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LI-YAU GRADIENT ESTIMATES ON CLOSED MANIFOLDS UNDER BAKRY-ÉMERY RICCI CURVATURE CONDITIONS

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Abstract. In this paper, motivated by the work of Q. S. Zhang in [25], we derive optimal Li-Yau gradient bounds for positive solutions of the f-heat equation on closed manifolds with Bakry-Émery Ricci curvature bounded below.

1. Introduction

Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space, where (M^n, g) is an n-dimensional complete Riemannian manifold, dv is the volume element of g, f is a smooth function on M (called the potential function), and $e^{-f}dv$ (for short, $d\mu$) is called the weighted volume element. The m-Bakry-Émery Ricci curvature ([1], [13]) associated to $(M^n, g, e^{-f}dv)$ is defined by

$$Ric_f^{m,n} := Ric + Hessf - \frac{1}{m-n} df \otimes df \ (m > n),$$

where Ric is the Ricci curvature of (M^n, g), $Hess$ is the Hessian with respect to the metric g. The m-Bakry-Émery Ricci curvature is a natural generalization of Ricci curvature on Riemannian manifolds. When $m = \infty$, we denote

$$Ric_f = Ric_f^{\infty,n} = Ric + Hessf,$$

which is called the Bakry-Émery Ricci curvature ([1]). Manifolds with constant Bakry-Émery Ricci curvature are so called gradient Ricci solitons, which play a crucial role in the singularity analysis of the Ricci flow ([3], [5], [14], [17]). With respect to the weighted volume element the natural self-adjoint Laplacian operator is the f-Laplacian

$$\Delta_f = \Delta - \langle \nabla f, \nabla \rangle .$$

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The f-heat equation is defined as

$$(\Delta_f - \partial_t)u = 0.$$

In [12], P. Li and S.-T. Yau showed that if (M^n, g) is a complete Riemannian manifold with $Ric \geq -K$ for some constant $K \geq 0$, then for any positive solution u of the heat equation $(\Delta - \partial_t)u = 0$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{2(\alpha-1)}, \ \forall \alpha > 1, \ t > 0. \tag{1}$$

In particular, when $Ric \geq 0$, one obtains the optimal Li-Yau bound

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}. \tag{2}$$

Many applications of (1) and (2) have been demonstrated, including the parabolic Harnack inequality, optimal Guassian estimates of the heat kernel, estimates of eigenvalues of the Laplace operator, and estimates of the Green's function. Moreover, (1) and (2) can even imply the Laplacian Comparison Theorem (see e.g. [6] page 394).

The estimate (2) is sharp since the equality is achieved by the heat kernel of $\mathbb{R}^n$. However, (1) is not sharp for $K > 0$. An open question asks if we can find sharp Li-Yau-type gradient estimates for $K > 0$. Many works were done to improve or generalize (1).

In [9], R. S. Hamilton discovered Li-Yau-type bound for the heat equation

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{\partial_t u}{u} \leq e^{4Kt}\frac{n}{2t}. \tag{3}$$

In [21], S.-T. Yau obtained the following estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t} + \sqrt{2nK}\sqrt{\frac{|\nabla u|^2}{u^2} + \frac{n}{2t} + 2nK}. \tag{4}$$

In [2], D. Bakry and Z. M. Qian obtained

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right)\frac{\partial_t u}{u} \leq \frac{n}{2t} + \frac{nK}{2}\left(1 + \frac{K}{3}t\right). \tag{5}$$

In [15], B. Qian improved the estimate (5) in the following form

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2K}{a(t)}\int_0^t a(s)ds\right)\frac{\partial_t u}{u} \leq \frac{nK}{2} + \frac{nK^2}{2a(t)}\int_0^t a(s)ds \\ + \frac{n}{8a(t)}\int_0^t \frac{a'(s)^2}{a(s)}ds,\end{aligned} \tag{6}$$

where $a(t) : (0, \infty) \to (0, \infty)$ is any C^1 positive function that satisfies the following two conditions:

(A1) For all $t > 0$, $a(t) > 0$, $a'(t) > 0$ and $\lim_{t\to 0} a(t) = \lim_{t\to 0} \frac{a(t)}{a'(t)} = 0$.

(A2) For any $L > 0$, $\frac{a'(t)^2}{a(t)}$ is continuous and integrable on the interval $[0, L]$.

In the above results, the constant α in (1) is replaced by functions in the form of $\alpha(t, K)$ which is strictly greater than 1 but converges to 1 as $t \to 0$. Recently, Q. S. Zhang [25] obtains that for all closed manifolds one can take $\alpha = 1$ for $K \geq 0$. It shows that if (M^n, g) is a closed Riemannian manifold with $Ric \geq -K$ for $K \geq 0$, and $diam_M$ is the diameter of M, then for any positive solution u of the heat equation $(\Delta - \partial_t)u = 0$,

$$t\left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \leq \frac{n}{2} + \sqrt{2nK(1+Kt)(1+t)}diam_M + \sqrt{K(1+Kt)(C_1 t + C_2 Kt)}, \tag{7}$$

where C_1 and C_2 are positive constants only depending on n. It is an improvement on Li-Yau gradient bound (1) on closed manifolds.

In [27,28], Q. S. Zhang and M. Zhu obtained Li-Yau-type gradient estimates under integral curvature assumptions. Moreover, Li-Yau-type bounds were also got for weighted manifolds with Bakry-Émery Ricci curvature bounded below [10]. More information about Li-Yau-type bounds can be found in [8,11,16,18, 22–24,26].

In this paper, we show optimal Li-Yau gradient bounds for f-heat equation on closed manifolds with either Bakry-Émery Ricci curvature or m-Bakry-Émery Ricci curvature bounded below, which both generalize (7).

More precisely, we show that:

Theorem 1.1. *Let (M^n, g) be a closed Riemannian manifold with $Ric_f \geq -K$ and $|\nabla f| \leq L$ for some constants $K, L \geq 0$. Let u be a positive solution of the f-heat equation on $M \times (0, +\infty)$, i.e., $(\Delta_f - \partial_t)u = 0$, and $diam_M$ the diameter of M. Then there exists a constant c depending only on n such that*

$$\begin{aligned} &t\left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \\ &\leq \frac{n}{2} + c(L+\sqrt{K})\sqrt{(1+Kt)(1+t)}diam_M \\ &\quad + c(L+\sqrt{K})\sqrt{(1+Kt)(t+Kt+L^2t+A^2t+A^2Kt)}, \end{aligned} \tag{8}$$

where $A = \sup_{x\in M}|f(x)|$.

Notice that (8) still holds when we add f by any constant. In particular, for some point $o \in M$ we may choose f such that $f(o)$ equals zero. Then (8) becomes

$$\begin{aligned} &t\left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \\ &\leq \frac{n}{2} + c(L+\sqrt{K})\sqrt{(1+Kt)(1+t)}diam_M \\ &\quad + c(L+\sqrt{K})\sqrt{(1+Kt)(t+Kt+L^2t+L^2diam_M^2 t+L^2 diam_M^2 Kt)}. \end{aligned} \tag{9}$$

If the potential function is constant, then from (9) we get a result similar to (7).

Corollary 1.2. *Let (M^n, g) be a closed Rimannian manifold with $Ric \geq -K$, where $K \geq 0$. Let u be a positive solution of the heat equation on $M \times (0, +\infty)$, i.e., $(\Delta - \partial_t)u = 0$, and $diam_M$ the diameter of M. Then there exists a constant c depending only on n such that*

$$t\left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \leq \frac{n}{2} + c\sqrt{K(1+Kt)(1+t)}diam_M + c\sqrt{K(1+Kt)(t+Kt)}. \tag{10}$$

For closed manifolds with m-Bakry-Émery Ricci curvature bounded below, we have a similar result. But there is no assumption on the potential function and the constants depend on m.

Theorem 1.3. *Let (M^n, g) be a closed Riemannian manifold with $Ric_f^{m,n} \geq -K$ for some constant $K \geq 0$. Let u be a positive solution of the f-heat equation on $M \times (0, +\infty)$, i.e., $(\Delta_f - \partial_t)u = 0$, and $diam_M$ the diameter of M. Then there exist constants $\widetilde{c_1}$ and $\widetilde{c_2}$ depending only on m such that*

$$t\left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \leq \frac{m}{2} + \sqrt{2mK(1+Kt)(\widetilde{c_1}+t)}diam_M + \widetilde{c_2}\sqrt{K(1+Kt)(t+Kt)}. \tag{11}$$

If the potential function is constant, then we can take $m = n$ and get a result similar to Corollary 1.2.

Remark 1.4. The constants c, $\widetilde{c_1}$ and $\widetilde{c_2}$ in Theorem 1.1 and Theorem 1.3 arise from the volume comparison theorem and upper and lower bounds for the f-heat kernel. Since the lower bound constants of the f-heat kernel cannot be written accurately, the constants c and $\widetilde{c_1}$ cannot be written as accurately as Q. S. Zhang does.

We prove Theorem 1.1 and Theorem 1.3 separately in Sections 2 and 3. The proofs follow a method of Q. S. Zhang [25]. It is sufficient to prove the optimal Li-Yau bound for the f-heat kernel, then the same bound holds for all positive solutions of the f-heat equation. The main tools in the proof of Theorem 1.1 are relative volume comparison [19] of G. Wei and W. Wylie, Y. Li's Hamilton type estimates [10], the upper and lower bounds for the f-heat kernel [20] and the Harnack inequality for the positive solutions of the f-heat equation [20] proved by J. Wu and P. Wu for manifolds with Bakry-Émery Ricci curvature bounded below. Here we generally assume that the potential function is bounded or the gradient of the function is bounded. The proof of Theorem 1.3 is similar to Theorem 1.1. However, in Theorem 1.3 for manifolds with m-Bakry-Émery Ricci curvature bounded below, there is no assumption on potential function and relative results can be found in [4, 10, 19].

2. Li-Yau gradient bounds on closed manifolds under Bakry-Émery Ricci curvature conditions

In this section, we prove Theorem 1.1. First we present some results as preparation.

Lemma 2.1 ([10], Hamilton type estimate)**.** *Suppose that (M^n, g) is a closed Riemannian manifold with $Ric_f \geq -K$, where $K \geq 0$. If u is a positive solution of $(\Delta_f - \partial_t)u = 0$ with $0 < u \leq B$ on $M \times (0,T]$ for some constant B, then*

$$\frac{|\nabla u|^2}{u^2} \leq (\frac{1}{t} + 2K) \ln \frac{B}{u},$$

on $M \times (0,T]$.

In [19], G. Wei and W. Wylie proved the volume comparison theorem under Bakry-Émery Ricci curvature conditions.

Lemma 2.2 ([19], Volume comparison)**.** *Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space with $Ric_f \geq (n-1)H$. For $p \in M$, If $|f(x)| \leq A$ for some constant A, then for $R \geq r > 0$ (assume $R \leq \frac{\pi}{4\sqrt{H}}$ if $H > 0$),*

$$\frac{V_f(B_p(R))}{V_f(B_p(r))} \leq \frac{V_H^{n+4A}(B(R))}{V_H^{n+4A}(B(r))},$$

where $V_H^n(B(r))$ is the volume of the geodesic ball with radius r in the model space M_H^n and $V_f(B_p(r))$ is the weighted volume of the geodesic ball with radius r in M centerd at p.

In [20], J. Wu and P. Wu showed that:

Lemma 2.3 ([20])**.** *Let $(M^n, g, e^{-f}dv)$ be a complete noncompact smooth metric measure space with $Ric_f \geq -(n-1)K$ for $K \geq 0$. For any point $o \in M$ and $R > 0$, denote $A(R) = \sup_{x \in B_o(3R)} |f(x)|$, $A^{'}(R) = \sup_{x \in B_o(3R)} |\nabla f(x)|$, and let $H(x,t,y)$ be the f-heat kernel. Then for any $\epsilon > 0$, there exist constants $c_3(n,\epsilon)$, $c_i(n)$, $4 \leq i \leq 8$, such that*

$$\begin{aligned}(12) \qquad \frac{c_3 e^{c_4 A + c_5(1+A)\sqrt{Kt}}}{V_f^{\frac{1}{2}}(B_x(\sqrt{t}))V_f^{\frac{1}{2}}(B_y(\sqrt{t}))} e^{\left(-\frac{d^2(x,y)}{(4+\epsilon)t}\right)} &\geq H(x,t,y) \\ &\geq \frac{c_6 e^{-c_7(A'^2+K)t}}{V_f(B_x(\sqrt{t}))} e^{\left(-\frac{d^2(x,y)}{c_8 t}\right)}\end{aligned}$$

for all $x, y \in B_o(\frac{1}{2}R)$ and $0 < t < \frac{R^2}{4}$, where $d(x,y)$ is the geodesic distance between x and y.

Note that for closed Riemannian manifolds with $Ric_f \geq -K$ and $|\nabla f| \leq L$, where $K, L \geq 0$, the same bound still holds for all $x, y \in M$ and $t > 0$.

Lemma 2.4 ([20])**.** *Let $(M^n, g, e^{-f}dv)$ be a complete noncompact smooth metric measure space with $Ric_f \geq -(n-1)K$ for $K \geq 0$. For any point $o \in M$ and $R > 0$, denote $A^{'}(R) = \sup_{y \in B_o(R+1)} |\nabla f(y)|$. There exists a constant $c(n)$ such that, for two positive solutions $u(x,s)$ and $u(y,t)$ of the f-heat equation in $B_o\left(\frac{R}{2}\right) \times (0,T)$, $0 < s < t < T$,*

$$\ln\left(\frac{u(x,s)}{u(y,t)}\right) \leq c(n)\left[\left(A'^2 + K + \frac{1}{R^2} + \frac{1}{s}\right)(t-s) + \frac{d^2(x,y)}{t-s}\right].$$

Note that for closed Riemannian manifolds with $Ric_f \geq -K$ and $|\nabla f| \leq L$, where $K, L \geq 0$, there exists a constant $c(n)$ such that, in $M \times (0, +\infty)$, $0 < s < t$,

$$u(x,s) \leq u(y,t) e^{c(n)\left[\left(L^2+K+\frac{1}{s}\right)(t-s)+\frac{d^2(x,y)}{t-s}\right]}. \tag{13}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u = u(x,t)$ be a positive solution of the f-heat equation on $M \times (0, +\infty)$. Denote

$$\begin{aligned} Y = Y(x,t) &= |\nabla \ln u|^2 - \partial_t(\ln u) \\ &= \left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) \\ &= -\Delta_f \ln u. \end{aligned}$$

The Bochner formula [10] for Ric_f is

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |Hess\ u|^2 + \langle \nabla \Delta_f u, \nabla u\rangle + Ric_f(\nabla u, \nabla u). \tag{14}$$

Using (14) and the Bakry-Émery Ricci curvature condition, we have

$$\begin{aligned} &(\Delta_f - \partial_t)Y + 2\langle \nabla Y, \nabla \ln u\rangle \\ &= 2|Hess \ln u|^2 + 2Ric_f(\nabla \ln u, \nabla \ln u) \\ &\geq \frac{2(\Delta \ln u)^2}{n} - 2K|\nabla \ln u|^2 \\ &= \frac{2(\Delta_f \ln u + \langle \nabla f, \nabla \ln u\rangle)^2}{n} - 2K|\nabla \ln u|^2 \\ &= \frac{2}{n}(\Delta_f \ln u)^2 + \frac{4}{n}(\Delta_f \ln u)\langle \nabla f, \nabla \ln u\rangle \\ &\quad + \frac{2}{n}|\langle \nabla f, \nabla \ln u\rangle|^2 - 2K|\nabla \ln u|^2 \\ &\geq \frac{2}{n}Y^2 - \frac{4}{n}\langle \nabla f, \nabla \ln u\rangle Y - \left(\frac{2}{n}L^2 + 2K\right)|\nabla \ln u|^2. \end{aligned} \tag{15}$$

Let $Y^+(x,t) = \max\{Y(x,t), 0\}$. Then the inequality (15) implies that Y^+ is a subsolution of the inequality in the weak sense: on $M \times (0, +\infty)$,

$$(\Delta_f - \partial_t)Y^+ + 2\left\langle \nabla Y^+, \nabla \ln u \right\rangle \geq \frac{2}{n}(Y^+)^2 - \frac{4}{n}\langle \nabla f, \nabla \ln u\rangle Y^+ - \left(\frac{2}{n}L^2 + 2K\right)|\nabla \ln u|^2. \tag{16}$$

For a positive integer j and a small positive number ϵ, multiplying both sides of (16) by $((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}$ and integrating on $M \times (0,T]$ for any $T > 0$, we obtain

$$\begin{aligned} &\frac{2}{n}\int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j+2} d\mu dt \\ \leq & \int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}(\Delta_f - \partial_t)Y^+ d\mu dt \\ &+ 2\int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}\left\langle \nabla Y^+, \nabla \ln u\right\rangle d\mu dt \\ &+ (\frac{2}{n}L^2 + 2K)\int_0^T \int_M |\nabla \ln u|^2((t-\epsilon)^+)^{2j+2}(Y^+)^{2j} d\mu dt \\ &+ \frac{4L}{n}\int_0^T \int_M |\nabla \ln u|((t-\epsilon)^+)^{2j+2}(Y^+)^{2j+1} d\mu dt \\ = & \, T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{17}$$

Here we need to estimate the upper bound of $(t-\epsilon)^+Y^+$ on $M \times (0,T]$. Then let us bound T_1, T_2, T_3 and T_4, respectively. Using integration by parts, we see that

$$\begin{aligned} T_1 = & -2j\int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j-1}|\nabla Y^+|^2 d\mu dt \\ & - \frac{1}{2j+1}\int_M \int_0^T ((t-\epsilon)^+)^{2j+2} d(Y^+)^{2j+1} d\mu \\ = & -2j\int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j-1}|\nabla Y^+|^2 d\mu dt \\ & - \frac{1}{2j+1}\int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j+1}|_0^T d\mu \\ & + \frac{2j+2}{2j+1}\int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j+1} d\mu dt \\ = & -2j\int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j-1}|\nabla Y^+|^2 d\mu dt \\ & - \frac{1}{2j+1}\int_M ((T-\epsilon)^+)^{2j+2}(Y^+)^{2j+1}(x,T) d\mu \end{aligned} \tag{18}$$

$$+\frac{2j+2}{2j+1}\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+1}d\mu dt.$$

Writing $(Y^+)^{2j}\nabla Y^+ = \frac{1}{2j+1}\nabla(Y^+)^{2j+1}$ and doing integration by parts, we deduce

$$\begin{aligned}
T_2 &= \frac{2}{2j+1}\int_0^T\int_M\left\langle\nabla(Y^+)^{2j+1},\nabla\ln u\right\rangle((t-\epsilon)^+)^{2j+2}d\mu dt \\
&= -\frac{2}{2j+1}\int_0^T\int_M(\Delta_f\ln u)(Y^+)^{2j+1}((t-\epsilon)^+)^{2j+2}d\mu dt \\
&\leq \frac{2}{2j+1}\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+2}d\mu dt.
\end{aligned}\tag{19}$$

Throwing away the non-positive terms of (18) and plugging (18) and (19) into (17), we arrive at

$$\begin{aligned}
&(\frac{2}{n}-\frac{2}{2j+1})\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+2}d\mu dt \\
\leq &\frac{2j+2}{2j+1}\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+1}d\mu dt \\
&+\underbrace{(\frac{2}{n}L^2+2K)\int_0^T\int_M|\nabla\ln u|^2((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}d\mu dt}_{T_3} \\
&+\underbrace{\frac{4}{n}L\int_0^T\int_M|\nabla\ln u|((t-\epsilon)^+)^{2j+2}(Y^+)^{2j+1}d\mu dt}_{T_4}.
\end{aligned}\tag{20}$$

This estimate holds for all positive solutions. Now we take, in particular $u = H(x,t,y)$ the f-heat kernel with pole at a fixed point $y\in M$. We will find upper bounds for T_3 and T_4, which rely on the Hamilton type estimate (Lemma 2.1).

For a time $t_0>0$, we consider the f-heat kernel $H(x,t+t_0,y)$ with $t\in[0,t_0]$.

According to Lemma 2.3 , we can choose $\epsilon=1$ and find the upper and lower bounds for f-heat kernel. There exist some positive constants C_1, C_2, C_3, C_4, C_5 and C_6 depending only on n such that

$$\begin{aligned}
&\frac{C_1e^{C_2A+C_3(1+A)\sqrt{K(t+t_0)}}}{V_f^{\frac{1}{2}}(B_x(\sqrt{t+t_0}))V_f^{\frac{1}{2}}(B_y(\sqrt{t+t_0}))}e^{\left(-\frac{d^2(x,y)}{5(t+t_0)}\right)} \\
\geq\ &H(x,t+t_0,y)\geq\frac{C_4e^{-C_5(L^2+K)(t+t_0)}}{V_f(B_x(\sqrt{t+t_0}))}e^{\left(-\frac{d^2(x,y)}{C_6(t+t_0)}\right)}.
\end{aligned}\tag{21}$$

This upper bound implies

$$B := \sup_{M\times(0,t_0)} H(x,t+t_0,y) \le \frac{C_1 e^{C_2A + C_3(1+A)\sqrt{2Kt_0}}}{\inf_{x\in M} V_f(B_x(\sqrt{t_0}))},$$

which yields, by the lower bound of $H(x,t+t_0,y)$, that

$$\frac{B}{H(x,t+t_0,y)} \le C_7 e^{\left(C_2A + C_3(1+A)\sqrt{2Kt_0} + C_5(L^2+K)(2t_0) + \frac{d^2(x,y)}{C_6 t_0}\right)} \times \frac{\sup_{y\in M} V_f(B_y(\sqrt{2t_0}))}{\inf_{x\in M} V_f(B_x(\sqrt{t_0}))}. \tag{22}$$

We notice that the infimum and supremum of the geodesic ball's volumes can be achieved by some points on M, say point p and point q, i.e.,

$$\frac{\sup_{y\in M} V_f(B_y(\sqrt{2t_0}))}{\inf_{x\in M} V_f(B_x(\sqrt{t_0}))} = \frac{V_f(B_q(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))}. \tag{23}$$

Here is the ratio of geodesic balls's volumes at different points. Notice that the relationship between the geodesic ball $B_q(\sqrt{2t_0})$ and the geodesic ball $B_p(\sqrt{2t_0}+d(p,q))$ is $B_q(\sqrt{2t_0}) \subset B_p(\sqrt{2t_0}+d(p,q))$. Then applying the volume comparison theorem (Lemma 2.2), we get

$$\begin{aligned}
\frac{V_f(B_p(\sqrt{2t_0}+d(p,q)))}{V_f(B_p(\sqrt{2t_0}))} &\le \frac{\int_0^{d(p,q)+\sqrt{2t_0}} \left(\sinh\left(\sqrt{\frac{K}{n-1}}r\right)\right)^{n+4A-1} dr}{\int_0^{\sqrt{2t_0}} \left(\sinh\left(\sqrt{\frac{K}{n-1}}r\right)\right)^{n+4A-1} dr} \\
&\le \left(\frac{d(p,q)}{\sqrt{2t_0}}+1\right)^{n+4A} e^{d(p,q)\sqrt{\frac{K}{n-1}}(n+4A-1)} \\
&= e^{(n+4A)\ln(\frac{d(p,q)}{\sqrt{2t_0}}+1) + d(p,q)\sqrt{\frac{K}{n-1}}(n+4A-1)} \\
&\le e^{(n+4A)\frac{d(p,q)}{\sqrt{2t_0}} + d(p,q)\sqrt{\frac{K}{n-1}}(n+4A-1)},
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\frac{V_f(B_p(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))} &\le \frac{\int_0^{\sqrt{2t_0}} \left(\sinh\left(\sqrt{\frac{K}{n-1}}r\right)\right)^{n+4A-1} dr}{\int_0^{\sqrt{t_0}} \left(\sinh\left(\sqrt{\frac{K}{n-1}}r\right)\right)^{n+4A-1} dr} \\
&\le 2^{\left(\frac{n+4A}{2}\right)} e^{[(\sqrt{2}-1)\sqrt{t_0}\sqrt{\frac{K}{n-1}}](n+4A-1)}.
\end{aligned} \tag{25}$$

Substituting (24) and (25) into (23), we find

$$\begin{aligned}
&\frac{\sup_{y\in M} V_f(B_y(\sqrt{2t_0}))}{\inf_{x\in M} V_f(B_x(\sqrt{t_0}))} \\
&= \frac{V_f(B_q(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))} \le \frac{V_f(B_p(\sqrt{2t_0}+d(p,q)))}{V_f(B_p(\sqrt{t_0}))}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{V_f(B_p(\sqrt{2t_0}+d(p,q)))}{V_f(B_p(\sqrt{2t_0}))}\frac{V_f(B_p(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))}\\
&\le 2^{\left(\frac{n+4A}{2}\right)}e^{[(\sqrt{2}-1)\sqrt{t_0}\sqrt{\frac{K}{n-1}}](n+4A-1)+(n+4A)\frac{d(p,q)}{\sqrt{2t_0}}+d(p,q)\sqrt{\frac{K}{n-1}}(n+4A-1)}.
\end{aligned}
$$

This and (22) imply that

$$
\begin{aligned}
&\ln\frac{B}{H(x,t+t_0,y)}\\
&\le \ln C_7 + C_2A + C_3\sqrt{2Kt_0} + C_3A\sqrt{2Kt_0} + 2C_5L^2t_0\\
&\quad + 2C_5Kt_0 + \frac{d^2(x,y)}{C_6t_0} + \frac{nd(p,q)}{\sqrt{2t_0}} + 2\sqrt{2}A\frac{d(p,q)}{\sqrt{t_0}} + d(p,q)\sqrt{K(n-1)}\\
&\quad + 4d(p,q)A\sqrt{\frac{K}{n-1}} + \sqrt{K(n-1)t_0} + 4A\sqrt{\frac{Kt_0}{n-1}} + \frac{\ln 2}{2}(n+4A).
\end{aligned}
$$

Using the following basic inequalities with

$$
\begin{aligned}
&\sqrt{Kt_0} \le \frac{1}{4} + Kt_0,\ \frac{d(p,q)}{\sqrt{t_0}} \le \frac{d^2(p,q)}{t_0} + \frac{1}{4},\ \frac{Ad(p,q)}{\sqrt{t_0}} \le \frac{d^2(p,q)}{t_0} + \frac{1}{4}A^2,\\
&d(p,q)\sqrt{K} \le \frac{d^2(p,q)}{t_0} + \frac{1}{4}Kt_0,\ Ad(p,q)\sqrt{K} \le \frac{d^2(p,q)}{t_0} + \frac{1}{4}A^2Kt_0,\\
&A\sqrt{Kt_0} \le \frac{1}{4}A^2 + Kt_0,
\end{aligned}
$$

we get

$$
\begin{aligned}
\ln\frac{B}{H(x,t+t_0,y)} \le C_8\Bigg(&1 + A + Kt_0 + AKt_0 + L^2t_0 + \frac{diam_M^2}{t_0}\\
&+ A^2 + A^2Kt_0\Bigg),\ \forall t \in (0,t_0].
\end{aligned}
$$

Using $A \le \frac{1}{4} + A^2$ gives that

$$
\ln\frac{B}{H(x,t+t_0,y)} \le C_9\left(1 + Kt_0 + L^2t_0 + \frac{diam_M^2}{t_0} + A^2 + A^2Kt_0\right).
$$

Hamilton type estimate (Lemma 2.1) implies that

$$
\begin{aligned}
t|\nabla_x \ln H(x,t+t_0,y)|^2 \le C_9(1+2Kt_0)\Bigg(&1 + Kt_0 + L^2t_0 + \frac{diam_M^2}{t_0}\\
&+ A^2 + A^2Kt_0\Bigg),\ t \in (0,t_0].
\end{aligned}
$$

We take $t = t_0$ and use the arbitrariness of t_0 to conclude

$$
\begin{aligned}
t|\nabla_x \ln H(x,t,y)|^2 \le C_{10}(1+Kt)\Bigg(&1 + Kt + L^2t + \frac{diam_M^2}{t}\\
&+ A^2 + A^2Kt\Bigg),\ \forall t > 0.
\end{aligned}
\tag{26}
$$

The bound is adequate for us when the time is short, say $t \leq 4$. When t is large, the f-heat kernel converges to the positive constant $\frac{1}{V_f(M)}$, where $V_f(M)$ is the weighted volume of M. In this case the above bound becomes inaccurate. Instead we will use a better bound based on the Harnack inequality (13).

Pick any time $t \geq 4$. Since $\int_M H(x, t+1, y) d\mu(x) = 1$, there is a point $x_1 \in M$ such that $H(x_1, t+1, y) = \frac{1}{V_f(M)}$. According to (13), there exists a dimensional constant $C_0 > 0$ such that

$$H(x, t, y) \leq H(x_1, t+1, y) e^{C_0(L^2+K+\frac{1}{t}+d^2(x,x_1))}.$$

Since $t \geq 4$, this implies

$$H(x, t, y) \leq \frac{1}{V_f(M)} e^{C_0(L^2+K+\frac{1}{4}+diam_M^2)} := B. \tag{27}$$

Similarly, there is a point x_2 such that $H(x_2, t-1, y) = \frac{1}{V_f(M)}$ and that

$$H(x_2, t-1, y) \leq H(x, t, y) e^{C_0(L^2+K+\frac{1}{t-1}+d^2(x,x_2))},$$

which infers

$$H(x, t, y) \geq \frac{1}{V_f(M)} e^{-C_0(L^2+K+\frac{1}{3}+diam_M^2)}. \tag{28}$$

Using (27) and (28), we find, for $t_0 \geq 4$, that

$$\ln \frac{B}{H(x, t+t_0, y)} \leq 2C_0(L^2 + K + 1 + diam_M^2),\ t \in (0, t_0].$$

This and Lemma 2.1 yield

$$t|\nabla_x \ln H(x, t+t_0, y)|^2 \leq 2C_0(1 + 2Kt_0)(L^2 + K + 1 + diam_M^2),\ t \in (0, t_0].$$

Therefore

$$t|\nabla_x \ln H(x, t, y)|^2 \leq 4C_0(1 + Kt)(L^2 + K + 1 + diam_M^2),\ t \geq 4. \tag{29}$$

Next plugging (26) for $t < 4$ and (29) for $t \geq 4$ into the term T_3 and T_4 in (20) with $u = H(x, t, y)$, we obtain

$$\begin{aligned}
T_3 \leq\ & \left(\frac{2}{n}L^2 + 2K\right) \left(\int_0^4 \int_M |\nabla \ln u|^2 t^2 ((t-\epsilon)^+ Y^+)^{2j} d\mu dt\right) \\
&+ \left(\frac{2}{n}L^2 + 2K\right) \left(\int_4^T \int_M |\nabla \ln u|^2 t^2 ((t-\epsilon)^+ Y^+)^{2j} d\mu dt\right) \\
\leq\ & C_{11} \left(L^2 + K\right) (1 + KT)(T + KT + L^2 T + A^2 T + A^2 KT + diam_M^2 \\
&+ T diam_M^2) \int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j} d\mu dt,
\end{aligned}$$

and

$$T_4 \leq \frac{4}{n} L \int_0^4 \int_M t|\nabla \ln u|((t-\epsilon)^+ Y^+)^{2j+1} d\mu dt$$

$$+\frac{4}{n}L\int_4^T\int_M t|\nabla\ln u|((t-\epsilon)^+Y^+)^{2j+1}d\mu dt$$
$$\le C_{12}L\sqrt{(1+KT)(T+KT+L^2T+A^2T+A^2KT+diam_M^2+Tdiam_M^2)}$$
$$\times\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+1}d\mu dt.$$

Denote $\lambda=\sqrt{(1+KT)(T+KT+L^2T+A^2T+A^2KT+diam_M^2+Tdiam_M^2)}$ and (20) becomes

$$\begin{aligned}(30)\qquad &\left(\frac{2}{n}-\frac{2}{2j+1}\right)\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+2}d\mu dt\\ &\le\left(\frac{2j+2}{2j+1}+C_{12}L\lambda\right)\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j+1}d\mu dt\\ &\quad+C_{11}\left(L^2+K\right)\lambda^2\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j}d\mu dt.\end{aligned}$$

Using the notation

$$A_{j,\epsilon}=\left(\int_0^T\int_M((t-\epsilon)^+Y^+)^j d\mu dt\right)^{\frac{1}{j}},$$

we can write (30) as

$$\begin{aligned}(31)\qquad &\left(\frac{2}{n}-\frac{2}{2j+1}\right)A_{2j+2,\epsilon}^{2j+2}\\ &\le\left(\frac{2j+2}{2j+1}+C_{12}L\lambda\right)A_{2j+1,\epsilon}^{2j+1}+C_{11}\left(L^2+K\right)\lambda^2A_{2j,\epsilon}^{2j}.\end{aligned}$$

By the Hölder inequality,

$$A_{2j+1,\epsilon}^{2j+1}\le A_{2j+2,\epsilon}^{2j+1}\left(\int_0^T\int_M d\mu dt\right)^{\frac{1}{2j+2}},\quad A_{2j,\epsilon}^{2j}\le A_{2j+2,\epsilon}^{2j}\left(\int_0^T\int_M d\mu dt\right)^{\frac{2}{2j+2}},$$

which imply, together with (31), that

$$\begin{aligned}(32)\qquad &\left(\frac{2}{n}-\frac{2}{2j+1}\right)A_{2j+2,\epsilon}^{2}\\ &\le\left(\frac{2j+2}{2j+1}+C_{12}L\lambda\right)A_{2j+2,\epsilon}\left(\int_0^T\int_M d\mu dt\right)^{\frac{1}{2j+2}}\\ &\quad+C_{11}\left(L^2+K\right)\lambda^2\left(\int_0^T\int_M d\mu dt\right)^{\frac{2}{2j+2}}.\end{aligned}$$

Letting $j \to \infty$, we arrive at

$$\frac{2}{n} A_{\infty,\epsilon}^2 \le (1 + C_{12} L\lambda)\, A_{\infty,\epsilon} + C_{11}(L^2 + K)\lambda^2, \tag{33}$$

where $A_{\infty,\epsilon} = \sup\limits_{M\times[0,T]} (t-\epsilon)^+ Y^+$.

This shows

$$\begin{aligned} &\sup_{M\times[0,T]} (t-\epsilon)^+ Y^+ \\ &\le \frac{n}{2} + C_{13}(L+\sqrt{K})\sqrt{(1+KT)(1+T)}diam_M \\ &\quad + C_{13}(L+\sqrt{K})\sqrt{(1+KT)(T+KT+L^2T+A^2T+A^2KT)}. \end{aligned} \tag{34}$$

Since $\epsilon > 0$ is arbitrary and $Y = -\Delta_f \ln H(x,t,y)$, we conclude that

$$\begin{aligned} &t\left(\frac{|\nabla H|^2}{H^2} - \frac{\partial_t H}{H}\right) \\ &\le \frac{n}{2} + C_{13}(L+\sqrt{K})\sqrt{(1+Kt)(1+t)}diam_M \\ &\quad + C_{13}(L+\sqrt{K})\sqrt{(1+Kt)(t+Kt+L^2t+A^2t+A^2Kt)},\ \forall t > 0. \end{aligned} \tag{35}$$

From (35), a short argument from [25] (see also [24]) implies that the same bound actually holds if one replaces the f-heat kernel by any positive solution of the f-heat equation.

This completes the proof of Theorem 1.1. □

3. Li-Yau gradient bounds on closed manifolds under m-Bakry-Émery Ricci curvature conditions

In this section, we prove Theorem 1.3. Since the proof of Theorem 1.3 is similar, so we only present the key steps.

Before starting the proof of Theorem 1.3, let us present some results needed. First of all, Y. Li showed Li-Yau gradient estimate for manifolds with m-Bakry-Émery Ricci curvature bounded below.

Lemma 3.1 ([10], Li-Yau gradient estimate)**.** *Let $(M^n, g, e^{-f}dv)$ be a complete Riemannian manifold with $Ric_f^{m,n} \ge -K$ for $K \ge 0$. Then any positive solution u of the f-heat equation $(\Delta_f - \partial_t)u = 0$ on $M \times (0,T]$ satisfies*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \le \frac{m\alpha^2 K}{\alpha - 1} + \frac{m\alpha^2}{2t}$$

for any $\alpha > 1$.

Based on the above lemma, we can easily get the Harnack inequality.

Corollary 3.2 (Harnack inequality). *Under the same hypotheses as Lemma 3.1, we have*

$$u(x,t_1) \le u(y,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{m\alpha}{2}} e^{\left(\frac{\alpha d^2(x,y)}{4(t_2-t_1)}+\frac{m\alpha K}{\alpha-1}(t_2-t_1)\right)}$$

for all $x, y \in M$ and $0 < t_1 < t_2 \le T$.

In [19], G. Wei and W. Wylie proved the volume comparision for manifolds with m-Bakry-Émery Ricci curvature bounded below (see also [7]).

Lemma 3.3 ([19], Volume comparison). *Let $(M^n, g, e^{-f}dv)$ be a complete Riemannian manifold with $Ric_f^{m,n} \ge -(m-1)H$ for $H \ge 0$. Then $\frac{V_f(B_p(R))}{V_H^m(B(R))}$ is nonincreasing in R.*

In [4], N. Charalambous, Z. Lu and J. Rowlett got the upper and lower bounds for the f-heat kernel.

Lemma 3.4 ([4]). *Let $(M^n, g, e^{-f}dv)$ be a complete Riemannian manifold with $Ric_f^{m,n} \ge -K$ on $B_o(4R+4) \subset M$ for $K \ge 0$, where o is a point on M and $R > 0$. Then for any $x, y \in B_o(\frac{R}{4})$, $0 < t < \frac{R^2}{4}$, and $\delta_1 \in (0,1)$*

$$H(x,t,y) \ge \widetilde{c_6}(\delta_1,m)V_f^{-\frac{1}{2}}(B_x(\sqrt{t}))V_f^{-\frac{1}{2}}(B_y(\sqrt{t}))e^{\left(-\widetilde{c_7}(\delta_1,m)\frac{d^2(x,y)}{t}-\widetilde{c_8}(m)Kt\right)}$$

and

$$\begin{aligned}&H(x,t,y)\\ &\le \widetilde{c_3}(\delta_1,m)V_f^{-\frac{1}{2}}(B_x(\sqrt{t}))V_f^{-\frac{1}{2}}(B_y(\sqrt{t}))e^{\left(-\lambda_{1,f}(M)t-\frac{d^2(x,y)}{\widetilde{c_4}(\delta_1,m)t}+\widetilde{c_5}(m)\sqrt{Kt}\right)}\end{aligned}$$

for some positive constants $\widetilde{c_3}(\delta_1,m)$, $\widetilde{c_4}(\delta_1,m)$, $\widetilde{c_5}(m)$, $\widetilde{c_6}(\delta_1,m)$, $\widetilde{c_7}(\delta_1,m)$ and $\widetilde{c_8}(m)$, where $\lambda_{1,f}(M)$ is the infimum of the weighted Rayleigh quotient on M.

Whenever $Ric_f^{m,n} \ge -K$ on M with $K \ge 0$, then the same bound also holds for all $x, y \in M$ and $t > 0$.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $u = u(x,t)$ be a positive solution of the f-heat equation on $M \times (0, +\infty)$. Write

$$Y = Y(x,t) = \left(\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right) = -\Delta_f \ln u.$$

The Bochner formula [10] for $Ric_f^{m,n}$ is

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla u|^2 &= |Hess\ u|^2 + \langle \nabla\Delta_f u, \nabla u\rangle + Ric_f^{m,n}(\nabla u, \nabla u)\\ &\quad + \frac{1}{m-n}|\langle \nabla f, \nabla u\rangle|^2\\ &\ge \frac{(\Delta_f u)^2}{m} + \langle \nabla\Delta_f u, \nabla u\rangle + Ric_f^{m,n}(\nabla u, \nabla u).\end{aligned} \tag{36}$$

Using (36) and the m-Bakry-Émery Ricci curvature condition, we have

$$
\begin{aligned}
(37)\quad (\Delta_f - \partial_t)Y + 2\langle \nabla Y, \nabla \ln u\rangle &= 2|Hess \ln u|^2 + 2Ric_f^{m,n}(\nabla \ln u, \nabla \ln u)\\
&\quad + \frac{2}{m-n}|\langle \nabla f, \nabla \ln u\rangle|^2\\
&\geq \frac{2(\Delta_f \ln u)^2}{m} - 2K|\nabla \ln u|^2\\
&= \frac{2}{m}Y^2 - 2K|\nabla \ln u|^2.
\end{aligned}
$$

Let $Y^+(x,t) = \max\{Y(x,t), 0\}$. Then the inequality (37) implies that Y^+ is a subsolution of the inequality in the weak sense: on $M \times (0, +\infty)$,

$$
(38)\qquad (\Delta_f - \partial_t)Y^+ + 2\left\langle \nabla Y^+, \nabla \ln u\right\rangle \geq \frac{2}{m}(Y^+)^2 - 2K|\nabla \ln u|^2.
$$

For a positive integer j and a small positive number ϵ, multiplying both sides of (38) by $((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}$ and integrating on $M \times (0, T]$ for any $T > 0$, we obtain

$$
\begin{aligned}
(39)\qquad &\frac{2}{m}\int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j+2} d\mu dt\\
&\leq \int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}(\Delta_f - \partial_t)Y^+ d\mu dt\\
&\quad + 2\int_0^T \int_M ((t-\epsilon)^+)^{2j+2}(Y^+)^{2j}\left\langle \nabla Y^+, \nabla \ln u\right\rangle d\mu dt\\
&\quad + 2K\int_0^T \int_M |\nabla \ln u|^2((t-\epsilon)^+)^{2j+2}(Y^+)^{2j} d\mu dt = T_1 + T_2 + T_3.
\end{aligned}
$$

Using integration by parts as (18) and (19) for T_1 and T_2, we have the similar inequality

$$
\begin{aligned}
(40)\qquad &\left(\frac{2}{m} - \frac{2}{2j+1}\right)\int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j+2} d\mu dt\\
&\leq \frac{2j+2}{2j+1}\int_0^T \int_M ((t-\epsilon)^+ Y^+)^{2j+1} d\mu dt\\
&\quad + \underbrace{2K\int_0^T \int_M |\nabla \ln u|^2((t-\epsilon)^+)^{2j+2}(Y^+)^{2j} d\mu dt}_{T_3}.
\end{aligned}
$$

Using the upper and lower bounds (Lemma 3.4) for the f-heat kernel $H(x,t,y)$ and Harnack inequality (Corollary 3.2), Hamilton type estimate (Lemma 2.1) and the volume comparison theorem (Lemma 3.3) for manifolds with the m-Bakry-Émery Ricci curvature bounded below, we can find the bound for $t|\nabla_x \ln H(x,t,y)|^2$.

For a time $t_0 > 0$, we consider the f-heat kernel $H(x, t+t_0, y)$ with $t \in [0, t_0]$ at a fixed point $y \in M$.

According to Lemma 3.4, we choose $\delta = \frac{1}{2}$ and there exist some positive constants $\widetilde{C_1}$, $\widetilde{C_2}$, $\widetilde{C_3}$, $\widetilde{C_4}$, $\widetilde{C_5}$ and $\widetilde{C_6}$ depending only on m, such that

$$\tag{41} \begin{aligned} &\frac{\widetilde{C_1} e^{[-\frac{d^2(x,y)}{\widetilde{C_2}(t+t_0)} + \widetilde{C_3}\sqrt{K(t+t_0)}]}}{V_f^{\frac{1}{2}}(B_x(\sqrt{t+t_0}))V_f^{\frac{1}{2}}(B_y(\sqrt{t+t_0}))} \\ &\geq H(x, t+t_0, y) \geq \frac{\widetilde{C_4} e^{[-\widetilde{C_5}\frac{d^2(x,y)}{(t+t_0)} - \widetilde{C_6} K(t+t_0)]}}{V_f^{\frac{1}{2}}(B_x(\sqrt{t+t_0}))V_f^{\frac{1}{2}}(B_y(\sqrt{t+t_0}))}. \end{aligned}$$

The upper bound implies

$$B := \sup_{M\times(0,t_0)} H(x, t+t_0, y) \leq \frac{\widetilde{C_7} e^{\widetilde{C_8} K t_0}}{\inf_{z\in M} V_f(B_z(\sqrt{t_0}))},$$

which yields, by the lower bound of $H(x, t+t_0, y)$, that

$$\tag{42} \frac{B}{H(x, t+t_0, y)} \leq \widetilde{C_9} e^{[\widetilde{C_{10}} K t_0 + \widetilde{C_5}\frac{d^2(x,y)}{t_0}]} \frac{\sup_{w\in M} V_f(B_w(\sqrt{2t_0}))}{\inf_{z\in M} V_f(B_z(\sqrt{t_0}))}.$$

We notice the infimum and supremum of the geodesic ball's volumes can be achieved by some points on M, say point p and point q, i.e.,

$$\tag{43} \frac{\sup_{w\in M} V_f(B_w(\sqrt{2t_0}))}{\inf_{x\in M} V_f(B_z(\sqrt{t_0}))} = \frac{V_f(B_q(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))}.$$

By Lemma 3.3, we get

$$\tag{44} \begin{aligned} \frac{V_f(B_p(\sqrt{2t_0} + d(p,q)))}{V_f(B_p(\sqrt{2t_0}))} &\leq \frac{\int_0^{d(p,q)+\sqrt{2t_0}} \left(sinh(\sqrt{\frac{K}{m-1}}r)\right)^{m-1} dr}{\int_0^{\sqrt{2t_0}} \left(sinh(\sqrt{\frac{K}{m-1}}r)\right)^{m-1} dr} \\ &\leq \left(\frac{d(p,q)}{\sqrt{2t_0}} + 1\right)^m e^{\sqrt{K(m-1)}d(p,q)} \\ &= e^{m\ln\left(\frac{d(p,q)}{\sqrt{2t_0}}+1\right) + \sqrt{K(m-1)}d(p,q)} \\ &\leq e^{m\frac{d(p,q)}{\sqrt{2t_0}} + \sqrt{K(m-1)}d(p,q)}, \end{aligned}$$

and

$$\frac{V_f(B_p(\sqrt{2t_0}))}{V_f(B_p(\sqrt{t_0}))} \leq \widetilde{C_{12}} e^{\widetilde{C_{11}}\sqrt{K t_0}}.$$

This and (42) imply that

$$\ln \frac{B}{H(x, t+t_0, y)} \leq \ln \widetilde{C_9} + \widetilde{C_{10}} K t_0 + \widetilde{C_5}\frac{diam_M^2}{t_0} + \ln \widetilde{C_{12}} + \widetilde{C_{11}}\sqrt{K t_0}$$

$$+ m\frac{diam_M}{\sqrt{2t_0}} + \sqrt{K(m-1)}diam_M.$$

Since the m-Bakry-Émery Ricci curvature bounded below, we can deduce Bakry-Émery Ricci curvature bounded below, so we can still use Lemma 2.1 and that

$$t|\nabla_x \ln H(x, t+t_0, y)|^2 \le \widetilde{C_{13}}(1+2Kt_0)\left(1 + Kt_0 + \frac{diam_M^2}{t_0}\right),\ t \in (0, t_0].$$

Then we conclude

$$t|\nabla_x \ln H(x,t,y)|^2 \le \widetilde{C_{14}}(1+Kt)\left(1 + Kt + \frac{diam_M^2}{t}\right),\ \forall t > 0. \tag{45}$$

The bound is adequate for us when then time is short, say $t \le 4$. For any time $t \ge 4$, since $\int_M H(x, t+1, y)d\mu(x) = 1$, there is a point $x_1 \in M$ such that $H(x_1, t+1, y) = \frac{1}{V_f(M)}$. According to Corollary 3.2 with $\alpha = 2$, $t_1 = t$, $t_2 = t+1$, we have

$$H(x,t,y) \le H(x_1, t+1, y)\left(\frac{t+1}{t}\right)^m e^{\left(2mK + \frac{d^2(x,x_1)}{2}\right)}.$$

Since $t \ge 4$, this implies

$$H(x,t,y) \le \left(\frac{5}{4}\right)^m \frac{1}{V_f(M)} e^{\left(2mK + \frac{diam_M^2}{2}\right)} := B. \tag{46}$$

Similarly, there is a point x_2 such that $H(x_2, t-1, y) = \frac{1}{V_f(M)}$ and that

$$H(x_2, t-1, y) \le H(x,t,y)\left(\frac{t}{t-1}\right)^m e^{\left(2mK + \frac{d^2(x,x_2)}{2}\right)},$$

which implies

$$H(x,t,y) \ge \left(\frac{3}{4}\right)^m \frac{1}{V_f(M)} e^{\left(-2mK - \frac{diam_M^2}{2}\right)},\ t \ge 4. \tag{47}$$

Using (46) and (47), we find, for $t_0 \ge 4$, that

$$\ln\frac{B}{H(x, t+t_0, y)} \le m\ln 2 + 4mK + diam_M^2,\ t \in (0, t_0].$$

This and Lemma 2.1 yield

$$t|\nabla_x \ln H(x, t+t_0, y)|^2 \le (1+2Kt_0)(m\ln 2 + 4mK + diam_M^2),\ t \in (0, t_0].$$

Therefore

$$t|\nabla_x \ln H(x,t,y)|^2 \le 2(1+Kt)(m\ln 2 + 4mK + diam_M^2),\ t \ge 4. \tag{48}$$

Next, plugging (45) for $t < 4$ and (48) for $t \ge 4$ into the term T_3 in (40) with $u = H(x,t,y)$, we obtain

$$T_3 \le K(1+KT)(\widetilde{C_{15}}(T+KT+diam_M^2)+Tdiam_M^2)\int_0^T\int_M((t-\epsilon)^+Y^+)^{2j}d\mu dt.$$

These are the key steps to prove Theorem 1.3 and the rest of the proof is similar to Theorem 1.1. □

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