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SUMMABILITY IN MUSIELAK–ORLICZ HARDY SPACES

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Abstract. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be a growth function and $H^\varphi(\mathbb{R}^n)$ the Musielak–Orlicz Hardy space defined via the non-tangential grand maximal function. A general summability method, the so-called θ-summability is considered for multi-dimensional Fourier transforms in $H^\varphi(\mathbb{R}^n)$. Precisely, with some assumptions on θ, the authors first prove that the maximal operator of the θ-means is bounded from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$. As consequences, some norm and almost everywhere convergence results of the θ-means, which generalizes the well-known Lebesgue's theorem, are then obtained. Finally, the corresponding conclusions of some specific summability methods, such as Bochner–Riesz, Weierstrass and Picard–Bessel summations, are also presented.

1. Introduction

The main purpose of this article is to investigate the so-called θ-summability, which is generated by a single function θ and includes many well-known summations, in the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$. Recall that the study on summability means was originally motivated by the convergence problem of the Dirichlet integrals which are defined by setting, for any $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$s_\delta f(x) := \int_{\mathbb{R}^n} \mathbf{1}_{\{z\in\mathbb{R}^n:|z|\le\delta\}}(\xi)\widehat{f}(\xi)e^{2\pi\imath x\cdot\xi}\,d\xi, \quad \forall\,\delta \in (0,\infty), \tag{1.1}$$

here and thereafter, $\imath := \sqrt{-1}$, $x \cdot \xi := \sum_{k=1}^n x_k\xi_k$ for any $x := (x_1,\ldots,x_n)$, $\xi := (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$, and $\widehat{f}$ denotes the *Fourier transform* of f, which is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi\imath x\cdot\xi}\,dx.$$

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First, Carleson [6] and Hunt [12] proved that, for any one-dimensional function $f \in L^p(\mathbb{R})$,

$$\lim_{\delta\to\infty} s_\delta f = f \qquad \text{almost everywhere,} \tag{1.2}$$

which now is known as one of the deepest results in harmonic analysis. Moreover, the convergence of (1.2) also holds true in the $L^p(\mathbb{R})$-norm (see, for instance, [11]). However, the convergence in (1.2) does not hold true for any higher dimensional function $f \in L^p(\mathbb{R}^n)$, except the norm convergence for $p = 2$ (see, for instance, Stein and Weiss [24] or Grafakos [11]). But more than this, the convergence in (1.2) does not hold true for $p = 1$ even when $n = 1$. This motivates one to replace the Dirichlet integrals by some summability means, which are defined via replacing the characteristic function in (1.1) by various functions with higher regularity [see (3.2) below]. Via doing this, one can extend (1.2) to the case $p \le 1$ for the classical Hardy spaces and also to the case $p = 1$ for the space $L^1(\mathbb{R}^n)$. This is just the main motivation for one to investigate the summability means.

It is well known that Stein, Taibleson and Weiss [22] proved for the Bochner–Riesz summability that the maximal operator σ_*^θ of the θ-means is bounded from the classical Hardy $H^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ with the index p greater than some constant p_0. This result has been extended to many other Hardy-type and other summability methods. For more progress about this topic, we refer the reader to [17, 18, 21, 22, 27, 29, 30] and references therein.

On the other hand, via the non-tangential grand maximal function, Ky [15] introduced the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which is a generalization of both the Orlicz–Hardy space (see [13, 25]) and the weighted Hardy space (see [10]), where $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a growth function (see Definition 2.2 below). It is worth noticing that some special Musielak–Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see, for instance, [3, 16]), and the endpoint estimates for both the div-curl lemma and the commutators of Calderón–Zygmund operators (see, for instance, [2, 14]). For more progress on the theory of Musielak–Orlicz-type spaces, we refer the reader to [1, 31] and references therein.

In this article, under some conditions on θ and φ, we show that the maximal operator σ_*^θ is bounded from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$. As a consequence, we prove some norm and almost everywhere convergence results for the θ-means. In this way, the well-known Lebesgue's theorem is generalized. As special cases of the θ-summation, we consider the Bochner–Riesz, Weierstrass and Picard–Bessel summations.

Finally, we make some conventions on notation. We always define $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbf{0}$ to be the *origin* of $\mathbb{R}^n$. For each $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and

$$\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The symbol C means a positive constant independent of the main parameters, but may vary in different settings. The *symbol* $f \lesssim h$ means $f \le Ch$ and, if $f \lesssim h \lesssim f$, then we write $f \sim h$. In addition, for any set $\Omega \subset \mathbb{R}^n$, we denote by $\mathbf{1}_\Omega$ its *characteristic function*, by $\Omega^{\complement}$ the set $\mathbb{R}^n \setminus \Omega$ and by $|\Omega|$ its *n-dimensional Lebesgue measure*. For any $t \in [1, \infty]$, we denote by t' its *conjugate index*, namely, $1/t + 1/t' = 1$ and by $\lfloor s \rfloor$ the *largest integer not greater than s* for any $s \in \mathbb{R}$.

2. Preliminaries

In this section, we recall the notion of the Musielak–Orlicz Hardy spaces via the non-tangential grand maximal functions.

We begin with the notion of the Orlicz functions (see, for instance, [15]). Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if it is non-decreasing, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and, for any $t \in (0, \infty)$, $\Phi(t) \in (0, \infty)$. The function Φ is said to be of *upper* (resp. *lower*) *type p* for some $p \in (-\infty, \infty)$ if there exists a positive constant C such that, for any $s \in [1, \infty)$ (resp. $s \in [0, 1)$) and $t \in [0, \infty)$,

$$\Phi(st) \le Cs^p \Phi(t).$$

For a given function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of *uniformly upper* (resp. *lower*) *type p* for some $p \in (-\infty, \infty)$ if there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $s \in [1, \infty)$ (resp. $s \in [0, 1)$) and $t \in [0, \infty)$,

$$\varphi(x, st) \le Cs^p \varphi(x, t).$$

Moreover, the *critical uniformly lower type index* $i(\varphi)$ of φ is defined by

$$i(\varphi) := \sup \{p \in (-\infty, \infty) : \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

We now recall the classes of uniformly Muckenhoupt weights; see [15].

Definition 2.1. Let $p \in [1, \infty)$. A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $p \in [1, \infty)$, denoted by $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$, if, when $p \in (1, \infty)$,

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{\frac{1}{|B|}\int_B \varphi(x, t)\, dx\right\} \left\{\frac{1}{|B|}\int_B [\varphi(y, t)]^{-\frac{1}{p-1}}\, dy\right\}^{p-1} < \infty$$

and, when $p = 1$,

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \left\{\frac{1}{|B|}\int_B \varphi(x, t)\, dx\right\} \left\{\operatorname*{ess\,sup}_{y\in B} [\varphi(y, t)]^{-1}\right\} < \infty,$$

where the first supremums are taken over all $t \in [0, \infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$. Moreover, let

$$\mathcal{A}_\infty(\mathbb{R}^n) := \bigcup_{p\in[1,\infty)} \mathcal{A}_p(\mathbb{R}^n).$$

The *critical weight index* of $\varphi \in \mathcal{A}_\infty(\mathbb{R}^n)$ is defined by

$$q(\varphi) := \inf\{p \in [1,\infty) : \varphi \in \mathcal{A}_p(\mathbb{R}^n)\}. \tag{2.2}$$

The following notion of growth functions comes from [15].

Definition 2.2. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if it satisfies the following conditions:

- (i) φ is a *Musielak–Orlicz function*, namely,
  - (i)$_1$ the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  - (i)$_2$ the function $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.
- (ii) $\varphi \in \mathcal{A}_\infty(\mathbb{R}^n)$.
- (iii) φ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Throughout this article, we always assume that φ is a growth function as in Definition 2.2 and, for any measurable subset $\Omega \subset \mathbb{R}^n$ and $t \in [0,\infty)$, define

$$\varphi(\Omega,t) := \int_\Omega \varphi(x,t)\,dx.$$

Recall also that the Musielak–Orlicz space $L^\varphi(\mathbb{R}^n)$ is defined to be the collection of all measurable functions f on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi(x,|f(x)|/\lambda)\,dx < \infty$ for some $\lambda \in (0,\infty)$, equipped with the *Luxemburg–Nakano quasi-norm* $\|\cdot\|_{L^\varphi(\mathbb{R}^n)}$, defined by setting, for any $f \in L^\varphi(\mathbb{R}^n)$,

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi(x,|f(x)|/\lambda)\,dx \le 1\right\}.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For any $\tau \in \mathbb{N}$, let

$$\mathcal{S}_\tau(\mathbb{R}^n) := \left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le \tau+1} \sup_{x \in \mathbb{R}^n} (1+|x|)^{(\tau+2)(n+1)} |\partial^\alpha \phi(x)| \le 1\right\}.$$

In what follows, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $t \in (0,\infty)$, $\phi_t(\cdot) := t^{-n}\phi(\frac{\cdot}{t})$. Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *non-tangential grand maximal function* f_τ^* of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f_\tau^*(x) := \sup_{\phi \in \mathcal{S}_\tau(\mathbb{R}^n)} \sup_{|y-x|<t,\, t \in (0,\infty)} |f * \phi_t(y)|.$$

The following notion of the Musielak–Orlicz Hardy spaces is just [15, Definition 5.1].

Definition 2.3. Let $\tau \in \mathbb{N}$ and φ be a growth function as in Definition 2.2. The *Musielak–Orlicz Hardy space* $H_\tau^\varphi(\mathbb{R}^n)$ is defined as

$$H_\tau^\varphi(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : f_\tau^* \in L^\varphi(\mathbb{R}^n)\}$$

and, for any $f \in H_\tau^\varphi(\mathbb{R}^n)$, let

$$\|f\|_{H_\tau^\varphi(\mathbb{R}^n)} := \|f_\tau^*\|_{L^\varphi(\mathbb{R}^n)}.$$

Remark 2.4. (i) Although the quasi-norm of $H^\varphi_\tau(\mathbb{R}^n)$ in Definition 2.3 depends on τ, it follows from [15, Theorems 5.1 and 5.2] that the space $H^\varphi_\tau(\mathbb{R}^n)$ is independent of the choice of τ as long as $\tau \in \mathbb{N} \cap [\tau(\varphi), \infty)$, Here and thereafter, for any given growth function φ,

$$\tau(\varphi) := \left\lfloor n \left[\frac{q(\varphi)}{i(\varphi)} - 1\right]\right\rfloor \tag{2.3}$$

with $q(\varphi)$ and $i(\varphi)$, respectively, as in (2.2) and (2.1). Thus, we always denote simply by $H^\varphi(\mathbb{R}^n)$ the Musielak–Orlicz Hardy space.

(ii) When $p \in (0, 1]$ and

$$\varphi(x, t) := t^p, \quad \forall\, x \in \mathbb{R}^n \text{ and } t \in (0, \infty),$$

the space $H^\varphi(\mathbb{R}^n)$ goes back to the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [8].

(iii) When $p \in (0, 1]$ and

$$\varphi(x, t) := w(x)t^p, \quad \forall\, x \in \mathbb{R}^n \text{ and } t \in [0, \infty),$$

where $w \in A_\infty(\mathbb{R}^n)$ is the Muckenhoupt weight, the space $H^\varphi(\mathbb{R}^n)$ coincides with the classical weighted Hardy space of García-Cuerva [10], which includes the classical Orlicz–Hardy space of Janson [13] as a special case.

3. Boundedness of maximal θ-operators in $H^\varphi(\mathbb{R}^n)$

Recall that, for any given $p \in [1, 2]$ and any $f \in L^p(\mathbb{R}^n)$, the Fourier inversion formula, namely,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(t) e^{2\pi\imath x\cdot t}\, dt, \quad \forall\, x \in \mathbb{R}^n,$$

holds true if $\widehat{f} \in L^1(\mathbb{R}^n)$. This motivates the succeeding definition of θ-summability of the Fourier transforms, which was considered in a great number of monographs and articles; see, for instance, Butzer and Nessel [5], Grafakos [11], Trigub and Belinsky [26] as well as Feichtinger and Weisz [9, 27–29] and references therein. We always assume that

$$\theta \in C_0(\mathbb{R}), \qquad \theta(|\cdot|) \in L^1(\mathbb{R}^n), \qquad \theta(0) = 1 \quad \text{and} \quad \theta \text{ is even}, \tag{3.1}$$

where $C_0(\mathbb{R})$ denotes the *set of all continuous functions f satisfying that*

$$\lim_{|x|\to\infty} |f(x)| = 0.$$

The *m-th θ-mean* of the function $f \in L^p(\mathbb{R}^n)$, with $p \in [1, 2]$, is defined by setting, for any $m \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\sigma^\theta_m f(x) := \int_{\mathbb{R}^n} \theta\left(\frac{|u|}{m}\right) \widehat{f}(u) e^{2\pi\imath x\cdot u}\, du. \tag{3.2}$$

This integral is well defined because $\theta \in L^p(\mathbb{R})$ with $p \in [1, 2]$ and $\widehat{f} \in L^{p'}(\mathbb{R}^n)$.

Let $\theta_0(x) := \theta(|x|)$ for any $x \in \mathbb{R}^n$ and assume that

$$\widehat{\theta_0} \in L^1(\mathbb{R}^n). \tag{3.3}$$

For an integrable function f, $m \in (0, \infty)$ and $x \in \mathbb{R}^n$, we can rewrite $\sigma_m^\theta f(x)$ as

$$\sigma_m^\theta f(x) = \int_{\mathbb{R}^n} f(x-t) K_m^\theta(t)\, dt = f * K_m^\theta(x), \tag{3.4}$$

where the *m-th θ-kernel* is defined by setting, for any $t \in \mathbb{R}^n$,

$$K_m^\theta(t) := \int_{\mathbb{R}^n} \theta\left(\frac{|u|}{m}\right) e^{2\pi\imath t\cdot u}\, du = m^n \widehat{\theta_0}(mt).$$

It is easy to check that these two definitions in (3.2) and (3.4) coincide for any $f \in L^p(\mathbb{R}^n)$ with $p \in [1, 2]$. We can extend the definition of the θ-means to any $f \in H^\varphi(\mathbb{R}^n)$ by setting, for any $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := f * K_m^\theta(x), \tag{3.5}$$

where $m \in (0, \infty)$. Furthermore, the *maximal θ-operator* σ_*^θ is defined by setting, for any $f \in H^\varphi(\mathbb{R}^n)$,

$$\sigma_*^\theta f := \sup_{m \in (0,\infty)} \left|\sigma_m^\theta f\right|.$$

The main result of this article is the following boundedness of maximal θ-operators from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.

Theorem 3.1. *Let θ and θ_0 be, respectively, as in* (3.1) *and* (3.3) *satisfying that there exists a positive constant $\beta \in (1, \infty)$ such that, for any $\alpha \in (\mathbb{Z}_+)^n$ and $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\left|\partial^\alpha \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)} |x|^{-\beta}, \tag{3.6}$$

where the positive constant $C_{(\alpha,\beta)}$ is independent of x. Assume that φ is a growth function such that

$$\frac{i(\varphi)}{q(\varphi)} \in \left(\frac{n}{\beta}, \infty\right), \tag{3.7}$$

where $i(\varphi)$ and $q(\varphi)$ are, respectively, as in (2.1) *and* (2.2). *Then there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

To show Theorem 3.1, we need some technical lemmas. We begin with recalling the notion of the space $L^q_\varphi(E)$. For any measurable subset $E \subset \mathbb{R}^n$, the *space* $L^q_\varphi(\Omega)$ is defined to be the set of all measurable functions f on Ω such that

$$\|f\|_{L^q_\varphi(\Omega)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\varphi(\Omega,t)} \displaystyle\int_\Omega |f(x)|^q \varphi(x,t)\, dx\right]^{1/q} < \infty & \text{if } q \in [1,\infty), \\ \|f\|_{L^\infty(\Omega)} < \infty & \text{if } q = \infty, \end{cases} \tag{3.8}$$

where φ is a given growth function.

The succeeding notions of both atoms and Musielak–Orlicz finite atomic Hardy space are from [15].

Definition 3.2. Let φ be a growth function as in Definition 2.2 and $q(\varphi)$ as in (2.2).

(i) An triplet (φ, q, s) is said to be admissible if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{Z}_+ \cap [\tau(\varphi), \infty)$, where $\tau(\varphi)$ is as in (2.3).

(ii) For a given admissible triplet (φ, q, s), a measurable function a on $\mathbb{R}^n$ is called a *Musielak–Orlicz* (φ, q, s)*-atom* [shortly, a (φ, q, s)*-atom*] if

(ii)$_1$ supp $a \subset B$, where $B \subset \mathbb{R}^n$ is a ball;

(ii)$_2$ $\|a\|_{L^q_\varphi(\mathbb{R}^n)} \le \|\mathbf{1}_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

(ii)$_3$ for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \le s$, $\int_{\mathbb{R}^n} a(x)x^\beta\, dx = 0$.

(iii) For a given admissible triplet (φ, q, s), the *Musielak–Orlicz finite atomic Hardy space* $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $I \in \mathbb{N}$, a sequence $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ, q, s)-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported respectively in $\{B_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{R}^n$ such that $f = \sum_{i=1}^I \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, for any $f \in H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\{\Lambda\left(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}}\right)\right\}, \tag{3.9}$$

where the infimum is taken over all finite decompositions of f as above and, for any $I \in \mathbb{N}$,

$$\Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) : \sum_{i\in[1,I]\cap\mathbb{N}} \varphi\left(B_i,\, \frac{|\lambda_i|}{\lambda\|\mathbf{1}_{B_i}\|_{L^\varphi(\mathbb{R}^n)}}\right) \le 1\right\}.$$

In addition, the space $L^\infty_{c,s}(\mathbb{R}^n)$, with $s \in \mathbb{N}$, is defined to be the set of all functions $f \in L^\infty(\mathbb{R}^n)$ with compact support such that, for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le s$, $\int_{\mathbb{R}^n} f(x)x^\gamma\, dx = 0$. Then we have some conclusions as follows, which can be found in [4].

Lemma 3.3. *Let φ be as in Definition 2.2, $q \in (q(\varphi), \infty)$ and $s \in \mathbb{N}\cap[\tau(\varphi), \infty)$, where $q(\varphi)$ and $\tau(\varphi)$ are as in (2.3). Then,*

(i) *for any $f \in L^\infty_{c,s}(\mathbb{R}^n)$, there exist some $I \in \mathbb{N}$, a sequence $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ, q, s)-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, such that*

$$f = \sum_{i=1}^I \lambda_i a_i$$

holds true both in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere, and

$$\Lambda\left(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}}\right) \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

(ii) *$L^\infty_{c,s}(\mathbb{R}^n)$ is dense in $H^\varphi(\mathbb{R}^n)$.*

The following items are just, respectively, [15, Lemma 4.5(i), Lemma 4.1(i) and Lemma 4,3(i)].

Lemma 3.4. *Let φ be a growth function as in Definition 2.2.*

(i) *If $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ with some $q \in [1, \infty)$, then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$, subset $\Omega \subset B$ and $t \in (0, \infty)$,*

$$\frac{\varphi(B,t)}{\varphi(\Omega,t)} \leq C \left[\frac{|B|}{|\Omega|}\right]^q.$$

(ii) *There exists a positive constant C such that, for any $\{(x, t_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}^n \times [0, \infty)$,*

$$\varphi\left(x, \sum_{i \in \mathbb{N}} t_i\right) \leq C \sum_{i \in \mathbb{N}} \varphi(x, t_i).$$

(iii) *For any given positive constant C, there exists a positive constant $\widetilde{C}$ such that, for any $t \in (0, \infty)$ and measurable function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{t}\right) dx \leq C \quad \textit{implies} \quad \|f\|_{L^{\varphi}(\mathbb{R}^n)} \leq \widetilde{C} t.$$

Recall that, for any locally integrable function f, the *Hardy–Littlewood maximal function* $M_{\mathrm{HL}}(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy, \tag{3.10}$$

where the supremum is taken over all balls $B \ni x$.

We also need the boundedness of M_{HL} on the space $L^{\varphi}(\mathbb{R}^n)$, which comes from [19, Corollary 2.8].

Lemma 3.5. *Let φ be a Musielak–Orlicz function with uniformly lower type p_{φ}^- and uniformly upper type p_{φ}^+ satisfying $q(\varphi) < p_{\varphi}^- \leq p_{\varphi}^+ < \infty$, where $q(\varphi)$ is as in* (2.2)*. Then the Hardy–Littlewood maximal operator M_{HL} is bounded on $L^{\varphi}(\mathbb{R}^n)$ and there exists a positive constants C such that, for any $f \in L^{\varphi}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi\left(x, M_{\mathrm{HL}} f(x)\right) dx \leq C \int_{\mathbb{R}^n} \varphi(x, |f(x)|) \, dx.$$

By [27, (5.19)] and [7, Lemma 3.2(ii)], we easily obtain the following conclusion; the details are omitted.

Lemma 3.6. *Assume that θ and θ_0 are, respectively, as in* (3.1) *and* (3.3)*. Let $q \in (1, \infty]$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any locally integrable function f and $t \in (0, \infty)$,*

$$\int_{\mathbb{R}^n} \left[\sigma_*^{\theta} f(x)\right]^q \varphi(x,t) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t) \, dx.$$

Next, we prove Theorem 3.1.

Proof of Theorem 3.1. Let all notation be the same as those in Theorem 3.1 and (φ, q, s) an admissible triplet. We show this theorem by three steps.

Step 1. In this step, we aim to prove that there exists a positive constant $C_{(\varphi,\,q,\,s)}$, depending on φ, q and s, such that, for any $\lambda \in (0, \infty)$ and (φ, q, s)-atom a supported in some ball $B \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi\left(x,\, \lambda \sigma_*^\theta(a)(x)\right)\, dx \le C_{(\varphi,\,q,\,s)} \varphi\left(B,\, \lambda \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right). \tag{3.11}$$

To this end, we rewrite

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x,\, \lambda\sigma_*^\theta(a)(x)\right)\, dx &= \int_{2B} \varphi\left(x,\, \lambda\sigma_*^\theta(a)(x)\right)\, dx + \int_{(2B)^\complement} \cdots \\ &=: \mathrm{I}_1 + \mathrm{I}_2,\end{aligned} \tag{3.12}$$

where $2B$ denotes the ball with the same center and with the twice radius of B.

When $q \in (q(\varphi), \infty)$, from the fact that φ is non-decreasing and of uniformly upper type 1, Lemma 3.4(i), the Hölder inequality, Lemma 3.6 and Definition 3.2(ii), it follows that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_{2B} \left[\frac{\sigma_*^\theta(a)(x)}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}} + 1\right] \varphi\left(x,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\, dx \\ &\lesssim \varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) + \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \left\{\int_{2B} \left[\sigma_*^\theta(a)(x)\right]^q \right. \\ &\quad \left. \times\, \varphi\left(x,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\, dx\right\}^{1/q} \left[\varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\right]^{(q-1)/q} \\ &\lesssim \varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) + \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}\, \|a\|_{L^q_\varphi(B)} \varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \\ &\lesssim \varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).\end{aligned} \tag{3.13}$$

To deal with I_2, by an argument similar to those used in the proofs of [21, (5.10)] and [20, (3.6)] with some slight modifications (see also [30] for the variable atoms), we conclude that, for any $x \in (2B)^\complement$,

$$\sigma_*^\theta(a)(x) \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}\left(\mathbf{1}_B\right)(x)\right]^{\beta/n}. \tag{3.14}$$

Note that $n/\beta < \frac{i(\varphi)}{q(\varphi)}$ [see (3.7)]. Then there exist two numbers $\kappa \in (q(\varphi), \infty)$ and $\nu \in (0, i(\varphi))$ such that $n\kappa/\beta < \nu$, $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^n)$ and φ is of uniformly lower type ν. Thus, $\widetilde{\varphi}(x, t) := \varphi(x, t^{\beta/n})$ is of uniformly lower type $\nu\beta/n$. Moreover, by (3.14), Lemma 3.5 and the fact that $\nu\beta/n > \kappa > q(\varphi)$, we find that

$$\begin{aligned}\mathrm{I}_2 &\lesssim \int_{(2B)^\complement} \widetilde{\varphi}\left(x,\, \lambda^{n/\beta}\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-n/\beta} M_{\mathrm{HL}}\left(\mathbf{1}_B\right)(x)\right)\, dx \\ &\lesssim \int_{\mathbb{R}^n} \widetilde{\varphi}\left(x,\, \lambda^{n/\beta}\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-n/\beta}\, \mathbf{1}_B(x)\right)\, dx \\ &\sim \varphi\left(B,\, \lambda\, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).\end{aligned} \tag{3.15}$$

This, combined with (3.12) and (3.13), finishes the proof of (3.11) for the case when $q \in (q(\varphi), \infty)$.

If $q = \infty$, then, for I_1, similarly to (3.13), for any $\lambda \in (0, \infty)$, we have

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \int_{2B} \left[\frac{\sigma_*^\theta(a)(x)}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}} + 1 \right] \varphi\left(x, \lambda \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) dx \\
&\lesssim \varphi\left(B, \lambda \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) + \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \|a\|_{L^\infty(\mathbb{R}^n)} \varphi\left(B, \lambda \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \\
&\lesssim \varphi\left(B, \lambda \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right).
\end{aligned}$$

Observe that (3.15) also holds true for $q = \infty$. The proof of (3.11) is completed.

Step 2. Let $q \in (q(\varphi), \infty)$, where $q(\varphi)$ is as in (2.2). In this step, we prove that there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ such that, for any $f \in L^\infty_{c,s}(\mathbb{R}^n)$,

$$\left\|\sigma_*^\theta f\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \|f\|_{H^\varphi(\mathbb{R}^n)}. \tag{3.16}$$

For this purpose, for any $f \in L^\infty_{c,s}(\mathbb{R}^n)$, by Lemma 3.3(i), we know that there exist some $I \in \mathbb{N}$, a sequence $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ, q, s)-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported respectively in $\{B_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{R}^n$, such that $f = \sum_{i=1}^I \lambda_i a_i$ holds true both in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere, and

$$\Lambda\left(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}}\right) \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)}$$

and

$$\sum_{i\in[1,I]\cap\mathbb{N}} \varphi\left(B_i, \frac{|\lambda_i| \|\mathbf{1}_{B_i}\|_{L^\varphi(\mathbb{R}^n)}^{-1}}{\Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}})}\right) = 1.$$

From this, Lemma 3.4(ii) and (3.11), we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\left(x, \frac{\sigma_*^\theta f(x)}{\Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}})}\right) dx &\lesssim \sum_{i\in[1,I]\cap\mathbb{N}} \int_{\mathbb{R}^n} \varphi\left(x, \frac{|\lambda_i| \sigma_*^\theta(a_i)(x)}{\Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}})}\right) dx \\
&\lesssim \sum_{i\in[1,I]\cap\mathbb{N}} \varphi\left(B_i, \frac{|\lambda_i| \|\mathbf{1}_{B_i}\|_{L^\varphi(\mathbb{R}^n)}^{-1}}{\Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}})}\right) \lesssim 1,
\end{aligned}$$

which, together with Lemma 3.4(iii), further implies that

$$\left\|\sigma_*^\theta f\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_i a_i\}_{i\in[1,I]\cap\mathbb{N}}) \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

This finishes the proof of Step 2.

Step 3. In this step, we show that (3.16) holds true for any $f \in H^\varphi(\mathbb{R}^n)$. To do this, let $f \in H^\varphi(\mathbb{R}^n)$. Then, by the density of the set $L^\infty_{c,s}(\mathbb{R}^n)$ in $H^\varphi(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^\varphi(\mathbb{R}^n)}$ (see Lemma 3.3(ii)), we conclude that there exists a Cauchy sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^\infty_{c,s}(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \|f_j - f\|_{H^\varphi(\mathbb{R}^n)} = 0.$$

From this, the linearity of σ_*^θ and the fact that (3.16) holds true on the space $L_{c,s}^\infty(\mathbb{R}^n)$, it follows that, as $j,\, i\to\infty$,

$$\left\|\sigma_*^\theta(f_j)-\sigma_*^\theta(f_i)\right\|_{H^\varphi(\mathbb{R}^n)}=\left\|\sigma_*^\theta(f_j-f_i)\right\|_{H^\varphi(\mathbb{R}^n)}\lesssim\|f_j-f_i\|_{H^\varphi(\mathbb{R}^n)}\to 0,$$

which implies that $\{\sigma_*^\theta(f_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^\varphi(\mathbb{R}^n)$. By this and the completeness of $H^\varphi(\mathbb{R}^n)$ (see [15, Proposition 5.2]), we find that there exists some $g\in H^\varphi(\mathbb{R}^n)$ such that $g=\lim_{j\to\infty}\sigma_*^\theta(f_j)$ in $H^\varphi(\mathbb{R}^n)$. Let $\sigma_*^\theta(f):=g$. Then we know from (3.16) that $\sigma_*^\theta(f)$ is well defined and, moreover, for any $f\in H^\varphi(\mathbb{R}^n)$,

$$\begin{aligned}\left\|\sigma_*^\theta(f)\right\|_{H^\varphi(\mathbb{R}^n)}&\lesssim\limsup_{j\to\infty}\left[\left\|\sigma_*^\theta(f)-\sigma_*^\theta(f_j)\right\|_{H^\varphi(\mathbb{R}^n)}+\left\|\sigma_*^\theta(f_j)\right\|_{H^\varphi(\mathbb{R}^n)}\right]\\&\lesssim\limsup_{j\to\infty}\left\|\sigma_*^\theta(f_j)\right\|_{H^\varphi(\mathbb{R}^n)}\\&\lesssim\lim_{j\to\infty}\|f_j\|_{H^\varphi(\mathbb{R}^n)}\sim\|f\|_{H^\varphi(\mathbb{R}^n)}.\end{aligned}$$

This proves that (3.16) holds true for any $f\in H^\varphi(\mathbb{R}^n)$ and hence finishes the proof of Theorem 3.1. ☐

Remark 3.7. If φ is as in Remark 2.4(ii), then $\frac{i(\varphi)}{q(\varphi)}=p$ and Theorem 3.1 goes back to the classical result with $\beta\in(n,\infty)$ and $p\in(n/\beta,\infty)$ (see Weisz [29]). The classical result was proved in a special case, namely, for the Bochner–Riesz means, in Stein et al. [23] and. For the same case, a counterexample was also given in [23] to illustrate that the same conclusion is not true for $p\in(0,n/\beta]$.

As applications of Theorem 3.1, we obtain some convergence results.

Corollary 3.8. *With the same assumptions as in Theorem 3.1, if $f\in H^\varphi(\mathbb{R}^n)$, then $\sigma_m^\theta f$ converges almost everywhere as well as in the $L^\varphi(\mathbb{R}^n)$ quasi-norm as $m\to\infty$.*

Proof. Assume first that $g\in L_{c,s}^\infty(\mathbb{R}^n)$. Then, it follows from (3.5) that, for any $x\in\mathbb{R}^n$,

$$\sigma_m^\theta g(x)=\int_{\mathbb{R}^n}g\left(x-\frac{t}{m}\right)\widehat{\theta_0}(t)\,dt.$$

Observe that $\lim_{m\to\infty}\frac{t}{m}=\mathbf{0}$ for any $t\in\mathbb{R}^n$, $\widehat{\theta_0}$ is integrable on $\mathbb{R}^n$ (see (3.3)) and $g\in L^\infty(\mathbb{R}^n)$. By the Lebesgue dominated convergence theorem, we conclude that, for almost every $x\in\mathbb{R}^n$,

$$\lim_{m\to\infty}\sigma_m^\theta g(x)=\int_{\mathbb{R}^n}g(x)\widehat{\theta_0}(t)\,dt=g(x)\theta_0(\mathbf{0})=g(x). \tag{3.17}$$

The above convergence also holds true in the $L^\varphi(\mathbb{R}^n)$ quasi-norm due to $g\in L_{c,s}^\infty(\mathbb{R}^n)\subset H^\varphi(\mathbb{R}^n)$ which implies $\sigma_*^\theta g\in L^\varphi(\mathbb{R}^n)$.

Note that the set $L_{c,s}^\infty(\mathbb{R}^n)$ is dense in $H^\varphi(\mathbb{R}^n)$ [see Lemma 3.3(ii)]. Therefore, for any given $f\in H^\varphi(\mathbb{R}^n)$ and any $\varepsilon\in(0,\infty)$, there exists a function

$g \in L^\infty_{c,s}(\mathbb{R}^n)$ such that

$$\|f-g\|_{H^\varphi(\mathbb{R}^n)} < \varepsilon. \tag{3.18}$$

For any $L \in (0,\infty)$ and $x \in \mathbb{R}^n$, let

$$h_L(x) := \sup_{m,r\in[L,\infty)} \left|\sigma_m^\theta f(x) - \sigma_r^\theta f(x)\right| \quad \text{and} \quad h(x) := \lim_{L\to\infty} h_L(x).$$

To show Corollary 3.8, it suffices to prove that $h = 0$ almost everywhere. To see this, for any $L \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_L(x) \le & \sup_{m\in[L,\infty)} \left|\sigma_m^\theta (f-g)(x)\right| \\ & + \sup_{m,r\in[L,\infty)} \left|\sigma_m^\theta g(x) - \sigma_r^\theta g(x)\right| + \sup_{r\in[L,\infty)} \left|\sigma_r^\theta (g-f)(x)\right|. \end{aligned}$$

This implies that, for any $x \in \mathbb{R}^n$,

$$h(x) \le 2\sigma_*^\theta (f-g)(x).$$

Combining this with Theorem 3.1 and (3.18), we obtain

$$\|h\|_{L^\varphi(\mathbb{R}^n)} \le 2\left\|\sigma_*^\theta (f-g)\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f-g\|_{H^\varphi(\mathbb{R}^n)} \lesssim \varepsilon.$$

Since $\varepsilon \in (0,\infty)$ is arbitrary, it follows that $h = 0$ almost everywhere, which completes the proof of Corollary 3.8. □

The following Corollary 3.9 can be deduced from Theorem 3.1 and an argument similar to that used in the proof of [21, Corollary 2.20]; the details are omitted.

Corollary 3.9. *With the same assumptions as in Theorem 3.1, if* $f \in H^\varphi(\mathbb{R}^n)$ *and there exists a subset* $I \subset \mathbb{R}^n$ *such that the restriction* $f|_I \in L^\Phi(I)$*, where* Φ *is some growth function with* $\frac{i(\Phi)}{q(\Phi)} \in [1,\infty)$*, then*

$$\lim_{m\to\infty} \sigma_m^\theta f(x) = f(x)$$

for almost every $x \in I$ *as well as in the* $L^\varphi(I)$ *quasi-norm.*

4. Some summability methods

As special cases, we consider some summability methods.

4.1. Bochner–Riesz summation

For any $\alpha \in (0,\infty)$ and $\gamma \in \mathbb{N}$, the *Bochner-Riesz summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := \begin{cases} (1-|t|^\gamma)^\alpha & \text{when } |t| \in [0,1), \\ 0 & \text{when } |t| \in [1,\infty). \end{cases} \tag{4.1}$$

The next lemma can be found in Stein and Weiss [24] (see also Weisz [29]).

Lemma 4.1. *Let θ_0 be as in* (4.1)*. Then the conditions* (3.1) *and* (3.3) *are satisfied if $\alpha \in (\frac{n-1}{2}, \infty)$ and, for any $\beta \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\left|\partial^\beta \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)} \, |x|^{-n/2-\alpha-1/2},$$

where $C_{(\alpha,\beta)}$ is a positive constant independent of x.

By Lemma 4.1 and Theorem 3.1, we immediately have the following conclusion, which is also obtained in [18]; the details are omitted.

Theorem 4.2. *Let φ be a growth function and θ_0 as in* (4.1)*. If*

$$\alpha \in \left(\frac{n-1}{2}, \infty\right) \quad \text{and} \quad \frac{i(\varphi)}{q(\varphi)} \in \left(\frac{n}{n/2+\alpha+1/2}, \infty\right),$$

where $i(\varphi)$ and $q(\varphi)$ are, respectively, as in (2.1) *and* (2.2)*, then there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \, \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

Remark 4.3. Let θ_0 be as in (4.1). Then, in this special case, the corresponding conclusions in Corollaries 3.8 and 3.9 hold true as well.

4.2. Weierstrass summation

The *Weierstrass summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := e^{-|t|^2/2}. \tag{4.2}$$

It is known that $\widehat{\theta_0}(x) = e^{-|x|^2/2}$ for any $x \in \mathbb{R}^n$. Then it is easy to verify the following result, which also can be found in [21, Lemma 2.27].

Lemma 4.4. *Let θ_0 be as in* (4.2)*. Then the conditions* (3.1) *and* (3.3) *are satisfied and, for any $\beta \in (1, \infty)$, $\alpha \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\left|\partial^\alpha \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)} \, |x|^{-\beta},$$

where $C_{(\alpha,\beta)}$ is a positive constant independent of x.

By this lemma and Theorem 3.1, we obtain the following Theorem 4.5; the details are omitted.

Theorem 4.5. *Let θ_0 be as in* (4.2)*. If φ is a growth function, then there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \, \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

Moreover, the corresponding conclusions in Corollaries 3.8 and 3.9 hold true as well.

4.3. Picard–Bessel summation

The *Picard–Bessel summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := \frac{1}{(1+|t|^2)^{\frac{n+1}{2}}}. \tag{4.3}$$

For more summability methods, we refer the reader to [27, 29] and their references.

Remark 4.6. Let θ_0 be as in (4.3). Then Lemma 4.4, Theorem 4.5 as well as the corresponding conclusions in Corollaries 3.8 and 3.9 hold true.

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