_0, u_0) is a singular point of $\psi(s,u) = \gamma(s) + u\overline{\gamma}(s)$ surface, we get equation (4.2). □

Example 4.5. *Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a Frenet-type framed curve defined by*

$$\gamma(s) = \left(\frac{3s^2}{2}, \frac{\sqrt{6}s^3}{3}, \frac{s^4}{4} \right)$$

(See Fig.1 (A)). The curve γ has a singular point at $s = 0$, so that it is not a regular curve. On the other hand, γ is a framed base curve with the mapping $\mathcal{T}(s) : \mathbb{R} \to S^2$

$$\mathcal{T}(s) = \left(\frac{3}{s^2+3}, \frac{\sqrt{6}s}{s^2+3}, \frac{s^2}{s^2+3} \right)$$

gives the generalized tangent vector and

$$\alpha(s) = s(s^2+3).$$

By a direct calculation, we get

$$\begin{aligned}
\mathcal{N}(s) &= \left(\frac{-\sqrt{6}s}{s^2+3}, \frac{-(s^2-3)}{s^2+3}, \frac{\sqrt{6}s}{s^2+3} \right), \\
\mathcal{B}(s) &= \left(\frac{s^2}{s^2+3}, \frac{-\sqrt{6}s}{s^2+3}, \frac{3}{s^2+3} \right), \\
\kappa(s) &= \frac{\sqrt{6}}{s^2+3}, \\
\tau(s) &= \frac{\sqrt{6}}{s^2+3}.
\end{aligned}$$

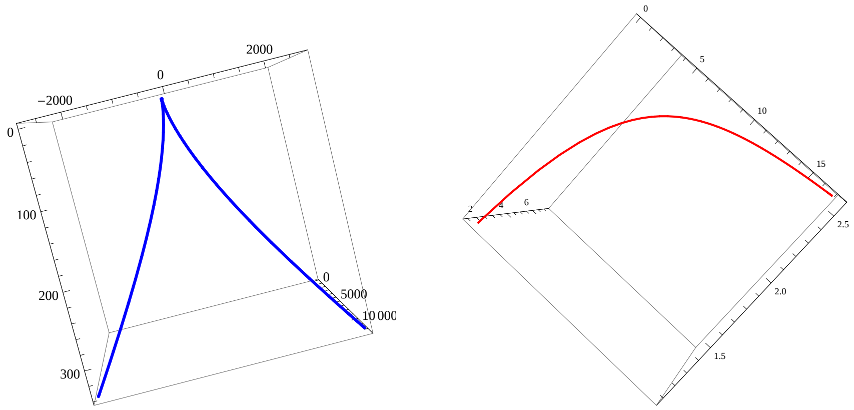
On the other hand, since $\sigma = 1$, the adjoint curve $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ of the Frenet-type framed curve $(\gamma, \mathcal{N}, \mathcal{B})$ is given by

$$\overline{\gamma}(s) = \int_{s_0}^{s} \mathcal{B}(s)ds = \left(s - \sqrt{3}\arctan\left(\frac{s}{\sqrt{3}}\right), -\frac{\sqrt{6}\log(s^2+3)}{2}, \sqrt{3}\arctan\left(\frac{s}{\sqrt{3}}\right) \right)$$

(See Fig.1 (B)). Also, the ruled surface with base curve γ and director curve $\overline{\gamma}$ is defined by

$$\begin{aligned}
\psi(s,u) &= \left(\frac{3s^2}{2}, \frac{\sqrt{6}s^3}{3}, \frac{s^4}{4} \right) \\
&+ u\left(s - \sqrt{3}\arctan\left(\frac{s}{\sqrt{3}}\right), -\frac{\sqrt{6}\log(s^2+3)}{2}, \sqrt{3}\arctan\left(\frac{s}{\sqrt{3}}\right) \right)
\end{aligned}$$

(See Fig.2).



(A) The framed curve γ

(B) The adjoint curve $\overline{\gamma}$

FIGURE 1. γ and $\overline{\gamma}$ curves

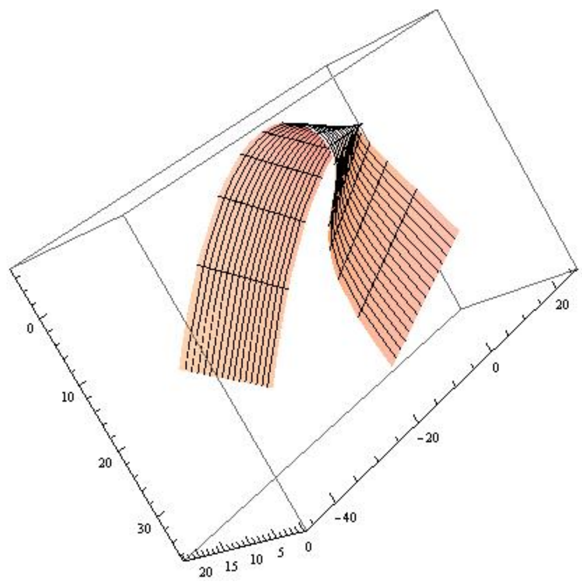


FIGURE 2. The ruled surface $\psi(s,u)$

4.3. Normal and binormal surfaces adjoint curves of Frenet-type framed curves

Now let us give normal and binormal surfaces related to adjoint curve. Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a Frenet-type framed curve and $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ adjoint curve. The normal surface of $\overline{\gamma}$ is defined by

$$M_1 : \varphi_{\overline{\gamma}}(s,u) = \overline{\gamma}(s) + u\overline{\mathcal{N}}(s) = \overline{\gamma}(s) + u(-\mathcal{N}(s)).$$

The binormal surface of $\overline{\gamma}$ is defined by

$$M_2 : \phi_{\overline{\gamma}}(s,u) = \overline{\gamma}(s) + u\overline{\mathcal{B}}(s) = \overline{\gamma}(s) + u\mathcal{T}(s).$$

Proposition 4.6. *Let $(\gamma, \mathcal{N}, \mathcal{B})$ be a Frenet-type framed curve and $(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ be an adjoint curve. The following characterizations can be given for the normal and binormal surfaces of γ and $\overline{\gamma}$:*

i. M_1 and M_2 surfaces are not developable.

ii. The set of singular points of the surface M_1 is

$$S = \{(s,u) | u\kappa(s) = 0, u\tau(s) = \sigma(s)\}.$$

iii. The set of singular points of the surface M_2 is

$$S = \{(s,u) | u\kappa(s) = 0, \sigma(s) = 0\}.$$

Proof. i. For the distribution parameter of M_1 surface, we get

$$\begin{aligned}\frac{det(\overline{\gamma}^{'}(s), -\mathcal{N}(s), -\mathcal{N}^{'}(s))}{\| -\mathcal{N}'(s)\|^2} &\Leftrightarrow \frac{det(\sigma(s)\mathcal{B}(s), -\mathcal{N}(s), \kappa(s)\mathcal{T}(s) - \tau(s)\mathcal{B}(s))}{\|\kappa(s)\mathcal{T}(s) - \tau(s)\mathcal{B}(s)\|^2} \\ &\Leftrightarrow \frac{\langle \sigma(s)\mathcal{B}(s), \kappa(s)\mathcal{B}(s) + \tau(s)\mathcal{T}(s)\rangle}{\kappa^2(s) + \tau^2(s)} \\ &\Leftrightarrow \frac{\sigma(s)\kappa(s)}{\kappa^2(s) + \tau^2(s)}.\end{aligned}$$

For the distribution parameter of M_2 surface, we get

$$\begin{aligned}\frac{det(\overline{\gamma}^{'}(s), \mathcal{T}(s), \mathcal{T}^{'}(s))}{\|\mathcal{T}'(s)\|^2} &\Leftrightarrow \frac{det(\sigma(s)\mathcal{B}(s), \mathcal{T}(s), \kappa(s)\mathcal{N}(s))}{\|\kappa(s)\mathcal{N}(s)\|^2} \\ &\Leftrightarrow \frac{\langle \sigma(s)\mathcal{B}(s), \kappa(s)\mathcal{B}(s)\rangle}{\kappa^2(s)} \\ &\Leftrightarrow \frac{\sigma(s)}{\kappa(s)}.\end{aligned}$$

Therefore, since $\kappa(s) > 0$ and $\sigma(s) \neq 0$ for some s, M_1 and M_2 surfaces are not developable.

ii. By a direct calculation, we have

$$\frac{\partial \varphi_{\overline{\gamma}}(s,u)}{\partial s} \wedge \frac{\partial \varphi_{\overline{\gamma}}(s,u)}{\partial u} = -\kappa(s)u\mathcal{B}(s) + (\sigma(s) - u\tau(s))\mathcal{T}(s).$$

Hence, (ii) is satisfied.

iii. By a direct calculation, we have

$$\frac{\partial \phi_{\overline{\gamma}}(s,u)}{\partial s} \wedge \frac{\partial \phi_{\overline{\gamma}}(s,u)}{\partial u} = \sigma(s)\mathcal{N}(s) - u\kappa\mathcal{B}(s).$$

Hence, (iii) is satisfied. □

Huang and Pei, in their study [10], divided the singular point set of principal normal surface and binormal surface into two sets. Now let us examine the types of singularity for M_1 and M_2 surfaces created with adjoint curves with similar logic. When $u \neq 0$ for the surface M_1, since $\kappa(s) = 0$, a contradiction is obtained. Therefore, the only singular set of points on the surface M_1 can be expressed as

$$S_1 = \{(s, 0) | \sigma(s) = 0\}.$$

Since the same situation is valid for the surface M_2, the singular point set S_1 is also the singular point set of the surface M_2. Therefore, let us give the type of singularity of the surfaces M_1 and M_2 with the following theorem:

Theorem 4.7. *i. Let M_1 be a normal surface of $\overline{\gamma}(s)$. Then, if $(s_1, u_1) \in S_1$ and $\sigma^{'}(s_1) \neq 0$, then M_1 surface is a cross-cap at the point $(s_1, 0)$.*

ii. Let M_2 be a binormal surface of $\overline{\gamma}(s)$. Then, if $(s_1, u_1) \in S_1$ and $\sigma^{'}(s_1) \neq 0$, then M_2 surface is a cross-cap at the point $(s_1, 0)$.

Proof. i. Assume that M_1 is normal surface of $\overline{\gamma}(s)$. Then, we get

$$\begin{aligned}
\frac{\partial \varphi_{\overline{\gamma}}}{\partial s}(s,u) &= u\kappa(s)\mathcal{T}(s) + (\sigma(s) - u\tau(s))\mathcal{B}(s), \\
\frac{\partial \varphi_{\overline{\gamma}}}{\partial u}(s,u) &= -\mathcal{N}(s), \\
\frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial u \partial s}(s,u) &= \kappa(s)\mathcal{T}(s) - \tau(s)\mathcal{B}(s), \\
\frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial s^2}(s,u) &= u\kappa^{'}(s)\mathcal{T}(s) + (u\kappa^2(s) + u\tau^2(s) - \sigma(s)\tau(s))\mathcal{N}(s) \\
&+ (\sigma^{'}(s) - u\tau^{'}(s))\mathcal{B}(s).
\end{aligned}$$

Then, we have

$$det\left(\frac{\partial \varphi_{\overline{\gamma}}}{\partial u}(s,u), \frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial u \partial s}(s,u), \frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial s^2}(s,u)\right) = -\kappa(s)u\tau^{'}(s) + \tau(s)u\kappa^{'}(s) + \kappa(s)\sigma^{'}(s).$$

If $(s_1, u_1) \in S_1$, then we have

$$det\left(\frac{\partial \varphi_{\overline{\gamma}}}{\partial u}(s_1,u_1), \frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial u \partial s}(s_1,u_1), \frac{\partial^2 \varphi_{\overline{\gamma}}}{\partial s^2}(s_1,u_1)\right) = \kappa(s_1)\sigma^{'}(s_1).$$

From [12] and since $\kappa(s) > 0$, this completes the proof.
ii. Suppose that M_2 is binormal surface of $\overline{\gamma}(s)$. Then, we get

$$\begin{aligned}
\frac{\partial \phi_{\overline{\gamma}}}{\partial s}(s,u) &= u\kappa(s)\mathcal{N}(s) + \sigma(s)\mathcal{B}(s), \\
\frac{\partial \phi_{\overline{\gamma}}}{\partial u}(s,u) &= \mathcal{T}(s), \\
\frac{\partial^2 \phi_{\overline{\gamma}}}{\partial u \partial s}(s,u) &= \kappa(s)\mathcal{N}(s), \\
\frac{\partial^2 \phi_{\overline{\gamma}}}{\partial s^2}(s,u) &= (u\kappa^{'}(s) - \sigma(s)\tau(s))\mathcal{N}(s) - u\kappa^2(s)\mathcal{T}(s) \\
&+ (\sigma^{'}(s) + u\kappa(s)\tau(s))\mathcal{B}(s).
\end{aligned}$$

Then, we have

$$det\left(\frac{\partial \phi_{\overline{\gamma}}}{\partial u}(s,u), \frac{\partial^2 \phi_{\overline{\gamma}}}{\partial u \partial s}(s,u), \frac{\partial^2 \phi_{\overline{\gamma}}}{\partial s^2}(s,u)\right) = \kappa(s)(\sigma^{'}(s) + u\kappa(s)\tau(s)).$$

If $(s_1, u_1) \in S_1$, then we have

$$det\left(\frac{\partial \phi_{\overline{\gamma}}}{\partial u}(s_1,u_1), \frac{\partial^2 \phi_{\overline{\gamma}}}{\partial u \partial s}(s_1,u_1), \frac{\partial^2 \phi_{\overline{\gamma}}}{\partial s^2}(s_1,u_1)\right) = \kappa(s_1)\sigma^{'}(s_1).$$

From [12] and since $\kappa(s) > 0$, this completes the proof.

□

Example 4.8. *Consider the Frenet-type framed curve*

$$\gamma(s) = \left(\cos^3 s, \sin^3 s, \cos 2s\right).$$

By a direct calculation, we get

$$\begin{aligned}
\mathcal{T}(s) &= \left(-\frac{3}{5}\cos s, \frac{3}{5}\sin s, -\frac{4}{5}\right), \\
\mathcal{N}(s) &= (\sin s, \cos s, 0), \\
\mathcal{B}(s) &= \left(\frac{4}{5}\cos s, -\frac{4}{5}\sin s, -\frac{3}{5}\right), \\
\alpha(s) &= 5\cos s \sin s, \\
\kappa(s) &= \frac{3}{5}, \quad \tau(s) = \frac{4}{5}.
\end{aligned}$$

On the other hand, a regular adjoint curve $(\overline{\gamma}_1, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ *of the framed curve* $(\gamma, \mathcal{N}, \mathcal{B})$ *is given by*

$$\overline{\gamma}_1(s) = \left(\frac{4}{5}\sin s, \frac{4}{5}\cos s, -\frac{3s}{5}\right),$$

where $\sigma = 1$. A singular adjoint curve $(\overline{\gamma}_2, \overline{\mathcal{N}}, \overline{\mathcal{B}})$ of the framed curve $(\gamma, \mathcal{N}, \mathcal{B})$ is given by

$$\overline{\gamma}_2(s) = \left(\frac{4s}{5}\sin s + \frac{4}{5}\cos s, \frac{4s}{5}\cos s - \frac{4s}{5}\sin s, -\frac{3s^2}{10}\right),$$

where $\sigma(s) = s$. Also, the normal and binormal surfaces of $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are defined by

$$\begin{aligned}
M_1 : \varphi_{\overline{\gamma}_1}(s,u) &= \left(\frac{4}{5}\sin s, \frac{4}{5}\cos s, -\frac{3s}{5}\right) + u(-\sin s, -\cos s, 0),\\
M_2 : \phi_{\overline{\gamma}_1}(s,u) &= \left(\frac{4}{5}\sin s, \frac{4}{5}\cos s, -\frac{3s}{5}\right) + u\left(-\frac{3}{5}\cos s, \frac{3}{5}\sin s, -\frac{4}{5}\right),\\
M_3 : \varphi_{\overline{\gamma}_2}(s,u) &= \left(\frac{4s}{5}\sin s + \frac{4}{5}\cos s, \frac{4s}{5}\cos s - \frac{4s}{5}\sin s, -\frac{3s^2}{10}\right)\\
&+ u(-\sin s, -\cos s, 0),\\
M_4 : \phi_{\overline{\gamma}_2}(s,u) &= \left(\frac{4s}{5}\sin s + \frac{4}{5}\cos s, \frac{4s}{5}\cos s - \frac{4s}{5}\sin s, -\frac{3s^2}{10}\right)\\
&+ u\left(-\frac{3}{5}\cos s, \frac{3}{5}\sin s, -\frac{4}{5}\right).
\end{aligned}$$

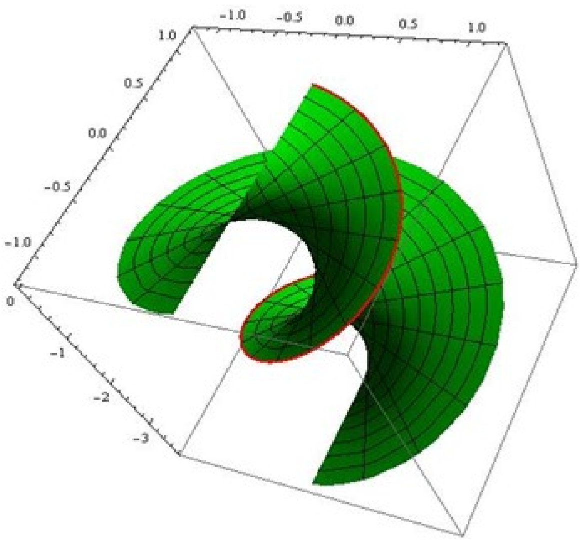


FIGURE 3. The normal surface M_1 of regular adjoint curve $\overline{\gamma}_1$

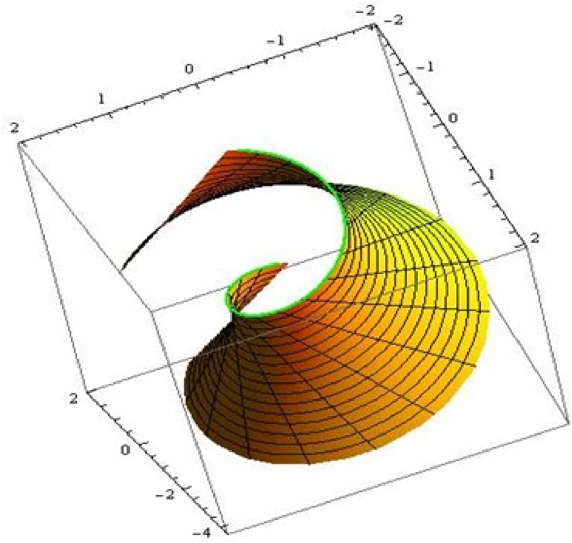


FIGURE 4. The binormal surface M_2 of regular adjoint curve $\overline{\gamma}_1$

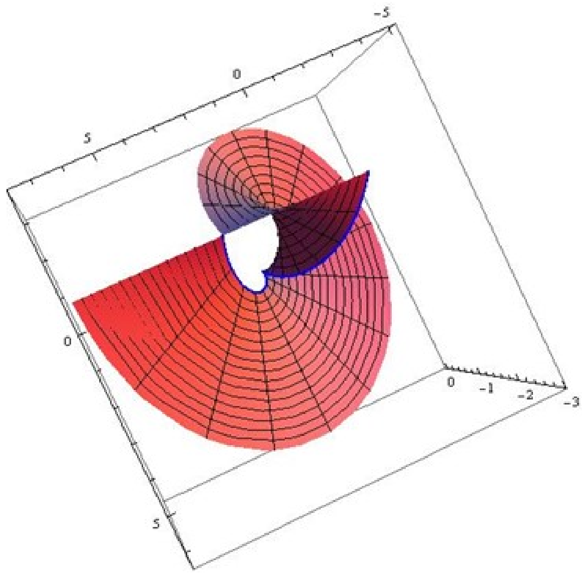


FIGURE 5. The normal surface M_3 of singular adjoint curve $\overline{\gamma}_2$

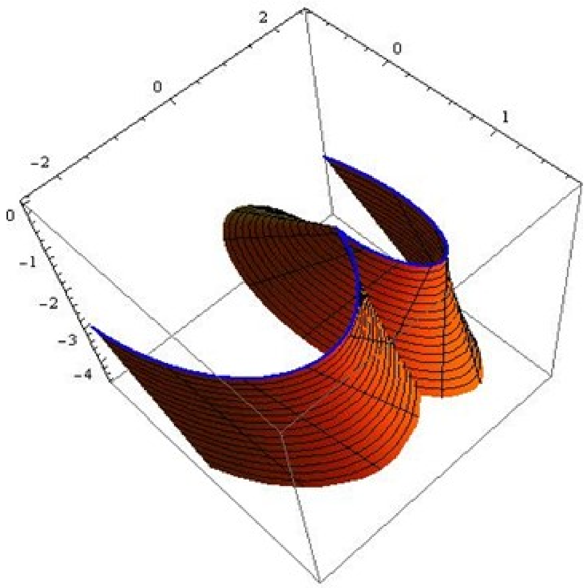


Figure 6. The binormal surface M_4 of singular adjoint curve $\overline{\gamma}_2$

Acknowledgement

The results presented in this paper were part of the PhD thesis of Bahar Doğan Yazıcı [17].

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