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INFINITE FAMILIES OF CONGRUENCES MODULO 2 FOR 2-CORE AND 13-CORE PARTITIONS

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Abstract. A partition of n is called a t-core partition if none of its hook number is divisible by t. In 2019, Hirschhorn and Sellers [5] obtained a parity result for 3-core partition function $a_3(n)$. Motivated by this result, both the authors [8] recently proved that for a non-negative integer α, $a_{3^\alpha m}(n)$ is almost always divisible by an arbitrary power of 2 and 3 and $a_t(n)$ is almost always divisible by an arbitrary power of p_i^j, where j is a fixed positive integer and $t = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with primes $p_i \geq 5$. In this article, by using Hecke eigenform theory, we obtain infinite families of congruences and multiplicative identities for $a_2(n)$ and $a_{13}(n)$ modulo 2 which generalizes some results of Das [2].

1. Introduction

A partition $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ of n is a non-increasing sequence of positive integers whose sum is n and the positive integers β_i are called parts of the partition β. A partition β of n can be represented by the Young diagram $[\beta]$ (also known as the Ferrers graph) which consists of the s number of rows such that the i^{th} row has β_i number of dots $\bullet$ and all the rows start in the first column. An illustration of the Young diagram for $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ is as follows.

$$[\beta] := \begin{array}{llllll} \bullet & \bullet & \cdots\cdots\cdots & \bullet & & \beta_1 \text{ dots} \\ \bullet & \bullet & \cdots\cdots & \bullet & & \beta_2 \text{ dots} \\ & \vdots & & & & \vdots \\ \bullet & \bullet & \cdots & \bullet & & \beta_r \text{ dots} \end{array}$$

For $1 \leq i \leqslant r$ and $1 \leq j \leq \beta_i$, the dot of $[\beta]$ which lies in the i^{th} row and j^{th} column is denoted by $(i,j)^{th}$-dot of β. Let $\beta_j^{'}$ denote the number of dots in j^{th} column. The hook number $H_{i,j}$ of $(i,j)^{th}$-dot is defined by $\beta_i + \beta_j^{'} - i - j + 1$. In other words, $H_{i,j} = 1 + h_0$, where h_0 is the sum of the number of dots lying right to the $(i,j)^{th}$-dot in the i^{th} row and the number of dots lying below the

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$(i,j)^{th}$-dot in the j^{th} column. Given a partition β of n, we say that it is a t-core partition if none of its hook number is divisible by t.

Example 1. The Young diagram of the partition $\beta = (6,3,1)$ of 10 is

$$\begin{array}{llllll} \bullet^8 & \bullet^6 & \bullet^5 & \bullet^3 & \bullet^2 & \bullet^1 \\ \bullet^4 & \bullet^2 & \bullet^1 & & & \\ \bullet^1 & & & & & \end{array}$$

where the superscript on each dot represents its hook number. It can be easily observed that this is a t-core partition of 10 for $t = 7$ and $t \geqslant 9$.

Example 2. There are no 3-core partitions of 7. This can be easily verified by looking at the Young diagram of each partition of 7.

For a positive integer n, let $a_t(n)$ denote the number of t-core partitions of n. Its generating function is given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)} = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1.1}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

In [3, Corollary 1], Garvan, Kim, Stanton obtained the congruence

$$a_p(p^j n - \delta_p) \equiv 0 \pmod{p^j}, \tag{1.2}$$

where $p \in \{5,7,11\}$, n, j are positive integers and $\delta_p = \frac{p^2-1}{24}$. In [4, Proposition 3], Granville and Ono proved similar congruences, namely

$$\begin{aligned} a_{5^j}(5^j n - \delta_{5,j}) &\equiv 0 \pmod{5^j}, \\ a_{7^j}(7^j n - \delta_{7,j}) &\equiv 0 \pmod{7^{\lfloor \frac{j}{2} \rfloor + 1}}, \\ a_{11^j}(11^j n - \delta_{11,j}) &\equiv 0 \pmod{11^j}, \end{aligned}$$

where n, j are positive integers and $\delta_{p,j} \equiv \frac{1}{24} \pmod{p^j}$ for $p \in \{5,7,11\}$.

In 2019, Hirschhorn and Sellers [5] proved a parity result for $a_3(n)$, i.e., for all $n \geq 0$,

$$a_3(n) = \begin{cases} 1 \pmod 2 & \text{if } n = 3r^2 + 2r \text{ for some integer } r, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Motivated by this result, both the authors proved that for a non-negative integer α, $a_{3^\alpha m}(n)$ is almost always divisible by an arbitrary power of 2 and 3. Moreover, they also proved that $a_t(n)$ is almost always divisible by an arbitrary power of p_i^j, where j is a fixed positive integer and $t = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with primes $p_i \geq 5$. In the following theorem, we obtain infinite families of congruences modulo 2 for $a_2(n)$ and $a_{13}(n)$ by using Hecke eigen form theory.

Theorem 1.1. *Let k and n be non-negative integers. For each $1 \leq i \leq k+1$, let $p_1, p_2, \ldots, p_{k+1}$ be prime numbers such that $p_i \geq 5$. Then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$, we have*

(i) $a_2\left(p_1^2p_2^2\cdots p_{k+1}^2 n+\frac{p_1^2p_2^2\cdots p_k^2 p_{k+1}(8j+p_{k+1})-1}{8}\right)\equiv 0 \pmod 2$,

(ii) $a_{13}\left(104p_1^2p_2^2\cdots p_{k+1}^2 n+13p_1^2p_2^2\cdots p_k^2p_{k+1}\left(\epsilon_p j+p_{k+1}\right)-7\right)\equiv 0 \pmod 2$, *where*

$$\epsilon_p=\begin{cases}1 & \text{if } p\not\equiv 1 \pmod 8,\\ 8 & \text{if } p\equiv 1 \pmod 8.\end{cases}$$

Corollary 1.1. *Let n and k be non-negative integers. For a prime $p\geq 5$ and an integer $j\not\equiv 0 \pmod p$, we have*

(i) $a_2\left(p^{2(k+1)}n+p^{2k+1}j+\frac{p^{2k+2}-1}{8}\right)\equiv 0 \pmod 2$,

(ii) $a_{13}\left(104p^{2k+2}n+13\epsilon_p p^{2k+1}j+13p^{2k+2}-7\right)\equiv 0 \pmod 2$.

Furthermore, we prove the following multiplicative formulae for 2-core partitions and 13-core partitions modulo 2.

Theorem 1.2. *Let k be a positive integer and p be a prime number such that $p\equiv 7 \pmod 8$. Let r be a non-negative integer such that p divides $8r+7$. Then*

(i) $a_2\left(p^{k+1}n+pr+\frac{7p-1}{8}\right)\equiv(-1)\left(\frac{-2}{p}\right)a_2\left(p^{k-1}n+\frac{8r+7-p}{8p}\right) \pmod 2$,

(ii) $a_{13}\left(104p^{k+1}n+104pr+91p-7\right)$

$$\equiv(-1)\left(\frac{-2}{p}\right)a_{13}\left(104p^{k-1}n+\frac{104r+91}{p}-7\right) \pmod 2.$$

Corollary 1.2. *Let k be a positive integer and p be a prime number such that $p\equiv 7 \pmod 8$. Then*

(i) $a_2\left(p^{2k}n+\frac{p^{2k}-1}{8}\right)\equiv(-1)^k\left(\frac{-2}{p}\right)^k a_2(n) \pmod 2$.

(ii) $a_{13}\left(104p^{2k}n+13p^{2k}-7\right)\equiv(-1)^k\left(\frac{-2}{p}\right)^k a_{13}(104n+6) \pmod 2$.

2. Preliminaries

We recall some basic facts and definition on modular forms. For more details, we refer to [6, 9]. We start with some matrix groups. We define

$$\Gamma:=\mathrm{SL}_2(\mathbb{Z})=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}: a,b,c,d\in\mathbb{Z}, ad-bc=1\right\},$$

$$\Gamma_\infty:=\left\{\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}: n\in\mathbb{Z}\right\}.$$

For a positive integer N, we define

$$\Gamma_0(N):=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathrm{SL}_2(\mathbb{Z}): c\equiv 0 \pmod N\right\},$$

$$\Gamma_1(N):=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\Gamma_0(N): a\equiv d\equiv 1 \pmod N\right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N},\ b \equiv c \equiv 0 \pmod{N} \right\}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if it contains $\Gamma(N)$ for some positive integer N and the smallest N with this property is called its level. Note that $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N, whereas $\mathrm{SL}_2(\mathbb{Z})$ and Γ_∞ are congruence subgroups of level 1. The index of $\Gamma_0(N)$ in Γ is

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where p runs over prime divisors of N.

Let $\mathbb{H}$ denote the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] z = \frac{az+b}{cz+d}$. We identify ∞ with $\frac{1}{0}$ and define $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \frac{r}{s} = \frac{ar+bs}{cr+ds}$, where $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and k is an integer, then define the slash operator $|_k$ by

$$(f|_k\gamma)(z) := (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z).$$

Definition 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of integer weight k on Γ if the following hold:

(1) For all $z \in \mathbb{H}$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$,

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z).$$

(2) If $\gamma \in SL_2(\mathbb{Z})$, then $(f|_k\gamma)(z)$ has a Fourier expansion of the form

$$(f|_k\gamma)(z) := \sum_{n \geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

For a positive integer k, the complex vector space of modular forms of weight k with respect to a congruence subgroup Γ is denoted by $M_k(\Gamma)$.

Definition 2.2 ([9, Definition 1.15])**.** Let χ be a Dirichlet character modulo N. We say that a modular form $f \in M_k(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_k(\Gamma_0(N), \chi)$.

The relevant modular forms for the results obtained in this article arise from eta-quotients. We recall the Dedekind eta-function $\eta(z)$ which is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \tag{2.1}$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) := \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N and r_δ are integers with $N > 0$.

Theorem 2.1 ([9, Theorem 1.64])**.** *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $k = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \textit{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for each $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s = \prod_{\delta|N} \delta^{r_\delta}$.

Theorem 2.2 ([9, Theorem 1.65])**.** *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. If f is an eta-quotient satisfying the conditions of Theorem 2.1 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

Suppose that $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.1 and that the associated weight k is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$. Theorem 2.2 gives the necessary criterion for determining orders of an eta-quotient at cusps. In the proofs of our results, we use Theorems 2.1 and 2.2 to prove that $f(z) \in M_k(\Gamma_0(N), \chi)$ for certain eta-quotients $f(z)$ we consider in the sequel.

We recall the definition of Hecke operators and a few relevant results. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \left(\sum_{d \,|\, \gcd(n,m)} \chi(d) d^{k-1} a\left(\frac{mn}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is a prime, we have

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{k-1} a\left(\frac{n}{p}\right) \right) q^n.$$

We note that $a(n) = 0$ unless n is a non-negative integer.

3. Proofs of Theorems 1.1 and 1.2

3.1. Prelude to the proofs

We define

$$\sum_{n=1}^{\infty} b(n)q^n = q(q^8;q^8)_\infty (q^{16};q^{16})_\infty \quad \text{and} \quad \sum_{n=0}^{\infty} c(n)q^n := (q;q)_\infty^3. \tag{3.1}$$

If $p \nmid n$, then we set $b\left(\frac{n}{p}\right) = 0$ and $c\left(\frac{n}{p}\right) = 0$. We have the following result.

Lemma 3.2. *For $n \geq 0$ and for a prime $p \not\equiv 1 \pmod 8$, we have*

$$b(pn) = (-1)\left(\frac{-2}{p}\right) b\left(\frac{n}{p}\right). \tag{3.2}$$

Further if $j \not\equiv 0 \pmod p$, then

$$b(p^2 n + pj) = 0. \tag{3.3}$$

Proof. Let p be a prime with $p \not\equiv 1 \pmod 8$. Using (2.1), we note that

$$\sum_{n=1}^{\infty} b(n)q^n = \eta(8z)\eta(16z).$$

By using Theorem 2.1, we obtain that $\eta(8z)\eta(16z) \in S_1(\Gamma_0(128), \left(\frac{-128}{\bullet}\right))$. Thus $\eta(8z)\eta(16z)$ has the Fourier expansion given by

$$\sum_{n=1}^{\infty} b(n)q^n = \eta(8z)\eta(16z) = q - q^9 - 2q^{17} + \cdots.$$

Therefore, $b(n) = 0$ for all $n \geq 0$ with $n \not\equiv 1 \pmod 8$. Since $\eta(8z)\eta(16z)$ is a Hecke eigenform, we obtain from [7, Table 1] that

$$\eta(8z)\eta(16z) \mid T_p = \sum_{n=1}^{\infty} \left(b(pn) + \left(\frac{-128}{p}\right) b\left(\frac{n}{p}\right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} b(n)q^n.$$

Note that $\left(\frac{-128}{p}\right) = \left(\frac{-2}{p}\right)$. Comparing the coefficients of q^n on both sides of the above equation, we get

$$b(pn) + \left(\frac{-2}{p}\right) b\left(\frac{n}{p}\right) = \lambda(p)b(n). \tag{3.4}$$

Since $b(1) = 1$ and $b(\frac{1}{p}) = 0$, by substituting $n = 1$ in the above expression, we get $b(p) = \lambda(p)$. Further, since $b(p) = 0$, we obtain that $\lambda(p) = 0$. Hence, we conclude from (3.4) that

$$b(pn) + \left(\frac{-2}{p}\right) b\left(\frac{n}{p}\right) = 0, \tag{3.5}$$

which proves (3.2). For $j \not\equiv 0 \pmod p$, replacing n by $pn + j$ in (3.5), we get $b(p^2n + pj) = 0$ which proves (3.3). □

Lemma 3.3. *For $n \geq 0$ and for a prime $p \equiv 1 \pmod 4$, we have*

$$c\left(p^2n + \frac{p^2-1}{8}\right) = pc(n). \tag{3.6}$$

If $p \nmid n$, then

$$c\left(pn + \frac{p^2-1}{8}\right) = 0. \tag{3.7}$$

Proof. From [1, Page 39, Entry 24(ii)], we have

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Thus

$$c(n) = \sum_{\substack{k=0 \\ \frac{k(k+1)}{2}=n}}^{\infty} (-1)^k(2k+1) = \sum_{\substack{k=0 \\ (2k+1)^2=8n+1}}^{\infty} (-1)^k(2k+1).$$

This implies that

$$c\left(pn + \frac{p^2-1}{8}\right) = \sum_{\substack{k=0 \\ (2k+1)^2=8pn+p^2}}^{\infty} (-1)^k(2k+1).$$

Note that if $(2k+1)^2 = 8pn + p^2$, then $p \mid (2k+1)$ and therefore, we can write $2k+1 = p(2k^{'}+1)$ for some positive integer $k^{'}$. Further for such k, we have $k = \frac{2k+1}{2} - \frac{1}{2} = \frac{p(2k^{'}+1)}{2} - \frac{1}{2} = pk^{'} + \frac{p-1}{2}$ which gives $(-1)^k = (-1)^{k^{'}}$. Hence

$$c\left(pn + \frac{p^2-1}{8}\right) = p \sum_{\substack{k=0 \\ (2k^{'}+1)^2=8\frac{n}{p}+1}}^{\infty} (-1)^{k^{'}}(2k^{'}+1) = pc\left(\frac{n}{p}\right).$$

Replacing n by pn, we obtain (3.6). Also, (3.7) follows since $c\left(\frac{n}{p}\right)=0$ if $p\nmid n$. This completes the proof. □

We recall the following identity for 13-core partitions obtained by Kuwali Das.

Lemma 3.4 ([2, Theorem 1])**.** *We have*

$$\sum_{n=0}^{\infty} a_{13}\left(104n+6\right)q^n \equiv (q;q)_\infty \left(q^2;q^2\right)_\infty \pmod 2.$$

Lemma 3.5. *For* $j \not\equiv 0 \pmod p$ *and* $n \geq 0$, *we have*

$$a_2\left(p^2n+pj+\frac{p^2-1}{8}\right) \equiv 0 \pmod 2, \tag{3.8}$$

$$a_2\left(p^2n+\frac{p^2-1}{8}\right) \equiv \delta_p a_2(n) \pmod 2, \tag{3.9}$$

$$a_{13}\left(104p^2n+13p(\epsilon_p j+p)-7\right) \equiv 0 \pmod 2, \tag{3.10}$$

$$a_{13}\left(104p^2n+13p^2-7\right) \equiv \delta_p a_{13}(104n+6) \pmod 2, \tag{3.11}$$

where

$$\epsilon_p = \begin{cases} 1 & \text{if } p \not\equiv 1 \pmod 8, \\ 8 & \text{if } p \equiv 1 \pmod 8, \end{cases} \quad \text{and} \quad \delta_p = \begin{cases} (-1)\left(\frac{-2}{p}\right) & \text{if } p \not\equiv 1 \pmod 8, \\ p & \text{if } p \equiv 1 \pmod 8. \end{cases}$$

Proof. We consider the two cases $p \not\equiv 1 \pmod 8$ and $p \equiv 1 \pmod 8$ separately as follows.
Case 1: $p \not\equiv 1 \pmod 8$.
From (1.1), we have

$$\sum_{n=0}^{\infty} a_2(n)q^n \equiv \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty} \equiv (q;q)_\infty(q^2;q^2)_\infty \pmod 2.$$

Thus using Lemma 3.4, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_2(n)q^{8n+1} &\equiv \sum_{n=0}^{\infty} a_{13}(104n+6)q^{8n+1} \\ &\equiv q(q^8;q^8)_\infty(q^{16};q^{16})_\infty \pmod 2.\end{aligned} \tag{3.12}$$

From (3.1) and (3.12), we get

$$a_2(n) \equiv a_{13}(104n+6) \equiv b(8n+1) \pmod 2. \tag{3.13}$$

Let $r \not\equiv 0 \pmod p$. From (3.3), we have

$$b(p^2n+pr)=0.$$

Replacing n by $8n-pr+1$, we obtain

$$b(8p^2n-p^3r+p^2+pr)=0.$$

Note that $8p^2n-p^3r+p^2+pr=8(p^2n-pr\frac{p^2-1}{8}+\frac{p^2-1}{8})+1$. Therefore, using (3.13), we obtain

$$\begin{aligned}(3.14)\qquad & a_2\left(p^2n-pr\frac{p^2-1}{8}+\frac{p^2-1}{8}\right)\\ &\equiv a_{13}\left(104p^2n-13pr(p^2-1)+13p^2-7\right)\\ &\equiv 0 \pmod 2.\end{aligned}$$

Since $\gcd(\frac{p^2-1}{8},p)=1$ and $\gcd(p^2-1,p)=1$, when r runs over a residue system excluding the multiples of p, so do $\frac{-r(p^2-1)}{8}$ and $-r(p^2-1)$. Thus for $j\not\equiv 0 \pmod p$, (3.14) can be written as

$$a_2\left(p^2n+pj+\frac{p^2-1}{8}\right)\equiv 0 \pmod 2$$

and

$$a_{13}\left(104p^2n+13pj+13p^2-7\right)\equiv 0 \pmod 2.$$

This proves (3.8) and (3.10) in the case of $p\not\equiv 1 \pmod 8$.

Next, replacing n by $8pn+p$ in (3.2), we obtain

$$b(8p^2n+p^2)=(-1)\left(\frac{-2}{p}\right)b\left(8n+1\right). \tag{3.15}$$

Note that $8p^2n+p^2=8(p^2n+\frac{p^2-1}{8})+1$. Therefore, using (3.13) in (3.15), we get

$$a_2\left(p^2n+\frac{p^2-1}{8}\right)\equiv(-1)\left(\frac{-2}{p}\right)a_2(n) \pmod 2$$

and

$$a_{13}\left(104p^2n+13p^2-7\right)\equiv(-1)\left(\frac{-2}{p}\right)a_{13}\left(104n+6\right) \pmod 2,$$

which proves (3.9) and (3.11) in the case of $p\not\equiv 1 \pmod 8$.

Case 2: $p\equiv 1 \pmod 8$.

From (1.1), we have

$$\sum_{n=0}^{\infty}a_2(n)q^n\equiv\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\equiv(q;q)_\infty^3 \pmod 2. \tag{3.16}$$

From Lemma 3.4, we have

$$\sum_{n=0}^{\infty}a_{13}(104n+6)q^n\equiv(q;q)_\infty(q^2;q^2)_\infty\equiv(q;q)_\infty^3 \pmod 2. \tag{3.17}$$

Invoking (3.1), (3.16) and (3.17), we have

$$a_2(n)\equiv a_{13}(104n+6)\equiv c(n) \pmod 2. \tag{3.18}$$

If $p \nmid n$, then from (3.7) and (3.18), we get

$$a_2\left(pn + \frac{p^2-1}{8}\right) \equiv a_{13}\left(104pn + 13p^2 - 7\right) \equiv c\left(pn + \frac{p^2-1}{8}\right) \equiv 0 \pmod{2}.$$

Next replacing n by $pn + j$ for $j \not\equiv 0 \pmod{p}$, we obtain

$$a_2\left(p^2n + pj + \frac{p^2-1}{8}\right) \equiv a_{13}\left(104p^2n + 104pj + 13p^2 - 7\right) \equiv 0 \pmod{2},$$

which proves (3.8) and (3.10) in the case of $p \equiv 1 \pmod{8}$.

Next using (3.6) and (3.18), we get

$$a_2\left(p^2n + \frac{p^2-1}{8}\right) \equiv a_{13}\left(104p^2n + 13p^2 - 7\right) \equiv c\left(p^2n + \frac{p^2-1}{8}\right) \equiv pc(n) \equiv pa_2(n) \equiv pa_{13}(104n+6) \pmod{2},$$

which proves (3.9) and (3.11) in the case of $p \equiv 1 \pmod{8}$. □

3.6. Proof of Theorem 1.1(i)

For $1 \le i \le k-1$, we note that

$$p_i^2 p_{i+1}^2 \cdots p_k^2 n + \frac{p_i^2 p_{i+1}^2 \cdots p_k^2 - 1}{8} = p_i^2\left(p_{i+1}^2 \cdots p_k^2 n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8}\right) + \frac{p_i^2 - 1}{8}.$$

Thus, for $1 \le i \le k-1$, using (3.9) for $p = p_i$, we have

$$a_2\left(p_i^2 p_{i+1}^2 \cdots p_k^2 n + \frac{p_i^2 p_{i+1}^2 \cdots p_k^2 - 1}{8}\right) \equiv \delta_{p_i} a_2\left(p_{i+1}^2 \cdots p_k^2 n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8}\right) \pmod{2}.$$

Also from (3.9), we have

$$a_2\left(p_k^2 n + \frac{p_k^2 - 1}{8}\right) \equiv \delta_{p_k} a_2(n) \pmod{2}.$$

Therefore, from the congruences in the above two displays, we get

$$a_2\left(p_1^2 p_2^2 \cdots p_k^2 n + \frac{p_1^2 p_2^2 \cdots p_k^2 - 1}{8}\right) \equiv \delta_{p_1}\delta_{p_2} \cdots \delta_{p_k} a_2(n) \pmod{2}.$$

Replacing n by $p_{k+1}^2 n + \frac{p_{k+1}(8j+p_{k+1})-1}{8}$ in the above expression and then using (3.8) for $p = p_{k+1}$, we get

$$a_2\left(p_1^2 p_2^2 \cdots p_k^2 p_{k+1}^2 n + \frac{p_1^2 p_2^2 \cdots p_k^2 p_{k+1}(8j + p_{k+1}) - 1}{8}\right)$$

$$\equiv \delta_{p_1}\delta_{p_2}\cdots\delta_{p_k}a_2\left(p_{k+1}^2 n + p_{k+1}j + \frac{p_{k+1}^2-1}{8}\right)$$
$$\equiv 0 \pmod 2,$$

when $j \not\equiv 0 \pmod{p_{k+1}}$. This completes the proof of Theorem 1.1(i).

3.7. Proof of Theorem 1.1(ii)

The proof is similar to the proof of Theorem 1.1(i). For $1 \le i \le k-1$, we note that

$$104p_i^2p_{i+1}^2\cdots p_k^2 n + 13p_i^2p_{i+1}^2\cdots p_k^2 - 7$$
$$= 104p_i^2\left(p_{i+1}^2\cdots p_k^2 n + \frac{p_{i+1}^2\cdots p_k^2-1}{8}\right) + 13p_i^2 - 7.$$

Thus, for $1 \le i \le k-1$, (3.11) implies

$$a_{13}(104p_i^2p_{i+1}^2\cdots p_k^2 n + 13p_i^2p_{i+1}^2\cdots p_k^2 - 7)$$
$$\equiv \delta_{p_i}a_{13}\left(104\left(p_{i+1}^2\cdots p_k^2 n + \frac{p_{i+1}^2\cdots p_k^2-1}{8}\right) + 6\right)$$
$$\equiv \delta_{p_i}a_{13}(104p_{i+1}^2\cdots p_k^2 n + 13p_{i+1}^2\cdots p_k^2 - 7) \pmod 2.$$

Also from (3.11), we have

$$a_{13}\left(104p_k^2 n + 13p_k^2 - 7\right) \equiv \delta_{p_k}a_{13}(104n+6) \pmod 2.$$

Therefore, from the above two congruences, we get

$$a_{13}(104p_1^2p_2^2\cdots p_k^2 n + 13p_1^2p_2^2\cdots p_k^2 - 7) \equiv \delta_{p_1}\delta_{p_2}\cdots\delta_{p_k}a_{13}(104n+6) \pmod 2.$$

Replacing n by $p_{k+1}^2 n + \frac{p_{k+1}(\epsilon_{p_{k+1}}j + p_{k+1})-1}{8}$ in the above expression and then using (3.10), we get

$$a_{13}(104p_1^2p_2^2\cdots p_k^2p_{k+1}^2 n + 13p_1^2p_2^2\cdots p_k^2p_{k+1}(\epsilon_{p_{k+1}}j + p_{k+1}) - 7)$$
$$\equiv \delta_{p_1}\delta_{p_2}\cdots\delta_{p_k}a_{13}(104p_{k+1}^2 n + 13(p_{k+1}(\epsilon_{p_{k+1}}j + p_{k+1}) - 1) + 6)$$
$$\equiv \delta_{p_1}\delta_{p_2}\cdots\delta_{p_k}a_{13}(104p_{k+1}^2 n + 13(p_{k+1}(\epsilon_{p_{k+1}}j + p_{k+1})) - 7)$$
$$\equiv 0 \pmod 2,$$

when $j \not\equiv 0 \pmod{p_{k+1}}$. This completes the proof of Theorem 1.1(ii).

3.8. Proof of Theorem 1.2

For any prime $p \equiv 7 \pmod 8$, we get from (3.2) that

$$b(pn) = (-1)\left(\frac{-2}{p}\right)b\left(\frac{n}{p}\right).$$

Let $r \not\equiv 0 \pmod p$. Replacing n by $8(p^k n + r) + 7$, we obtain

$$b(8(p^{k+1}n + pr) + 7p) = (-1)\left(\frac{-2}{p}\right)b\left(\frac{8(p^k n + r) + 7}{p}\right),$$

which can be rewritten as

$$b\left(8\left(p^{k+1}n+pr+\frac{7p-1}{8}\right)+1\right) = (-1)\left(\frac{-2}{p}\right)b\left(8\left(p^{k-1}n+\frac{8r+7-p}{8p}\right)+1\right). \tag{3.19}$$

We note here that $\frac{7p-1}{8}$ and $\frac{8r+7-p}{8p}$ are integers. Therefore, using (3.13) and (3.19), we get

$$a_2\left(p^{k+1}n+pr+\frac{7p-1}{8}\right) \equiv (-1)\left(\frac{-2}{p}\right)a_2\left(p^{k-1}n+\frac{8r+7-p}{8p}\right) \pmod 2, \tag{3.20}$$

and

$$a_{13}\left(104p^{k+1}n+104pr+91p-7\right) \equiv (-1)\left(\frac{-2}{p}\right)a_{13}\left(104p^{k-1}n+\frac{104r+91}{p}-7\right) \pmod 2. \tag{3.21}$$

3.9. Proof of Corollary 1.2

Let p be a prime such that $p \equiv 7 \pmod 8$. Choose a non negative integer r such that $8r+7=p^{2k-1}$. Replacing k by $2k-1$ in (3.20), we obtain

$$a_2\left(p^{2k}n+\frac{p^{2k}-1}{8}\right) \equiv (-1)\left(\frac{-2}{p}\right)a_2\left(p^{2k-2}n+\frac{p^{2k-2}-1}{8}\right) \equiv \cdots \equiv (-1)^k\left(\frac{-2}{p}\right)^k a_2(n) \pmod 2.$$

Replacing k by $2k-1$ in (3.21), we obtain

$$a_{13}\left(104p^{2k}n+13p^{2k}-7\right) \equiv (-1)\left(\frac{-2}{p}\right)a_{13}\left(104p^{2k-2}n+13p^{2k-2}-7\right) \equiv \cdots \equiv (-1)^k\left(\frac{-2}{p}\right)^k a_{13}(104n+6) \pmod 2.$$

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References

[1] B. C. Berndt, *Ramanujan's notebooks. Part III*, Springer, New York, 1991. https://doi.org/10.1007/978-1-4612-0965-2

[2] K. Das, *Parity results for 13-core partitions*, Mat. Vesnik **68** (2016), no. 3, 175–181.

[3] F. Garvan, D. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), no. 1, 1–17. https://doi.org/10.1007/BF01231493
[4] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 331–347. https://doi.org/10.1090/S0002-9947-96-01481-X
[5] M. D. Hirschhorn and J. A. Sellers, *Parity results for partitions wherein each part appears an odd number of times*, Bull. Aust. Math. Soc. **99** (2019), no. 1, 51–55. https://doi.org/10.1017/S0004972718001041
[6] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, second edition, Graduate Texts in Mathematics, 97, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-0909-6
[7] Y. Martin, *Multiplicative η-quotients*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4825–4856. https://doi.org/10.1090/S0002-9947-96-01743-6
[8] N. K. Meher and A. Jindal, *Arithmetic density and congruences for t-core partitions*, https://arxiv.org/pdf/2302.11830.pdf.
[9] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

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