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TOPOLOGICALLY STABLE POINTS AND UNIFORM LIMITS

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Abstract. In this paper we study a pointwise version of Walters topological stability in the class of homeomorphisms on a compact metric space. We also show that if a sequence of homeomorphisms on a compact metric space is uniformly expansive with the uniform shadowing property, then the limit is expansive with the shadowing property and so topologically stable. Furthermore, we give examples to illustrate our results.

1. Introduction and preliminaries

The concepts of the shadowing property, expansivity and topological stability play an important role in the qualitative theory of dynamical systems (see [1, 13]). Walters proved that every expansive homeomorphism with the shadowing property is topologically stable (see [15, Theorme 4]). Variant basic notions of dynamical systems were studied in the pointwise viewpoint (see [2, 3, 5–7, 11]). Morales [11] introduced the notion of a shadowable point and proved that the shadowing property is equivalent to all points to be shadowable in the class of homeomorphisms on a compact metric space. Also Koo *et al.* [7] decomposed the topological stability (in the sense of Walters in [15]) into the corresponding notion for points and showed that every shadowable point of an expansive homeomorphism of a compact metric space is topologically stable. To present well known notions of dynamical systems and our main results that is to extend the topological stability in the sense of Walters to pointswise viewpoint in the class of homeomorphisms on compact metric spaces, let us recall basic notions of dynamical systems which is used in sequential.

Let X be a compact metric space with metric d and $f : X \to X$ be a homeomorphism.

For any $x \in X$, the set $\{f^n(x)\}_{n\in\mathbb{Z}}$ is called the *orbit* of x under f denoted by $O_f(x)$. For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is said to be a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is called to be *ε-traced* (or *shadowed*) by a point $z \in X$ if $d(f^i(z), x_i) < \varepsilon$ for

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all $i \in \mathbb{Z}$. A subset B of X is said to be *f-invariant* for a homeomorphism f if $f(B) = B$.

We say that a homeomorphism $f : X \to X$ has the *shadowing property through* a subset K of X if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f *through* K (i.e., $x_0 \in K$) can be ε-shadowed (see [13]). We say that a homeomorphism $f : X \to X$ has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is ε-shadowed by a point in X. The theory of the shadowing property has been widely studied for the class of homeomorphisms on a compact metric space (see [1, 13]).

We recall some basic notions of dynamical systems.

First we introduce the notion of shadowable points for homeomorphisms which is defined to be a point such that the shadowing lemma holds for pseudo orbits through the point.

We say that a point $x \in X$ is a *shadowable point* if for every $\epsilon > 0$ there exists $\delta_x > 0$ such that every δ_x-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ with $x_0 = x$ can be ϵ-shadowed by some point (see [11, Definition 1.1]). The set of shadowable points of f is denoted by $\mathrm{Sh}(f)$. We see that a homeomorphism $f : X \to X$ has the shadowing property if and only if $\mathrm{Sh}(f) = X$ (see [11, Theorem 1.1(2)]).

We recall that a homeomorphism $f : X \to X$ is *expansive* [14] if there is $e > 0$ (called the *expansivity constant*) such that if $d(f^n(x), f^n(y)) \le e$ for all $n \in \mathbb{Z}$, whenever $x, y \in X$, then $x = y$.

Next we introduce the notion of expansivity from pointwise viewpoint.

We say that a point $x \in X$ is *uniformly expansive* if there exist a neighborhood U of x and a constant $e > 0$ such that if $d(f^n(z), f^n(y)) \le e$ for all $n \in \mathbb{Z}$, whenever $y, z \in U$, then $y = z$. Denote by $\mathrm{Exp}_u(f)$ the set of uniformly expansive points. Clearly, a homeomorphism $f : X \to X$ is expansive if and only if every point of X is uniformly expansive (see [3]).

The *C^0-distance* between maps $f, g : A \subset X \to X$ is defined by

$$d_{C^0}(f, g) = \sup_{x\in A} d(f(x), g(x)).$$

We recall that a homeomorphism $f : X \to X$ is *topologically stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \le \delta$ there is a continuous map $k : X \to X$ such that $d_{C^0}(k, i_X) \le \varepsilon$ and $f \circ k = k \circ g$ (see [7]). Here i_X is the identity map of X.

Also we give the notion of topologically stable points of a homeomorphism on a compact metric space (see [7, Definition 2.3]).

We say that a point $x \in X$ is a *topologically stable point* of a homeomorphism $f : X \to X$ of a metric space X if x satisfies the following property:

(P) For every $\varepsilon > 0$ there is $\delta_x > 0$ such that for every homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \le \delta_x$ there is a continuous map $h : \overline{O_g(x)} \to X$ such that $d_{C^0}(h, i_{\overline{O_g(x)}}) \le \varepsilon$ and $f \circ h = h \circ g$, where $\overline{O_g(x)}$ denotes the orbit closure (see [7, Remark 2.2]).

We note that a necessary condition for a homeomorphism $f : X \to X$ of a metric space X to be topologically stable is that every $x \in X$ satisfies the above property (P). Hereafter, $\mathrm{T}(f)$ will denote the set of topologically stable points of f. We see that if f is topologically stable, then every point of X is topologically stable. But we do not know whether the converse holds (see [7, Example 2.4]).

To discuss the dynamics of the limit of sequences of homeomorphisms on a compact metric space, we need the following notations.

We denote by $H(X)$ the set of all homeomorphisms $f : X \to X$ from a compact metric space X to itself. Then we see that the space $H(X)$ is complete under the metric ρ_0 given by $\rho_0(f,g) = \max\{d_{C^0}(f,g), d_{C^0}(f^{-1},g^{-1})\}$ for any $f, g \in H(X)$ (see [13]).

Arbieto and Rego [3] introduced the notion of the uniform shadowing property for a sequence of continuous maps and obtained positive entropy of the limit. We give the notions of the uniform shadowing property and the uniform expansiveness for sequences in the class of homeomorphisms as in the similar ways in [3].

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $H(X)$.

We say that $\{f_n\}_{n\in\mathbb{N}}$ has the *uniform shadowing property* if each f_n has the shadowing property and the constants δ_n are the same, $\delta_n = \delta$ for each $n \in \mathbb{N}$.

Similarly, we say that $\{f_n\}_{n\in\mathbb{N}}$ is *uniformly expansive* if each f_n is expansive with the expansive constant c_n and the expansive constants c_n are the same, i.e., $c_n = c$ for each $n \in \mathbb{N}$.

The purpose of our work is to study a pointwise version of well-known Walters stability theorem [15] and the uniform limit of a sequence of homeomorphisms.

In this paper we investigate a pointwise version of topological stability in the class of homeomorphisms on a compact metric space. We also study the dynamical behaviour of the limit of a sequence of homeomorphisms on a compact metric space that are expansive with the shadowing property. More precisely, we state our main results.

Theorem 1.1. *Every uniformly expansive and shadowable point of each homeomorphism on a compact metric space is a topologically stable point.*

Theorem 1.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $H(X)$ which converges uniformly to $f \in H(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly expansive with the uniform shadowing property, then the limit f is topologically stable.*

2. Proof of main results

In this section we introduce some results that are used to show our main theorems. Then we give proofs of our main theorems.

We recall that a point $x \in X$ is an ϵ_0-*shadowable point* of f if for given $\epsilon_0 > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ with

$x_0 = x$ can be ϵ_0-shadowed by a point in X. The set of ϵ_0-shadowable points of f is denoted by $\mathrm{Sh}(f, \epsilon_0)$.

We recall that a homeomorphism $f : X \to X$ has the *finite shadowing property through* a subset K of X if for every $\varepsilon > 0$ there is $\delta > 0$ such that every finite δ-pseudo orbit of f through K can be ε-shadowed by a point in X.

Lemma 2.1. *Let $x \in \mathrm{Exp}_u(f) \cap \mathrm{Sh}(f)$ and δ be given by the shadowableness of x for $f \in H(X)$. Then every δ-pseudo orbit of f through $\{x\}$ is ε-shadowed by a unique point in X.*

Proof. Let U be a neighborhood of x. Let e be a uniformly expansive constant of x such that $0 < \epsilon < \frac{e}{2}$. Let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a δ-pseudo orbit of f with $x_0 = x$. We can take $y, z \in B_d(x, \epsilon)$ that ϵ-shadowed ξ. Then one has

$$d(f^i(y), f^i(z)) \leq d(f^i(y), x_i) + d(x_i, f^i(z)) < 2\varepsilon \leq e \ \text{ for all}\, i \in \mathbb{Z}.$$

Since x is a uniformly expansive point of f, we get $y = z$. □

Lemma 2.2. *The following properties hold for each $f \in H(X)$:*

(1) $\mathrm{Sh}(f), \mathrm{Exp}_u(f)$ *and* $\mathrm{T}(f)$ *are f-invariant sets.*
(2) $\mathrm{Sh}(f) = \mathrm{Sh}(f^{-1})$, $\mathrm{Exp}_u(f) = \mathrm{Exp}_u(f^{-1})$, $\mathrm{T}(f) = \mathrm{T}(f^{-1})$, *where f^{-1} is the inverse homeomorphism of f.*

Proof. (1) From [11, Theorem 1.1] and [7, Theorem 2.8], respectively, we see that $\mathrm{Sh}(f)$ and $\mathrm{T}(f)$ are f-invariant.

Now we claim that $\mathrm{Exp}_u(f)$ is f-invariant. Let $y \in f(\mathrm{Exp}_u(f))$. Then there exists a point $x \in \mathrm{Exp}_u(f)$ such that $f(x) = y$. Since x is a uniformly expansive point, there exist a positive constant $e > 0$ and a neighborhood V of x such that for each pair of distinct points $z_1, z_2 \in V$ we have $d(f^i(z_1), f^i(z_2)) > e$ for a nonzero integer $i \in \mathbb{Z}$. Since f is continuous, $U = f^{-1}(B_d(y, e))$ is an open set containing x, where $B_d(y, e)$ is the e-ball centered at y. Set $W = U \cap V$. Then we see that every pair of distinct points in W must be e-apart for $m > 1$. Thus we obtain $y \in \mathrm{Exp}_u(f)$. Hence we have $f(\mathrm{Exp}_u(f)) \subset \mathrm{Exp}_u(f)$. By the similar argument for f^{-1}, we can verify that the converse inclusion is true. Hence $f(\mathrm{Exp}_u(f)) = \mathrm{Exp}_u(f)$.

(2) Also we see that $\mathrm{Sh}(f) = \mathrm{Sh}(f^{-1})$ and $\mathrm{T}(f) = \mathrm{T}(f^{-1})$ follow from [11, Lemma 2.6] and [7, Theorem 2.8(1)], respectively.

Next we claim that $\mathrm{Exp}_u(f) = \mathrm{Exp}_u(f^{-1})$. Let $x \in \mathrm{Exp}_u(f)$. Then there exit a neighborhood U of x and $e > 0$ such that if

$$d(f^i(z), f^i(y)) < e \text{ for all } i \in \mathbb{Z}, \tag{2.1}$$

whenever $y, z \in U$, then we have $y = z$. Putting $i = -j$ for all $i \in \mathbb{Z}$ in the above inequality (2.1), for each $i \in \mathbb{Z}$, we have

$$\begin{aligned} d(f^i(y), f^i(z)) &= d((f^{-1})^{-i}(y), (f^{-1})^{-i}(z)) \\ &= d((f^{-1})^{j}(y), (f^{-1})^{j}(z)) < e \text{ for all } j \in \mathbb{Z}. \end{aligned}$$

Thus we see that if $d((f^{-1})^j(y), (f^{-1})^j(z)) < e$ for all $j \in \mathbb{Z}$, whenever $y, z \in U$, then we also have $y = z$. Thus $x \in \mathrm{Exp}_u(f^{-1})$. So $\mathrm{Exp}_u(f) \subset \mathrm{Exp}_u(f^{-1})$.

Similarly, we can prove that the converse inclusion is true. Hence $\mathrm{Exp}_u(f) = \mathrm{Exp}_u(f^{-1})$. □

Let X and Y be compact metric spaces. We say that two homeomorphisms $f : X \to X$ and $g : Y \to Y$ are *topologically conjugated* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, where the homeomorphism h is called a *topological conjugacy* between f and g.

Lemma 2.3. *Let X and Y be compact metric spaces with metrics d and d', respectively. If two homeomorphisms $f : X \to X$ and $g : Y \to Y$ are topologically conjugated under a topological conjugacy $h : X \to Y$. Then we obtain the following properties*:

$$\mathrm{Sh}(g) = h(\mathrm{Sh}(f)),\ \mathrm{Exp}_u(g) = h(\mathrm{Exp}_u(f)),\ \mathrm{T}(g) = h(\mathrm{T}(f)).$$

Proof. First we claim that $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$. Let $y \in h(\mathrm{Sh}(f))$ and $\varepsilon > 0$. Then there exists $x \in \mathrm{Sh}(f)$ such that $h(x) = y$. It follows from uniform continuity of h that there is $\varepsilon_1 > 0$ such that $d(x, x') < \varepsilon_1$ implies $d'(h(x), h(x')) < \varepsilon$. Since $x \in X$ is shadowable, there exists $\delta_1 > 0$ such that each δ_1-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f with $x_0 = x$ is ε_1-shadowed by a point in X. Then it is enough to show that each δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ of g with $y_0 = y$ is ε-shadowed by a point in Y. Putting $x_i = h^{-1}(y_i)$ for all $i \in \mathbb{Z}$, since $d'(g(y_i), y_i) < \delta$ for all $i \in \mathbb{Z}$, we have

$$d(f(x_i), x_{i+1}) = d((f \circ h^{-1})(y_i), h^{-1}(y_{i+1})) = d((h^{-1} \circ g)(y_i), h^{-1}(y_{i+1})) < \delta_1$$

for all $i \in \mathbb{Z}$ and $x_0 = x = h^{-1}(y)$. Thus $\{x_i\}_{i\in\mathbb{Z}}$ is a δ_1-pseudo orbit of f with $x_0 = x$ and so it is ε_1-shadowed by a point $z' \in X$. Hence we have

$$d'((h \circ f^i)(z'), h(x_i)) = d'((g^i \circ h)(z'), y_i) < \varepsilon$$

for all $i \in \mathbb{Z}$. Thus the δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ of g with $y_0 = y$ is ε-shadowed by a point $z = h(z') \in Y$. Hence $y \in \mathrm{Sh}(g)$. This verifies that $h(\mathrm{Sh}(f)) \subset \mathrm{Sh}(g)$. By applying the similar argument for h^{-1}, we can verify that the converse inclusion is true. Hence we have $\mathrm{Sh}(g) = h(\mathrm{Sh}(f))$.

Also we see that $\mathrm{Exp}_u(g) = h(\mathrm{Exp}_u(f))$ (see [6, Proposition 2.3(2) and 2.3(3)]) and $\mathrm{T}(g) = h(\mathrm{T}(f))$ (see [7, Theorem 2.8]). The proof is complete. □

For the proof of Theorem 1.1, we also need the following result.

Lemma 2.4. *Let $x \in X$ be a uniformly expansive point with the expansive constant e and U be a neighborhood of x. For each $N \in \mathbb{N}$ with $N > 1$ there exists $\delta > 0$ that $d(y, z) < \delta$ implies $d(f^n(y), f^n(z)) < e$ for all n satisfying $|n| < N$, whenever $y, z \in U$. Conversely, given $e > 0$, there exists $N > 1$ such that $d(f^n(y), f^n(z)) < e$ for all n with $|n| < N$, whenever $y, z \in U$ implies $d(y, z) < \delta$.*

Proof. Let $N > 1$ be given. The continuity of f implies that $\delta > 0$ can be chosen as in the statement.

Conversely, let $\delta > 0$ be given. If no N can be chosen with the property stated, then for each $N > 1$ there exist $x_N, y_N \in U$ with $d(f^n(x_N), f^n(y_N)) \le e$ for all n with $|n| < N$ and $d(x_N, y_N) \ge \delta$. Since X is compact, we suppose that there are points $x, y \in X$ such that $x_{N_i} \to x$ and $y_{N_i} \to y$ as $i \to \infty$. Then $d(x, y) \ge \delta$ and also $d(f^n(x), f^n(y)) < e$ for all $n \in \mathbb{Z}$. This contradicts the uniformly expansive property of x. This completes the proof. □

Now we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that $x \in \mathrm{Exp}_u(f) \cap \mathrm{Sh}(f)$. Since x is uniformly expansive, there are $e > 0$ and a neighborhood U of x such that if $d(f^i(z), f^i(y)) \le e$ for all $i \in \mathbb{Z}$, whenever $y, z \in U$, then $y = z$. Let $0 < \varepsilon \le \frac{e}{4}$ be such that $B_d(x, \varepsilon) \subset U$ and $\delta > 0$ be given by the shadowableness of x. Let $g : X \to X$ be a homeomorphism satisfying $d_{C^0}(f, g) \le \delta$. Since $d((f \circ g^n)(x), g^{n+1}(x)) < \delta$ for all $n \in \mathbb{Z}$, $\{g^n(x)\}_{n\in\mathbb{Z}}$ is a δ-pseudo orbit of f. By Lemma 2.1, there is a unique point denoted by $h(x)$ whose f-orbit ε-shadowed $\{g^n(x)\}_{n\in\mathbb{Z}}$. Then we have the map $h : X \to X$ satisfying

$$d(f^n(h(x)), g^n(x)) < \epsilon \ \text{ for all } n \in \mathbb{Z}.$$

In particular, we have $d(h(x), x) < \epsilon$ for all $x \in X$. To prove that this map h is well defined, we claim that $g^n(x) = g^m(x)$ for $n, m \in \mathbb{Z}$. Then $g^{r+n}(x) = g^{r+m}(x)$ for every $r \in \mathbb{Z}$, and so

$$\begin{aligned} d(f^r(f^n(y)), f^r(f^m(y))) &\le d(f^{r+n}(y), g^{r+n}(x)) + d(g^{r+n}(x), g^{r+m}(x)) \\ &\quad + d(f^{r+m}(y), g^{r+m}(x)) \\ &= d(f^{r+n}(y), g^{r+n}(x)) + d(f^{r+m}(y), g^{r+m}(x)) \\ &\le 2\varepsilon < e. \end{aligned}$$

The uniform expansivity of x implies $f^n(y) = f^m(y)$ proving the assertion. It follows that the map h is well defined. Since

$$\begin{aligned} (f \circ h)(g^n(x)) &= f(f^n(y)) = f^{n+1}(y) = h(g^{n+1}(x)) \\ &= h(g(g^n(x))) = (h \circ g)(g^n(x)) \end{aligned}$$

for all $n \in \mathbb{Z}$ and

$$d(h(g^n(x)), g^n(x)) = d(f^n(y), g^n(x)) \le \epsilon \ \text{ for all } n \in \mathbb{Z},$$

one would have

$$f \circ h = h \circ g \quad \text{and} \quad d_{C^0}(h, i_{O_g(x)}) \le \varepsilon.$$

Let us prove that h is uniformly continuous. Fix $\varepsilon_1 > 0$. Since e is a uniformly expansive constant of x and X is compact, it follows from Lemma 2.4 that there is $N \in \mathbb{N}$ such that $d(a, b) \le \delta$ whenever $a, b \in U$ satisfies $d(f^n(a), f^n(b)) \le e$ for every n with $-N \le n \le N$. Since g is continuous and X is compact, we see that g is uniformly continuous. Hence, there is $\rho > 0$

such that $d(g^n(a), g^n(b)) \le \frac{e}{2}$ for all n with $-N \le n \le N$, whenever $a, b \in U$ satisfies $d(a,b) \le \rho$. Now take $a, b \in O_g(x)$ with $d(a,b) \le \rho$. It follows that

$$\begin{aligned} d(f^n(h(a)), f^n(h(b))) &= d(h(g^n(a)), h(g^n(b)) \\ &\le d(h(g^n(a)), g^n(a)) + d(g^n(a), g^n(b)) + d(h(g^n(b)), g^n(b)) \\ &\le 2\varepsilon + \frac{e}{2} \\ &< e \text{ for all } -N \le n \le N. \end{aligned}$$

By the choice of N we conclude that $d(h(a), h(b)) \le \delta$, and so h is uniformly continuous. Then we can extend h continuously to the orbit closure $\overline{O_g(x)}$ to obtain a continuous map, still denoted by $h : \overline{O_g(x)} \to X$. Since $\varepsilon < e$, then $d_{C^0}(h, i_{O_g(x)}) \le \varepsilon$ can be extended to $d_{C^0}(h, i_{\overline{O_g(x)}}) \le \varepsilon$. Since ε is arbitrary, we have $x \in \mathrm{T}(f)$. So $\mathrm{Exp}_u(f) \cap \mathrm{Sh}(f) \subset \mathrm{T}(f)$. This completes the proof. ☐

By Theorem 1.1, we immediately obtain the following result of Corollary 3.16 in [7].

Corollary 2.5. *Every shadowable point of an expansive homeomorphism of a compact metric space is topologically stable.*

We recall that a point $x \in X$ is *periodic* if $f^n(x) = x$ for some positive integer n. Moreover, a point $x \in X$ is *closable* (or satisfies the C^0-closing lemma) if for every $\delta > 0$ there is a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \le \delta$ such that x is a periodic point of g (see [10]). Denote by $\mathrm{CL}(f)$ the set of closable points of f and by $\mathrm{Per}(f)$ the set of periodic points of f. Given $\varepsilon > 0$, a finite sequence $\{x_0, \ldots, x_k\}$ is an *ε-chain* from x to y if $x_0 = x, x_k = y$ and $d(f(x_n), x_{n+1}) \le \varepsilon$ for every n with $0 \le n \le k-1$. We say that $x \in X$ is *chain recurrent* if for every ε there is an ε-chain from x to itself. The set $\mathrm{CR}(f)$ of chain recurrent points of f is called the *chain recurrent set*.

Corollary 2.6 ([7]). *Let $f : X \to X$ be a homeomorphism on a compact metric space X. Then $\mathrm{CL}(f) \cap \mathrm{Sh}(f) \cap \mathrm{Exp}_u(f) \subset \overline{\mathrm{Per}(f)}$.*

Proof. It follows from Theorem 1.1 that $\mathrm{Sh}(f) \cap \mathrm{Exp}_u(f) \subset T(f)$. Since $\mathrm{CL}(f) \cap T(f) \subset \overline{\mathrm{Per}(f)}$, it is easy to see that $\mathrm{CL}(f) \cap \mathrm{Sh}(f) \cap \mathrm{Exp}_u(f) \subset \overline{\mathrm{Per}(f)}$. ☐

We say that a homeomorphism $f : X \to X$ of a compact metric space X is *pointwise topologically stable* if $T(f) = X$ (see [7, Definition 3.3]).

Putting $\mathrm{Sh}(f) = X$ and $\mathrm{Exp}_u(f) = X$ in Theorem 1.1, we obtain the pointwise version of the Walters stability theorem as the following corollary (see [15, Theorem 4]).

Corollary 2.7. *Every expansive homeomorphism with the shadowing property is pointwise topologically stable.*

Proof. Let $f : X \to X$ be an expansive homeomorphism with the shadowing property. We see that $\mathrm{Sh}(f) = X$ and $\mathrm{Exp}_u(f) = X$. Since $\mathrm{Exp}_u(f) \cap \mathrm{Sh}(f) =$

X and $\mathrm{Exp}_u(f) \cap \mathrm{Sh}(f) \subset T(f)$ by Theorem 1.1, we obtain

$$X \subset T(f)$$

and so $T(f) = X$. Hence f is pointwise topologically stable. □

Remark 2.8. We give the pointwise version concerning the converse of Walters stability theorem (see [15, Theorem 4]).

(1) Every topologically stable homeomorphism of a compact manifold of $\dim \geq 2$ has the shadowing property (see [15, Theorem 11]). Also Aoki and Hiraide mentioned that Theorem 11 in [15] is also true for the case where $\dim = 1$ [12].
(2) Every topologically stable point of a homeomorphism of a compact manifold of $\dim \geq 2$ is shadowable (see [7, Lemma 3.11]).
(3) Let $f : X \to X$ be a homeomorphism of a compact manifold X of $\dim(X) \geq 2$. If every point of X is topologically stable, then $T(f) = X$ and so $Sh(f) = X$. Furthermore, f has the shadowing property (see [7, Corollary 3.12]).

Lemma 2.9. *Let* $\mathrm{Sh}(f)$ *be a dense subset of* X *for some* $f \in H(X)$. *Then* f *has the shadowing property through* $\mathrm{Sh}(f)$ *if and only if* f *has the finite shadowing property through* $\mathrm{Sh}(f)$.

Proof. Suppose that f has the shadowing property through $\mathrm{Sh}(f)$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that every δ-pseudo orbit of f through $\mathrm{Sh}(f)$ can be ε-shadowed. Let $\Gamma = \{x_0, x_1, \ldots, x_m\}$ be a δ-pseudo orbit of f with $x_0 \in \mathrm{Sh}(f)$. Then the sequence $\hat{\Gamma} = \{y_n \mid n \in \mathbb{Z}\}$ given by

$$y_n = \begin{cases} f^n(x_0) & \text{if } n < 0, \\ x_n & \text{if } 0 \leq n \leq m, \\ f^n(x_m) & \text{if } n \geq m+1, \end{cases}$$

is a δ-pseudo orbit $\hat{\Gamma}$ of f through $\mathrm{Sh}(f)$. Since f has the shadowing property through $\mathrm{Sh}(f)$, then there is a point $y \in X$ such

$$d(f^n(y), y_n) < \varepsilon \ \text{ for all } n \in \mathbb{Z}.$$

Thus every δ-finite pseudo orbit through $\mathrm{Sh}(f)$ can be ε-shadowed.

To prove the converse, given $\varepsilon > 0$ and $\delta > 0$ be the corresponding to definition of the finite shadowing property through $\mathrm{Sh}(f)$. Let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a $\frac{\delta}{2}$-pseudo orbit of f through $\mathrm{Sh}(f)$. By uniform continuity of f, there exists $\eta > 0$ with $\eta < \min\{\frac{\delta}{2}, \frac{\varepsilon}{2}\}$ such that $d(x, y) < \eta$ implies $d(f(x), f(y)) < \frac{\delta}{2}$. Fix $m \in \mathbb{N}$ and $x'_i = x_{i-m}$. Since $x_{-m} \in \overline{\mathrm{Sh}(f)} = X$, there is a point $y_m \in \mathrm{Sh}(f)$ such that

$$d(x'_0, y_m) = d(x_{-m}, y_m) < \eta.$$

Then $\Gamma = \{y_{-m}, x_{-m+1}, \dots, x_0, x_1, \dots, x_m\}$ is a finite δ-pseudo orbit of f through $\mathrm{Sh}(f)$ because

$$\begin{aligned} d(f(y_{-m}), x_{-m+1}) &\le d(x_{-m}, f(y_{-m})) + d(x_{-m}, x_{-m+1}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Since f has the finite shadowing property through $\mathrm{Sh}(f)$, there is a point $z_m \in X$ such that

$$d(f^i(z_m), x'_i) < \varepsilon \ \text{ for } \ 0 \le i \le 2m.$$

Setting $w_m = f^m(y_m)$, then we obtain

$$d(f^i(w_m), x_i) < \varepsilon \ \text{ for } \ -m \le i \le m. \tag{2.2}$$

By compactness of X, we can assume that there exists a limit point w of $\{w_n\}_{n\in\mathbb{N}} \subset X$. Passing to the limit as $m \to \infty$ in (2.2), we have that

$$d(f^i(w), x_i) \le \varepsilon \ \text{ for all } i \in \mathbb{Z}.$$

Thus ξ is 2ε-shadowed by w. Hence f has the shadowing property through $\mathrm{Sh}(f)$. This completes the proof. □

We recall that a point $x \in X$ is *finitely shadowable* of $f \in H(X)$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every finite set $\{x_0, x_1, \dots, x_k\}$ satisfying $x_0 = x$ and $d(f(x_n), x_{n+1}) < \delta$ for every n with $0 \le n \le k-1$ there is $y \in X$ such that $d(f^n(y), x_n) < \varepsilon$ for every $0 \le n \le k-1$.

Lemma 2.10 ([7, Lemma 3.10])**.** *Every finitely shadowable point of a homeomorphism of a compact metric space is shadowable.*

Lemma 2.11 ([11, Lemma 2.3])**.** *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then f is the shadowing property through a closed subset $K \subset X$ if and only if $K \subset \mathrm{Sh}(f)$.*

Proposition 2.12. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $\mathrm{Sh}(f)$ is a dense subset of X and $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property on $\mathrm{Sh}(f)$, then f also has the shadowing property.*

Proof. By assumptions, $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property and $X = \overline{\mathrm{Sh}(f)}$, it follows from [4, Lemma 3.1] that $f : X \to X$ has the shadowing property. □

Proposition 2.13 ([9, Lemma 1])**.** *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Suppose that there is a sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that every δ_k-pseudo orbit through $Sh(f)$ can be $\frac{1}{k}$-shadowed for every $k \in \mathbb{N}$. Then $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property.*

Proof. It follows from [9, Lemma 1] that $\mathrm{Sh}(f)$ is a closed subset of a compact metric space X and so a compact subspace of X. Hence $f|_{\mathrm{Sh}(f)} : \mathrm{Sh}(f) \to \mathrm{Sh}(f)$ has the shadowing property by Lemma 2.11. □

We recall that a homeomorphism $f : X \to X$ of a compact metric space X has the *almost shadowing property* if $\mathrm{Sh}(f)$ is dense in X (see [11]). From Lemma 2.3, we obtain the following.

Proposition 2.14 ([8, Theorem 1.1])**.** *Let X and Y be compact metric spaces with metrics d and d', respectively. If two homeomorphisms $f : X \to X$ and $g : Y \to Y$ are topologically conjugated, then f has the almost shadowing property if and only if so is g.*

For the proof of Theorem 1.2, we need the following lemmas.

Lemma 2.15 ([3, Lemma 8])**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $H(X)$ which converges uniformly to $f \in H(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ has the uniform shadowing property, then f has the shadowing property.*

Lemma 2.16. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $H(X)$ which converges uniformly to $f \in H(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly expansive with the expansive constants $e_n = e$ for each $n \in \mathbb{N}$, then the limit f is expansive with the expansive constant $\frac{e}{3}$.*

Proof. Fix $e > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, there exists $N_0 \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \frac{e}{3}$ for every $x \in X$ and each $n \geq N_0$. Note that if $\{f_n\}_{n\in\mathbb{N}}$ converges to f, then $\{f_n^i\}_{n\in\mathbb{N}}$ converges to f^i for each $i \in \mathbb{N}$. On the other hand, each f^i is homeomorphic for all $i \in \mathbb{N}$. Taking $n = N_0$ sufficiently large, we obtain

$$d(f^i(y), f^i_{N_0}(y)) < \frac{e}{3} \text{ for all } i \in \mathbb{N}.$$

We claim that there exists an expansive constant $\frac{e}{3}$ such that for each pair $x, y \in X$ satisfying $d(f^n(x), f^n(y)) \leq \frac{e}{3}$ for each $n \in \mathbb{Z}$ we have $x = y$. In fact, we have

$$\begin{aligned} d(f^n_{N_0}(x), f^n_{N_0}(y)) &\leq d(f^n_{N_0}(x), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(y), f^n_{N_0}(y)) \\ &\leq e \text{ for every } n \in \mathbb{Z}. \end{aligned}$$

Thus we obtain

$$d(f^n_{N_0}(x), f^n_{N_0}(y)) \leq e \text{ for every } n \in \mathbb{Z}.$$

From the expansivity of f_{N_0}, we obtain $x = y$. Hence the limit f is expansive. This completes the proof. □

For the proof of Theorem 1.2, we also need well-known Walters stability theorem (see [15]). Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that a sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly expansive with the uniform shadowing property. From Lemma 2.15, the limit $f : X \to X$ has the shadowing property. Also, it follows from Lemma 2.16 that f is expansive, so f is topologically stable by Theorem 4 in [15]. This completes the proof. □

Corollary 2.17. *Under the assumptions of Theorem 1.2, the following properties hold for each $f \in H(X)$.*

(1) $\mathrm{CR}(f) = \Omega(f) = \overline{\mathrm{Per}(f)}$, *where $\Omega(f)$ is the nonwandering set (see* [1, p. 92]*).*
(2) *The limit $f : X \to X$ is pointwise topologically stable.*

Proof. The proof of (1) immediately follows from [1, Theorem 3.1.8]. From Theorem 1.2 and Corollary 2.7, we also can verify that (2) holds. □

3. Examples

In this section we give some examples related our main results and their key notions.

First we give an example that there are a compact metric space X and a homeomorphism $f : X \to X$ such that $\mathrm{Sh}(f)$ is a nonempty noncompact subset of X.

Example 3.1 ([11, Example 2.1])**.** Let $X = C \cup [1, 2]$ be the compact metric subspace of $\mathbb{R}$, where C is the ternary Cantor set of $[0, 1]$. Take $f : X \to X$ as the identity map of X. Then we obtain that $\mathrm{Sh}(f) = C \setminus \{1\}$ is a non-empty and non-compact subset of X.

Next the following example shows that the set of topologically stable points of a homeomorphism need not be closed in general.

Example 3.2 ([7, Example 2.7])**.** Let $X = S(1) \cup (\cup_{n\in\mathbb{N}} S(1 + (\frac{1}{n})))$ be the compact metric space of the Euclidean space $\mathbb{R}^2$, where $S(r)$ is a circle of radius r of $\mathbb{R}^2$ centered at $(0, 0)$. Define a map $f : X \to X$ by setting $f_n = f|_{S(1+(\frac{1}{n}))}$ be a Morse-Smale diffeomorphism with $2+4(n-1)$ alternating hyperbolic fixed points and $f|_{S(1)} = i_{S(1)}$. Then we can see that f is a homeomorphism and a proper subset $\mathrm{T}(f) = \cup_{n\in\mathbb{N}} S(1 + (\frac{1}{n}))$ of X is not closed.

Also, we give an example to explain some properties about the set of uniformly expansive points of homeomorphisms.

Example 3.3. Let S^1 be the circle of the Euclidean space $\mathbb{R}^2$ with metric d_1 and $f : S^1 \to S^1$ be a homeomorphism with a fixed point $p \in S^1$ in Figure 1. Since there are no expansive homeomorphisms on S^1, we see that $\mathrm{Exp}_u(f) \neq S^1$ and moreover $\mathrm{Exp}_u(f) = \emptyset$. Let $(\Sigma_2 = \{0, 1\}^{\mathbb{Z}}, \sigma)$ be the shift system with metric d_2 and $X = S^1 \cup \Sigma_2$ be the compact metric space with the metric ρ given by

$$\rho(x, y) = \begin{cases} d_1(x, y) & \text{if } x, y \in S^1, \\ d_2(x, y) & \text{if } x, y \in \Sigma_2, \\ k & \text{if } x \in S^1,\ y \in \Sigma_2, \end{cases}$$

where $k \geq 3$. Define a homeomorphism $g : X \to X$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in S^1, \\ \sigma(x) & \text{if } x \in \Sigma_2. \end{cases}$$

Then the homeomorphism g is not expansive. Thus we have $\emptyset \neq \mathrm{Exp}_u(g) = \Sigma_2$.

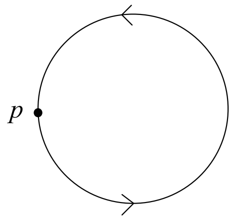

FIGURE 1. Dynamics of a homeomorphism f on S^1

The one-point compactification of a locally compact Hausdorff space satisfying an expansive homeomorphism may not admit an expansive homeomorphism.

Example 3.4. Note that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is expansive. Since $\mathbb{R}$ is a locally compact Hausdorff space, we see that there is a one-point compactification $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ that is a compact metric space and homeomorphic to the circle S^1. Define a homeomorphism $g : \mathbb{R}_\infty \to \mathbb{R}_\infty$ by

$$g(x) = \begin{cases} x & \text{if } x = \infty, \\ f(x) & \text{otherwise.} \end{cases}$$

Then we see that $\mathrm{Exp}_u(f) = \mathbb{R}$. Since $\mathrm{Exp}_u(f)$ is a dense subset of S^1 and there are no expansive homeomorphisms on the circle, thus the homeomorphism $g : \mathbb{R}_\infty \to \mathbb{R}_\infty$ is not expansive. We remark that though the notion of metrics is a topological property, the expansivity may not be preserved by the equivalent metrics.

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