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THE FIRST POSITIVE AND NEGATIVE DIRAC EIGENVALUES ON SASAKIAN MANIFOLDS

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Abstract. Using the results in the paper [12], we give an estimate for the first positive and negative Dirac eigenvalue on a 7-dimensional Sasakian spin manifold. The limiting case of this estimate can be attained if the manifold under consideration admits a Sasakian Killing spinor. By imposing the eta-Einstein condition on Sasakian manifolds of higher dimensions $2m+1 \geq 9$, we derive some new Dirac eigenvalue inequalities that improve the recent results in [12, 13].

1. Introduction

Let (M^n, g) be an n-dimensional closed Riemannian spin manifold. Consider the Riemann Dirac operator D of (M^n, g) defined by

$$D\psi = \sum_{u=1}^{n} E_u \cdot \nabla_{E_u} \psi,$$

where $(E_1, \dots, E_n)$ is a local orthonormal frame on (M^n, g), $\nabla\psi$ is the Levi-Civita connection acting on sections $\psi \in \Gamma(\Sigma(M))$ of the spinor bundle $\sum(M)$ over M^n and the dot "$\cdot$" denotes the Clifford multiplication [4, 5, 7]. Since D is a first-order self-adjoint elliptic operator, the spectrum Spec(D) of D is an unbounded discrete subset of $\mathbb{R}$ and can be written as

$$\cdots \leq \lambda_2^- \leq \lambda_1^- \leq 0 \leq \lambda_1^+ \leq \lambda_2^+ \leq \cdots,$$

where the eigenvalues have finite multiplicity, tend to $+\infty$ and $-\infty$ and zero "0" does not necessarily belong to Spec(D). In this sequence, the nonzero eigenvalues $\lambda_1^- \neq 0$ and $\lambda_1^+ \neq 0$ are called the *first negative eigenvalue* and the *first positive eigenvalue*, respectively. The eigenvalue $\lambda_1 \in$ Spec(D) with $|\lambda_1| = \min\{|\lambda_1^-|, |\lambda_1^+|\}$ is called the *first eigenvalue*. It is well-known [3, 9] that Spec(D) is symmetric with respect to zero if $n \not\equiv 3$ mod 4. In the case where $n \equiv 3$ mod 4, Spec(D) is generally asymmetric with respect to zero, and it is

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a problem of fundamental interest to find an optimal estimate for λ_1^- and that for λ_1^+.

Recall that a Sasakian manifold is an odd-dimensional Riemannian manifold (M^{2m+1}, g) equipped with a Sasakian structure (ϕ, ξ, η, g), where ϕ, ξ, η are a (1,1)-tensor field, Killing vector field of unit length and 1-form such that

$$\eta(\xi) = 1, \qquad \phi^2(X) = -X + \eta(X)\xi,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X$$

hold for all vector fields $X, Y \in \Gamma(T(M))$. A Sasakian manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is said to be *eta-Einstein* if the scalar curvature S is constant and the Ricci tensor Ric is given by

$$\mathrm{Ric} = \left(\frac{S}{2m} - 1\right) g + \left(2m + 1 - \frac{S}{2m}\right) \eta \otimes \eta.$$

In [10,11], we proved that if $(M^3, \phi, \xi, \eta, g)$ is a 3-dimensional closed Sasakian spin manifold and the minimum $S_{\min}$ of the scalar curvature satisfies $S_{\min} > -\frac{3}{2}$, then the first negative and positive Dirac eigenvalues satisfy

$$\lambda_1^- \leq \frac{1 - \sqrt{2S_{\min} + 4}}{2} \tag{1}$$

and

$$\lambda_1^+ \geq \begin{cases} \frac{S_{\min}+6}{8} & \text{for } \; -\frac{3}{2} < S_{\min} \leq 30, \\ \frac{1+\sqrt{2S_{\min}+4}}{2} & \text{for } \; S_{\min} \geq 30, \end{cases} \tag{2}$$

respectively. Furthermore, if $(M^5, \phi, \xi, \eta, g)$ is a 5-dimensional closed Sasakian spin manifold, we proved in [13] that the first Dirac eigenvalue satisfies

$$|\lambda_1| \geq \begin{cases} -\frac{1}{2} + \sqrt{\frac{S_{\min}+4}{3} + \frac{(S_{\min}+28)^2}{2304}} & \text{for } \; -4 < S_{\min} \leq 164, \\ -\frac{1}{2} + \sqrt{\frac{S_{\min}-20}{2}} & \text{for } \; S_{\min} \geq 164. \end{cases} \tag{3}$$

In this paper, we continue to study the first Dirac eigenvalue problem on Sasakian manifolds in dimensions $2m + 1 \geq 7$. Making use of the results in [12], we prove three main theorems: namely, Theorem 3.2 in Section 3 and Theorems 4.4 and 4.9 in Section 4.

We remark that inequalities (1)-(3) and all inequalities presented in this paper obviously enhance the Friedrich inequality

$$|\lambda_1| \geq \sqrt{\frac{nS_{\min}}{4(n-1)}},$$

the limiting case of which is characterized by the existence of a Killing spinor [6, 8]. We refer to [5, 8] for a detailed exposition of the first Dirac eigenvalue problem, especially the improvements of the Friedrich inequality for manifolds with special holonomy.

This paper is organized as follows. In the next section we collect the basic notions associated with three special classes of spinor fields, namely, Sasakian twistor spinors, Sasakian duos and Sasakian Killing spinors. It turns out that the Sasakian Killing spinors generalize to the Sasakian duos, which play a crucial role when we investigate the limiting case of an estimate for the first Dirac eigenvalue. In Section 3 we establish a new estimate for the first positive and negative Dirac eigenvalues on a 7-dimensional Sasakian spin manifold (see Theorem 3.2). In Section 4 we focus on eta-Einstein Sasakian manifolds of higher dimensions $2m+1 \geq 9$ and prove Theorems 4.4 and 4.9. More precisely, we prove that inequalities (39)-(41) in Theorem 4.4 improve (6.15)-(6.17) in [12] while (50) in Theorem 4.9 improves (3.17) in [13]. What is worth noting about these improvements is that we replaced the excessive restrictions imposed on the scalar curvature with more reasonable ones.

2. Three special classes of spinor fields on Sasakian manifolds

Throughout the paper we denote by $(M^{2m+1}, \phi, \xi, \eta, g)$ a Sasakian spin manifold of dimension $2m+1 \geq 3$ and by $\Sigma = \Sigma(M)$ the spinor bundle over M^{2m+1}. Let $\xi^{\perp}$ denote the orthogonal complement of the Killing vector field ξ in the tangent bundle $T(M)$ of $(M^{2m+1}, \phi, \xi, \eta, g)$. Consider a natural deformation [1, 11] of the Levi-Civita connection ∇ in the subbundle $\xi^{\perp} \subset T(M)$

$$\overline{\nabla}_V \psi = \nabla_V \psi - \frac{1}{2}\phi(V) \cdot \xi \cdot \psi, \quad V \in \Gamma(\xi^{\perp}),$$

where $\nabla\psi$ is the Levi-Civita connection acting on sections $\psi \in \Gamma(\Sigma(M))$ of the spinor bundle. Define non-elliptic, first-order operators $\overline{C}, \overline{Q}, D_+, D_-$ by

$$\overline{C}\psi = \sum_{u=1}^{2m} E_u \cdot \overline{\nabla}_{E_u}\psi, \qquad \overline{Q}\psi = \sum_{u=1}^{2m} \phi(E_u) \cdot \overline{\nabla}_{E_u}\psi,$$

$$D_+\psi = \frac{1}{2}\overline{C}\psi + \frac{\sqrt{-1}}{2}\overline{Q}\psi, \qquad D_-\psi = \frac{1}{2}\overline{C}\psi - \frac{\sqrt{-1}}{2}\overline{Q}\psi,$$

where $(E_1, \dots, E_{2m}, E_{2m+1} = \xi)$ is a local orthonormal frame on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then the operator identities

$$\overline{C} \circ \overline{Q} + \overline{Q} \circ \overline{C} = 0, \qquad \overline{C}^2 = \overline{Q}^2$$

and

$$D_+ \circ D_+ = D_- \circ D_- = 0$$

hold. These identities allow us to establish a Sasakian version of twistor spinors in a similar way as done for the Kählerian version [4, 8, 15, 16].

In [12, 13], we introduced a two-parameter generalization of the Riemann Dirac operator

$$D_{ab} := \overline{C} + aD_0 + \left(b - \frac{1}{2}\right)\xi \circ \Phi,$$

where $a, b \in \mathbb{R}$ are real numbers, $\Phi := g(\,\cdot\,, \phi(\cdot))$ is the fundamental form and $D_0 := \xi \circ \nabla_\xi - \frac{1}{2}\xi \circ \Phi$. The generalized Dirac operator D_{ab} coincides with the Riemann Dirac operator D if $(a = 1, b = 0)$ and with the so-called *cubic Dirac operator* [1, 2] if $\left(a = 1, b = \frac{1}{2}\right)$. When $(M^{2m+1}, \phi, \xi, \eta, g)$ is a closed manifold and $a \neq 0$ is non-zero, D_{ab} is a self-adjoint elliptic operator of first-order and hence its spectrum $\mathrm{Spec}(D_{ab})$ is discrete and real. In that case we can define the first negative eigenvalue λ_1^-, the first positive eigenvalue λ_1^+ and the first eigenvalue λ_1 of $\mathrm{Spec}(D_{ab})$ just as in the case of $\mathrm{Spec}(D)$.

Denote $P_+(V), P_-(V)$ the complex vector fields

$$P_+(V) = \frac{1}{2}\left[V + \sqrt{-1}\phi(V)\right], \quad P_-(V) = \frac{1}{2}\left[V - \sqrt{-1}\phi(V)\right], \quad V \in \Gamma(\xi^\perp).$$

Then it holds that

$$\begin{aligned}
&(P_+ \circ P_+)(V) = P_+(V),\\
&(P_- \circ P_-)(V) = P_-(V),\\
&(P_+ \circ P_-)(V) = (P_- \circ P_+)(V) = 0,\\
&P_+(V) \cdot P_+(V) = P_-(V) \cdot P_-(V) = 0,\\
&\sum_{u=1}^{2m} P_+(E_u) \cdot P_-(E_u) = -m + \sqrt{-1}\Phi,\\
&\sum_{u=1}^{2m} P_-(E_u) \cdot P_+(E_u) = -m - \sqrt{-1}\Phi
\end{aligned}$$

and

$$\begin{aligned}
D_+ &= \frac{1}{2}\overline{C} + \frac{\sqrt{-1}}{2}\overline{Q} = \sum_{u=1}^{2m} P_+(E_u) \cdot \overline{\nabla}_{E_u} = \sum_{u=1}^{2m} E_u \cdot \overline{\nabla}_{P_-(E_u)},\\
D_- &= \frac{1}{2}\overline{C} - \frac{\sqrt{-1}}{2}\overline{Q} = \sum_{u=1}^{2m} P_-(E_u) \cdot \overline{\nabla}_{E_u} = \sum_{u=1}^{2m} E_u \cdot \overline{\nabla}_{P_+(E_u)}.
\end{aligned}$$

Under the Clifford action of the fundamental form Φ, the spinor bundle Σ splits into the orthogonal direct sum $\Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_m$, where $\sum_r$ denotes the rank-$\binom{m}{r}$ eigenbundle associated with the eigenvalue $\sqrt{-1}(2r - m)$ of Φ [14]. Since $\overline{\nabla}_V$ commutes with Φ, we have

$$\overline{\nabla}_V(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_r), \quad D_+(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_{r+1}), \quad D_-(\Gamma(\Sigma_r)) \subset \Gamma(\Sigma_{r-1}),$$

where we use the convention that Σ_s is the zero subbundle of Σ if $s \notin \{0, 1, \ldots, m\}$.

Definition. Let $\varphi_r \in \Gamma(\Sigma_r)$ and $\varphi_{r+1} \in \Gamma(\Sigma_{r+1})$, $r \in \{0, 1, \ldots, m-1\}$, be non-trivial spinor fields on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then,

(i) φ_r and φ_{r+1} are called a *left* and a *right Sasakian twistor spinor* if, for all vector fields $V \in \Gamma(\xi^\perp)$,

$$\overline{\nabla}_V \varphi_r + \frac{1}{2r+2} P_-(V) \cdot D_+ \varphi_r = 0$$

and

$$\overline{\nabla}_V \varphi_{r+1} + \frac{1}{2m-2r} P_+(V) \cdot D_- \varphi_{r+1} = 0$$

hold, respectively.

(ii) A pair $(\varphi_r, \varphi_{r+1})$ is called a *Sasakian duo* with characteristic numbers (c_+, c_-) if, for all vector fields $V \in \Gamma(\xi^\perp)$, the system of two differential equations

$$\overline{\nabla}_V \varphi_r + \frac{c_+}{2r+2} P_-(V) \cdot \varphi_{r+1} = 0,$$
$$\overline{\nabla}_V \varphi_{r+1} + \frac{c_-}{2m-2r} P_+(V) \cdot \varphi_r = 0$$

hold, where $c_+, c_- \in \mathbb{C}$ are complex numbers such that the product $c_+ c_- \in \mathbb{R}$ is a real number.

(iii) When m is odd and $(\varphi_l, \varphi_{l+1})$, $l := \frac{m-1}{2}$, is a Sasakian duo with characteristic numbers (c, c), $0 \neq c \in \mathbb{R}$, the spinor field $\varphi := \varphi_l + \varphi_{l+1}$ is a *Sasakian Killing spinor* with characteristic number c, i.e., is a solution to the *Sasakian Killing equation*

$$\overline{\nabla}_V \varphi = -\frac{c}{2(m+1)} V \cdot \varphi - (-1)^{l+1} \frac{c}{2(m+1)} \phi(V) \cdot \xi \cdot \varphi.$$

Recall [9] that there exists such a complex-antilinear mapping $j_1 : \Sigma(M) \to \Sigma(M)$ on the spinor bundle $\Sigma(M)$ that satisfies

$$j_1(\Gamma(\Sigma_r)) = \Gamma(\Sigma_{m-r}), \quad r \in \{0, 1, \ldots, m\}. \tag{4}$$

Since j_1 commutes with the spinor derivative ∇ and the relations

$$j_1 \circ P_\pm(V) = (-1)^{m+1} P_\mp(V) \circ j_1, \quad D_\pm \circ j_1 = (-1)^{m+1} j_1 \circ D_\mp \tag{5}$$

hold, there exists a natural one-to-one correspondence, $\varphi_r \mapsto j_1 \varphi_r$, between the left Sasakian twistor spinors and the right Sasakian twistor spinors.

Lemma 2.1 (see [13])**.** *Let $\varphi_r \in \Gamma(\Sigma_r)$, $r \in \{0, 1, \ldots, m-1\}$, be a left Sasakian twistor spinor on $(M^{2m+1}, \phi, \xi, \eta, g)$. If $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein and $r \neq 0$, then it holds that*

$$D_0 \varphi_r = (-1)^{m+r} \frac{2r+1-m}{8m(m+1)} (S + 2m) \varphi_r \tag{6}$$

and

$$\overline{C}^2(\varphi_r) = \frac{(r+1)(m-r)}{m(m+1)} (S + 2m) \varphi_r. \tag{7}$$

Lemma 2.2 (see [13]). *Let $\varphi_{r+1} \in \Gamma(\Sigma_{r+1})$, $r \in \{0, 1, \dots, m-1\}$, be a right Sasakian twistor spinor on $(M^{2m+1}, \phi, \xi, \eta, g)$. If $(M^{2m+1}, \phi, \xi, \eta, g)$ is eta-Einstein and $r \neq m-1$, then it holds that*

$$D_0 \varphi_{r+1} = (-1)^{m+r+1} \frac{2r+1-m}{8m(m+1)} (S + 2m) \varphi_{r+1} \tag{8}$$

and

$$\overline{C}^2(\varphi_{r+1}) = \frac{(r+1)(m-r)}{m(m+1)} (S + 2m) \varphi_{r+1}. \tag{9}$$

In what follows, we fix the notation $\mathrm{E}^{\lambda}(D_{ab})$ to denote the space of all eigenspinors of the generalized Dirac operator D_{ab} with eigenvalue λ.

Definition. Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold. For each $\lambda \in \mathrm{Spec}(D_{ab})$, at least one of the following two statements is true.

(i) There exists a non-trivial eigenspinor $\psi \in \mathrm{E}^{\lambda}(D_{ab})$, called a *one-component eigenspinor* of D_{ab}, such that $\psi = \psi_p \in \Gamma(\Sigma_p)$ is a section in Σ_p for some $p \in \{0, 1, \dots, m\}$.

(ii) There exists an eigenspinor $\psi \in \mathrm{E}^{\lambda}(D_{ab})$, called a *two-component eigenspinor* of D_{ab}, such that $\psi = \psi_r + \psi_{r+1} \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$ is a section in $\Sigma_r \oplus \Sigma_{r+1}$ for some $r \in \{0, 1, \dots, m-1\}$ and all of ψ_r, ψ_{r+1}, $D_+\psi_r$, $D_-\psi_{r+1}$ are non-trivial.

It is convenient to fix the notations

$$\begin{aligned} \mathrm{Spec}(D_{ab}, \mathrm{I}) &:= \{\lambda \in \mathrm{Spec}(D_{ab}) \,|\, \text{There exists a one-componrnt eigenspinor} \\ &\qquad \psi = \psi_p \in \mathrm{E}^{\lambda}(D_{ab})\}, \\ \mathrm{Spec}(D_{ab}, \mathrm{II}) &:= \{\lambda \in \mathrm{Spec}(D_{ab}) \,|\, \text{There exists a two-componrnt eigenspinor} \\ &\qquad \psi = \psi_r + \psi_{r+1} \in \mathrm{E}^{\lambda}(D_{ab})\}. \end{aligned}$$

Recall that, for any $\lambda \in \mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$, $a \neq 0$, there exists a two-component eigenspinor $\psi = \psi_r + \psi_{r+1} \in \mathrm{E}^{\lambda}(D_{ab})$ such that

$$D_0 \psi_r = (-1)^{m+r} \nu \psi_r, \quad D_0 \psi_{r+1} = (-1)^{m+r+1} \nu \psi_{r+1}$$

hold for some real number $\nu \in \mathbb{R}$. For brevity, let us introduce the notations

$$(-1)^{m+r} \sigma_+ := (-1)^{m+r} \lambda + \frac{1}{2} - b + a\nu - \left(\frac{1}{2} - b\right)(m - 1 - 2r), \tag{10}$$

$$(-1)^{m+r} \sigma_- := (-1)^{m+r} \lambda + \frac{1}{2} - b - a\nu + \left(\frac{1}{2} - b\right)(m - 1 - 2r). \tag{11}$$

Then it holds that

$$D_+ \psi_r = \sigma_+ \psi_{r+1}, \quad D_- \psi_{r+1} = \sigma_- \psi_r, \quad D_+ \psi_{r+1} = D_- \psi_r = 0$$

and

$$\overline{C}^2 \psi_r = \sigma_+ \sigma_- \psi_r, \quad \overline{C}^2 \psi_{r+1} = \sigma_+ \sigma_- \psi_{r+1}.$$

Lemma 2.3 (see [13]). *Let $(\varphi_r, \varphi_{r+1})$ be a Sasakian duo with characteristic numbers (c_+, c_-) and let $\varphi = \varphi_r + \varphi_{r+1} \in \mathrm{E}^\lambda(D_{ab})$. Then all the possible values for c_+, c_- are:*

(a)
$$\begin{aligned}(-1)^{m+r}c_\pm &= \mp \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right](m-1-2r) \\ &\quad + \sqrt{\tfrac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right]^2 (m-1-2r)^2}.\end{aligned}$$

In this case λ is equal to

$$(-1)^{m+r}\lambda = b - \frac{1}{2} + \sqrt{\tfrac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right]^2 (m-1-2r)^2}.$$

(b)
$$\begin{aligned}(-1)^{m+r}c_\pm &= \mp \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right](m-1-2r) \\ &\quad - \sqrt{\tfrac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right]^2 (m-1-2r)^2}.\end{aligned}$$

In this case λ is equal to

$$(-1)^{m+r}\lambda = b - \tfrac{1}{2} - \sqrt{\tfrac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\tfrac{a(S+2m)}{8m(m+1)} + \tfrac{1}{2} - b\right]^2 (m-1-2r)^2}.$$

The following proposition is easy to verify, we omit the proof.

Proposition 2.4. *Let $\psi = \psi_r + \psi_{r+1} \in \mathrm{E}^\lambda(D_{ab})$ be a two-component eigenspinor of D_{ab} such that*

$$D_0\psi_r = (-1)^{m+r}\nu\psi_r, \quad D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$$

hold for some real number $\nu \in \mathbb{R}$. Then the spinor field $\psi^* = \psi_r^* + \psi_{r+1}^*$ defined by

$$\psi_r^* := \sigma_-\psi_r, \quad \psi_{r+1}^* := -\sigma_+\psi_{r+1}$$

is an eigenspinor of D_{ab} satisfying

$$D_{ab}\psi^* = \lambda^*\psi^*, \qquad D_0\psi_r^* = (-1)^{m+r}\nu^*\psi_r^*, \quad D_0\psi_{r+1}^* = (-1)^{m+r+1}\nu^*\psi_{r+1}^*,$$
$$D_+\psi_r^* = \sigma_+^*\psi_{r+1}^*, \quad D_-\psi_{r+1}^* = \sigma_-^*\psi_r^*,$$

where

$$\lambda^* = -\lambda + (-1)^{m+r}(2b-1)$$

and

$$\nu^* = \nu, \quad \sigma_+^* = -\sigma_-, \quad \sigma_-^* = -\sigma_+.$$

In particular, if (ψ_r, ψ_{r+1}) is a Sasakian duo with characteristic numbers (σ_+, σ_-), then (ψ_r^*, ψ_{r+1}^*) is a Sasakian duo with characteristic numbers (σ_+^*, σ_-^*).

Proposition 2.4 implies that if $\varphi = \varphi_r + \varphi_{r+1} \in \mathrm{E}^{\lambda}(D_{ab})$ is the spinor field described in the part (a) (resp. part (b)) of Lemma 2.3, then $\varphi^* = \varphi_r^* + \varphi_{r+1}^* \in \mathrm{E}^{\lambda^*}(D_{ab})$ is the spinor field described in the part (b) (resp. part (a)). The following lemma is useful when we study a limiting case of Dirac eigenvalue inequalities (see Theorems 4.4 and 4.9).

Lemma 2.5 (see [13]). *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold with constant scalar curvature $S + 2m > 0$ and let $\psi := \psi_r + \psi_{r+1} \in \mathrm{E}^{\lambda}(D_{ab})$, $r \in \{0, 1, \dots, m-1\}$. Assume that ψ_r, ψ_{r+1} satisfy*

$$D_0\psi_r = (-1)^{m+r}\nu\psi_r \text{ and } D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$$

with $\nu = \frac{2r+1-m}{8m(m+1)}(S + 2m)$.

(a) *If ψ_r is a left Sasakian twistor spinor, then ψ_{r+1} is a right Sasakian twistor spinor and hence (ψ_r, ψ_{r+1}) becomes a Sasakian duo with characteristic numbers (c_+, c_-) described in Lemma 2.3.*

(b) *If ψ_{r+1} is a right Sasakian twistor spinor, then ψ_r is a left Sasakian twistor spinor and hence (ψ_r, ψ_{r+1}) becomes a Sasakian duo with characteristic numbers (c_+, c_-) described in Lemma 2.3.*

3. Estimates on a 7-dimensional Sasakian manifold

In the following lemma, we summarize what is established for Dirac eigenvalue inequalities in [12,13]. To clarify the context, from now on, we denote by $\check{\lambda}$ an element of $\mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$ and by $\hat{\lambda}$ an element of $\mathrm{Spec}(D_{ab}, \mathrm{I})$.

Lemma 3.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold.*

(a) *Then, for any $\check{\lambda} \in \mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$, $a \neq 0$, there exists some real number $\nu \in \mathbb{R}$ and $r \in \{0, 1, \dots, m-1\}$ such that the three inequalities*

$$\begin{aligned} &\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2} - b\right]^2 \\ &- \left[a\nu + (m-1-2r)\left(\frac{m+1}{am} - \frac{1}{2} + b\right)\right]^2 \\ &\geq \frac{m+1}{4m}(S_{\min} + 2m) - \frac{m+1}{am}\left(\frac{m+1}{am} - 1 + 2b\right)(m-1-2r)^2 \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2} - b\right]^2 \\ &- \left[a\nu - \left(\frac{1}{2} - b\right)(m-1-2r) + \frac{2(r+1)(m-2r)}{a(2r+1)}\right]^2 \\ &\geq \frac{r+1}{2(2r+1)}(S_{\min} + 2m) - \frac{4(r+1)^2(m-2r)^2}{a^2(2r+1)^2} \end{aligned}$$

$$+\frac{2(1-2b)(r+1)}{a(2r+1)}(m-2r)(m-1-2r) \tag{13}$$

and

$$\begin{aligned}&\left[(-1)^{m+r}\check{\lambda}+\frac{1}{2}-b\right]^2\\ &-\left[a\nu-\left(\frac{1}{2}-b\right)(m-1-2r)+\frac{2(m-r)(m-2-2r)}{a(2m-2r-1)}\right]^2\\ \geq\ &\frac{m-r}{2(2m-2r-1)}(S_{\min}+2m)-\frac{4(m-r)^2(m-2-2r)^2}{a^2(2m-2r-1)^2}\\ &+\frac{2(1-2b)(m-r)}{a(2m-2r-1)}(m-1-2r)(m-2-2r)\end{aligned} \tag{14}$$

hold.

(b) *For any* $\hat{\lambda}\in\mathrm{Spec}(D_{ab},\mathrm{I})$, $a\neq 0$, *there exists some* $p\in\{0,1,\dots,m\}$ *such that the inequality*

$$-\frac{1}{a}(-1)^{m+p}(m-2p)\hat{\lambda}\geq\frac{1}{8}\left(S_{\min}+2m\right)+\frac{1}{a}\left(\frac{1}{2}-b\right)(m-2p)^2 \tag{15}$$

holds.

Let us now consider a 7-dimensional closed Sasakian spin manifold (M^7,ϕ,ξ,η,g) and try to find an optimal estimate for the first positive and negative eigenvalues of $\mathrm{Spec}(D_{ab})$ by comparing the four inequalities (12)-(15). The spinor bundle Σ of (M^7,ϕ,ξ,η,g) splits into the orthogonal direct sum $\Sigma=\Sigma_0\oplus\Sigma_1\oplus\Sigma_2\oplus\Sigma_3$. Since the complex-antilinear mapping $j_1:\Sigma\longrightarrow\Sigma$ satisfies $j_1(\Sigma_0)=\Sigma_3$, $j_1(\Sigma_1)=\Sigma_2$ and $D_{ab}\circ j_1=j_1\circ D_{ab}$, it suffices to consider the dependence of (12)-(15) on r only in the subset $\{0,1\}$ of $\{0,1,2,3\}$.

Let $\check{\lambda}\in\mathrm{Spec}(D_{ab},\mathrm{II})\setminus\mathrm{Spec}(D_{ab},\mathrm{I})$ and let $\psi=\psi_r+\psi_{r+1}\in\mathrm{E}^{\check{\lambda}}(D_{ab})$, $r\in\{0,1\}$, be a two-component eigenspinor of D_{ab} such that

$$D_0\psi_r=(-1)^{m+r}\nu\psi_r,\quad D_0\psi_{r+1}=(-1)^{m+r+1}\nu\psi_{r+1}$$

hold for some real number $\nu\in\mathbb{R}$. Choose the free parameters a,b of D_{ab} to be

$$a=1,\quad b=0.$$

Inserting ($a=1$, $b=0$) and $m=3$, $r=0$ into (13) and (14), we obtain

$$\left(-\check{\lambda}+\frac{1}{2}\right)^2-(\nu+5)^2\geq\frac{1}{2}(S_{\min}+6)-24, \tag{16}$$

$$\left(-\check{\lambda}+\frac{1}{2}\right)^2-\left(\nu+\frac{1}{5}\right)^2\geq\frac{3}{10}(S_{\min}+6)+\frac{24}{25}. \tag{17}$$

Inserting ($a=1$, $b=0$) and $m=3$, $r=1$ into (12), we obtain

$$\left(\check{\lambda}+\frac{1}{2}\right)^2-\nu^2\geq\frac{1}{3}(S_{\min}+6). \tag{18}$$

Let $\check{\lambda}^+ \in \mathrm{Spec}(D_{ab},\,\mathrm{II})\backslash\mathrm{Spec}(D_{ab},\,\mathrm{I})$ and $\check{\lambda}^- \in \mathrm{Spec}(D_{ab},\,\mathrm{II})\backslash\mathrm{Spec}(D_{ab},\,\mathrm{I})$ be, respectively, a positive and negative eigenvalue of $\mathrm{Spec}(D_{ab},\,\mathrm{II})\setminus\mathrm{Spec}(D_{ab},\,\mathrm{I})$. Assume that $S_{\min}+6>48$. Then, from (16)-(18) it follows that

$$\begin{aligned}\check{\lambda}^+ \geq \min\Big\{ &-\frac{1}{2}+\sqrt{\frac{1}{3}(S_{\min}+6)},\\ &\max\Big\{\frac{1}{2}+\sqrt{\frac{1}{2}(S_{\min}+6)-24},\ \frac{1}{2}+\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}}\Big\}\Big\},\end{aligned}\tag{19}$$

$$\begin{aligned}\check{\lambda}^- \leq \max\Big\{ &-\frac{1}{2}-\sqrt{\frac{1}{3}(S_{\min}+6)},\\ &\min\Big\{\frac{1}{2}-\sqrt{\frac{1}{2}(S_{\min}+6)-24},\ \frac{1}{2}-\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}}\Big\}\Big\},\end{aligned}\tag{20}$$

where

$$\begin{aligned}&\max\Big\{\frac{1}{2}+\sqrt{\frac{1}{2}(S_{\min}+6)-24},\ \frac{1}{2}+\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}}\Big\}\\ &= \begin{cases}\frac{1}{2}+\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}} & \text{for } \ 48<S_{\min}+6\leq\frac{624}{5},\\ \frac{1}{2}+\sqrt{\frac{1}{2}(S_{\min}+6)-24} & \text{for } \ S_{\min}+6\geq\frac{624}{5}\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\min\Big\{\frac{1}{2}-\sqrt{\frac{1}{2}(S_{\min}+6)-24},\ \frac{1}{2}-\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}}\Big\}\\ &= \begin{cases}\frac{1}{2}-\sqrt{\frac{3}{10}(S_{\min}+6)+\frac{24}{25}} & \text{for } \ 48<S_{\min}+6\leq\frac{624}{5},\\ \frac{1}{2}-\sqrt{\frac{1}{2}(S_{\min}+6)-24} & \text{for } \ S_{\min}+6\geq\frac{624}{5}.\end{cases}\end{aligned}$$

Next, let $\hat{\lambda}^+ \in \mathrm{Spec}(D_{ab},\,\mathrm{I})$ and $\hat{\lambda}^- \in \mathrm{Spec}(D_{ab},\,\mathrm{I})$ be, respectively, a positive and negative eigenvalue of $\mathrm{Spec}(D_{ab},\,\mathrm{I})$. Inserting $(a=1,\,b=0)$, $m=3$ and $p\in\{0,1\}$ into (15), we obtain

$$\hat{\lambda}^+ \geq \frac{1}{24}(S_{\min}+6)+\frac{3}{2},\tag{21}$$

$$\hat{\lambda}^- \leq -\frac{1}{8}(S_{\min}+6)-\frac{1}{2}.\tag{22}$$

Comparing (19)-(20) with (21)-(22) leads us to (23)-(24) in the following theorem. For the last statement of the theorem, we refer to the proof of Proposition 2.3 and that of Theorem 2.3 in [13].

Theorem 3.2. *Let (M^7,ϕ,ξ,η,g) be a 7-dimensional closed Sasakian spin manifold. Assume that the free parameters a,b of D_{ab} are chosen to be*

$$a=1,\quad b=0$$

and the scalar curvature S satisfies

$$S_{\min} > -\frac{21}{4}.$$

Then the first positive eigenvalue λ_1^+ of D_{ab} satisfies

$$\lambda_1^+ \geq -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min} + 6)} \tag{23}$$

and the first negative eigenvalue λ_1^- of D_{ab} satisfies

$$\lambda_1^- \leq \begin{cases} \frac{1}{2} - \sqrt{\frac{3}{10}(S_{\min}+6) + \frac{24}{25}} & \textit{for } \frac{3}{4} < S_{\min} + 6 \leq \frac{624}{5}, \\ \frac{1}{2} - \sqrt{\frac{1}{2}(S_{\min}+6) - 24} & \textit{for } \frac{624}{5} \leq S_{\min} + 6 \leq 324 + \sqrt{14976}, \\ -\frac{1}{2} - \sqrt{\frac{1}{3}(S_{\min}+6)} & \textit{for } S_{\min} + 6 \geq 324 + \sqrt{14976}. \end{cases} \tag{24}$$

The equality $\lambda_1^+ = -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min} + 6)}$ in (23) (*resp.* $\lambda_1^- = -\frac{1}{2} - \sqrt{\frac{1}{3}(S_{\min} + 6)}$ *in* (24)) *occurs if and only if $(M^7, \phi, \xi, \eta, g)$ admits a Sasakian Killing spinor. If $(M^7, \phi, \xi, \eta, g)$ is simply-connected and admits a Sasakian Killing spinor, then $(M^7, \phi, \xi, \eta, g)$ must be eta-Einstein and isometric to either a* 7*-dimensional spherical Sasakian space form or Tanno deformation of a* 7*-dimensional* 3*-Sasakian manifold.*

Remark 3.3. Consider the *Tanno deformation* [13, 14] of Sasakian structure $(\phi_0, \xi_0, \eta_0, g_0)$

$$\phi := \phi_0, \quad \xi := t^2\xi_0, \quad \eta := t^{-2}\eta_0, \quad g := t^{-2}g_0 + (t^{-4} - t^{-2})\eta_0 \otimes \eta_0,$$

where $t > 0$ is a positive real number. Let $(M^7, \phi_0, \xi_0, \eta_0, g_0)$ be the 7-dimensional standard sphere or a 7-dimensional 3-Sasakian manifold carrying a Killing spinor $\psi = \psi_1 + \psi_2 \in \Gamma(\Sigma_1 \oplus \Sigma_2)$ (see Theorem 2.3 in [13] and Theorem A.4.3 in [8]). Then, using the part (i) of Lemma 6.9 in [14], one easily verifies that the corresponding spinor $\widetilde{\psi} = \widetilde{\psi_1} + \widetilde{\psi_2}$ on $(M^7, \phi, \xi, \eta, g)$ is a Sasakian Killing spinor satisfying the equality $\lambda_1^+ = -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min} + 6)}$ in (23). Moreover, it follows from Proposition 2.4 that the conjugate $\widetilde{\psi}^* = \widetilde{\psi_1} - \widetilde{\psi_2}$ of $\widetilde{\psi}$ is a Sasakian Killing spinor satisfying the equality $\lambda_1^- = -\frac{1}{2} - \sqrt{\frac{1}{3}(S_{\min} + 6)}$ in (24). It is still an open question whether the equality $\lambda_1^- = \frac{1}{2} - \sqrt{\frac{3}{10}(S_{\min} + 6) + \frac{24}{25}}$ or $\lambda_1^- = \frac{1}{2} - \sqrt{\frac{1}{2}(S_{\min} + 6) - 24}$ in (24) can actually be attained.

Let $\check{\lambda} \in \operatorname{Spec}(D_{ab}, \mathrm{II}) \setminus \operatorname{Spec}(D_{ab}, \mathrm{I})$ and $\hat{\lambda} \in \operatorname{Spec}(D_{ab}, \mathrm{I})$. Then, from (19)-(22) it follows that

$$
|\check{\lambda}| \geq \begin{cases} -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min}+6)} & \text{for } \frac{3}{4} < S_{\min}+6 \leq \frac{144}{5}, \\ -\frac{1}{2} + \sqrt{\frac{3}{10}(S_{\min}+6) + \frac{24}{25}} & \text{for } \frac{144}{5} < S_{\min}+6 \leq \frac{624}{5}, \\ -\frac{1}{2} + \sqrt{\frac{1}{2}(S_{\min}+6) - 24} & \text{for } \frac{624}{5} \leq S_{\min}+6 \leq 144, \\ -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min}+6)} & \text{for } S_{\min}+6 \geq 144 \end{cases} \tag{25}
$$

and

$$
|\hat{\lambda}| \geq \begin{cases} \frac{1}{8}(S_{\min}+6) + \frac{1}{2} & \text{for } 0 < S_{\min}+6 \leq 12, \\ \frac{1}{24}(S_{\min}+6) + \frac{3}{2} & \text{for } S_{\min}+6 \geq 12. \end{cases} \tag{26}
$$

Putting (25) with (26) together, we deduce:

Theorem 3.4. *Let $(M^7, \phi, \xi, \eta, g)$ be a 7-dimensional closed Sasakian spin manifold. Assume that the free parameters a, b of D_{ab} are chosen to be*

$$a = 1, \quad b = 0$$

and the scalar curvature S satisfy

$$S_{\min} > -\frac{21}{4}.$$

Then the first eigenvalue λ_1 of D_{ab} satisfies

$$
|\lambda_1| \geq \begin{cases} -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min}+6)} & \textit{for } \frac{3}{4} < S_{\min}+6 \leq \frac{144}{5}, \\ -\frac{1}{2} + \sqrt{\frac{3}{10}(S_{\min}+6) + \frac{24}{25}} & \textit{for } \frac{144}{5} < S_{\min}+6 \leq \frac{624}{5}, \\ -\frac{1}{2} + \sqrt{\frac{1}{2}(S_{\min}+6) - 24} & \textit{for } \frac{624}{5} \leq S_{\min}+6 \leq 144, \\ -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min}+6)} & \textit{for } S_{\min}+6 \geq 144. \end{cases} \tag{27}
$$

The equality $\lambda_1 = -\frac{1}{2} + \sqrt{\frac{1}{3}(S_{\min}+6)}$ in (27) occurs if and only if $(M^7, \phi, \xi, \eta, g)$ admits a Sasakian Killing spinor. This limiting case is achieved if $(M^7, \phi, \xi, \eta, g)$ is simply-connected and isometric to either a 7-dimensional spherical Sasakian space form or Tanno deformation of a 7-dimensional 3-Sasakian manifold.

4. Estimates on eta-Einstein Sasakian manifolds

In this section, we focus our attention on eta-Einstein Sasakian spin manifolds. In [12], we proved inequality (28) in the following lemma.

Lemma 4.1. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 2$, be a closed eta-Einstein Sasakian spin manifold. Let $\check{\lambda} \in \operatorname{Spec}(D_{ab}, \mathrm{II}) \setminus \operatorname{Spec}(D_{ab}, \mathrm{I})$, $a \neq 0$, and let $\psi =$*

$\psi_r + \psi_{r+1} \in \mathrm{E}^{\check{\lambda}}(D_{ab})$, $r \in \{0, 1, \ldots, m-1\}$, *be a two-component eigenspinor of* D_{ab} *such that*

$$D_0\psi_r = (-1)^{m+r}\nu\psi_r, \quad D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$$

hold for some real number $\nu \in \mathbb{R}$. *If*

$$[4r + 4 + a(1-2b)(m-1-2r)]\,[4m - 4r - a(1-2b)(m-1-2r)] > 0,$$

then the inequality

$$\begin{aligned}&\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2} - b\right]^2 - a^2\nu^2 \\ &\geq \frac{8(r+1)(m-r) + a(1-2b)(m-1-2r)^2}{8m(m+1)} \cdot (S + 2m) \\ &+ \left(\frac{1}{2} - b\right)^2 (m-1-2r)^2 \end{aligned} \tag{28}$$

holds.

Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold and let $\varphi = \varphi_r + \varphi_{r+1} \in \Gamma(\Sigma_r \oplus \Sigma_{r+1})$, $r \in \{0, 1, \ldots, m-1\}$, be a spinor field with $D_-\varphi_r = D_+\varphi_{r+1} = 0$. Then, making use of identities (6.1)-(6.2) in [12], one establishes the identities

$$\begin{aligned}&2\sum_{u=1}^{2m}\left(\overline{\nabla}_{P_+E_u}{}^*\,\overline{\nabla}_{P_+E_u}\right)(\varphi_r) \\ &= \overline{C}^2(\varphi_r) - \frac{m-r}{2m}(S+2m)\varphi_r - 4(-1)^{m+r}(m-r)D_0\varphi_r,\end{aligned} \tag{29}$$

$$\begin{aligned}&2\sum_{u=1}^{2m}\left\{\overline{\nabla}_{P_-E_u} + tP_-(E_u)\circ D_+\right\}^*\left\{\overline{\nabla}_{P_-E_u} + tP_-(E_u)\circ D_+\right\}(\varphi_r) \\ &= \left\{1 - 4t + 4(r+1)t^2\right\}\overline{C}^2(\varphi_r) - \frac{r}{2m}(S+2m)\varphi_r + 4(-1)^{m+r}rD_0\varphi_r,\end{aligned} \tag{30}$$

$$\begin{aligned}&2\sum_{u=1}^{2m}\left(\overline{\nabla}_{P_-E_u}{}^*\,\overline{\nabla}_{P_-E_u}\right)(\varphi_{r+1}) \\ &= \overline{C}^2(\varphi_{r+1}) - \frac{r+1}{2m}(S+2m)\varphi_{r+1} - 4(-1)^{m+r}(r+1)D_0\varphi_{r+1},\end{aligned} \tag{31}$$

$$\begin{aligned}&2\sum_{u=1}^{2m}\left\{\overline{\nabla}_{P_+E_u} + sP_+(E_u)\circ D_-\right\}^*\left\{\overline{\nabla}_{P_+E_u} + sP_+(E_u)\circ D_-\right\}(\varphi_{r+1}) \\ &= \left\{1 - 4s + 4(m-r)s^2\right\}\overline{C}^2(\varphi_{r+1}) - \frac{m-1-r}{2m}(S+2m)\varphi_{r+1} \\ &+ 4(-1)^{m+r}(m-1-r)D_0\varphi_{r+1},\end{aligned} \tag{32}$$

where $(\cdot)^*$ denotes the adjoint operator of $(\cdot)$ with respect to the L^2-Hermitian product and $t, s \in \mathbb{R}$ are real parameters.

Remark 4.2. (a) Inserting $t = \frac{1}{2r+2}$ into (30), we obtain

$$\begin{aligned}&2\sum_{u=1}^{2m}\left\{\overline{\nabla}_{P_-E_u} + \frac{1}{2r+2}P_-(E_u)\circ D_+\right\}^* \\ &\times\left\{\overline{\nabla}_{P_-E_u} + \frac{1}{2r+2}P_-(E_u)\circ D_+\right\}(\varphi_r) \\ = &\frac{r}{r+1}\left[\overline{C}^2(\varphi_r) - \frac{r+1}{2m}(S+2m)\varphi_r + 4(-1)^{m+r}(r+1)D_0\varphi_r\right].\end{aligned}$$

In particular, if one chooses $r = 0$, then

$$\overline{\nabla}_{P_-(V)}\varphi_0 + \frac{1}{2}P_-(V)\cdot D_+\varphi_0 = 0$$

for all $V \in \Gamma(\xi^\perp)$ and $\varphi_0 \in \Gamma(\Sigma_0)$.

(b) Inserting $s = \frac{1}{2m-2r}$ into (32), we obtain

$$\begin{aligned}&2\sum_{u=1}^{2m}\left\{\overline{\nabla}_{P_+E_u} + \frac{1}{2m-2r}P_+(E_u)\circ D_-\right\}^* \\ &\times\left\{\overline{\nabla}_{P_+E_u} + \frac{1}{2m-2r}P_+(E_u)\circ D_-\right\}(\varphi_{r+1}) \\ = &\frac{m-1-r}{m-r}\left[\overline{C}^2(\varphi_{r+1}) - \frac{m-r}{2m}(S+2m)\varphi_{r+1} + 4(-1)^{m+r}(m-r)D_0\varphi_{r+1}\right].\end{aligned}$$

In particular, if one chooses $r = m-1$, then

$$\overline{\nabla}_{P_+(V)}\varphi_m + \frac{1}{2}P_+(V)\cdot D_-\varphi_m = 0$$

for all $V \in \Gamma(\xi^\perp)$ and $\varphi_m \in \Gamma(\Sigma_m)$.

Lemma 4.3. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 2$, be a closed eta-Einstein Sasakian spin manifold. Let $\psi = \psi_r + \psi_{r+1} \in \mathrm{E}^{\check{\lambda}}(D_{ab})$, $r \in \{0, 1, \dots, m-1\}$, be a two-component eigenspinor of D_{ab} such that $D_0\psi_r = (-1)^{m+r}\nu\psi_r$ and $D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$ hold for some real number $\nu \in \mathbb{R}$. Then, with the notations of* (10)-(11)*, the following three inequalities hold*:

$$\sigma_+\sigma_- - \frac{m-r}{2m}(S+2m) - 4(m-r)\nu \geq 0, \tag{33}$$

$$\sigma_+\sigma_- - \frac{r+1}{2m}(S+2m) + 4(r+1)\nu \geq 0, \tag{34}$$

$$\sigma_+\sigma_- - \frac{m+1}{4m}(S+2m) - 2(m-1-2r)\nu \geq 0. \tag{35}$$

Proof. Let μ denote the volume form of $(M^{2m+1}, \phi, \xi, \eta, g)$. Then, applying (29) to

$$\sum_{u=1}^{2m} \int_M \left(\overline{\nabla}_{P_+ E_u}\psi_r,\ \overline{\nabla}_{P_+ E_u}\psi_r\right) \mu \geq 0$$

and (31) to

$$\sum_{u=1}^{2m} \int_M \left(\overline{\nabla}_{P_- E_u}\psi_{r+1},\ \overline{\nabla}_{P_- E_u}\psi_{r+1}\right) \mu \geq 0,$$

we obtain (33) and (34), respectively. Inequality (35) follows from

$$\begin{aligned} &\sum_{u=1}^{2m} \int_M \left(\overline{\nabla}_{P_+ E_u}\psi_r,\ \overline{\nabla}_{P_+ E_u}\psi_r\right) \mu \\ &+ \frac{r+1}{r} \sum_{u=1}^{2m} \int_M \Big(\overline{\nabla}_{P_- E_u}\psi_r + \frac{1}{2r+2} P_-(E_u)\cdot D_+\psi_r, \\ &\qquad\qquad \overline{\nabla}_{P_- E_u}\psi_r + \frac{1}{2r+2} P_-(E_u)\cdot D_+\psi_r\Big)\mu \\ &= \left[\sigma_+\sigma_- - \frac{m+1}{4m}(S+2m) - 2(m-1-2r)\nu\right] \int_M (\psi_r,\ \psi_r)\mu \end{aligned} \tag{36}$$

if $r \neq 0$ or

$$\begin{aligned} &\sum_{u=1}^{2m} \int_M \left(\overline{\nabla}_{P_- E_u}\psi_{r+1},\ \overline{\nabla}_{P_- E_u}\psi_{r+1}\right) \mu \\ &+ \frac{m-r}{m-1-r} \sum_{u=1}^{2m} \int_M \Big(\overline{\nabla}_{P_+ E_u}\psi_{r+1} + \frac{1}{2m-2r} P_+(E_u)\cdot D_+\psi_{r+1}, \\ &\qquad\qquad \overline{\nabla}_{P_+ E_u}\psi_{r+1} + \frac{1}{2m-2r} P_+(E_u)\cdot D_+\psi_{r+1}\Big)\mu \\ &= \left[\sigma_+\sigma_- - \frac{m+1}{4m}(S+2m) - 2(m-1-2r)\nu\right] \int_M (\psi_{r+1},\ \psi_{r+1})\mu \end{aligned} \tag{37}$$

if $r \neq m-1$. □

Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 2$, be a closed eta-Einstein Sasakian spin manifold and let $\hat{\lambda} \in \mathrm{Spec}(D_{ab}, \mathrm{I})$. Assume that the scalar curvature satisfies $S + 2m > 0$ and the parameters a, b of D_{ab} are chosen to be $(a = 1, b = 0)$. In that case, (15) becomes

$$|\hat{\lambda}| \geq \frac{S+2m}{8m} + \frac{m}{2}, \tag{38}$$

since $p = 0$ or $p = m$ holds (see Proposition 6.1 in [12]). We are now ready to prove:

Theorem 4.4. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 7$, $m \equiv 1 \bmod 2$. Assume that the free parameters a, b of D_{ab} are chosen to be*

$$a = 1, \quad b = 0$$

and the scalar curvature S satisfy

$$S \geq -\frac{m(2m+1)}{m+1}.$$

Then,

(a) *the first eigenvalue λ_1 of D_{ab} satisfies*

$$|\lambda_1| \geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}. \tag{39}$$

(b) *The first negative eigenvalue λ_1^- of D_{ab} satisfies*

$$\lambda_1^- \leq \frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)} \tag{40}$$

and the first positive eigenvalue λ_1^+ of D_{ab} satisfies

$$\lambda_1^+ \geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}. \tag{41}$$

The limiting case of (39) (*resp.* (40), (41)) *occurs if and only if one of the following is true*:
Case 1. $(M^{2m+1}, \phi, \xi, \eta, g)$ *is Einstein and admits a Killing spinor.*
Case 2. $(M^{2m+1}, \phi, \xi, \eta, g)$ *is eta-Einstein and admits a Sasakian Killing spinor.*

Proof. Let $\psi = \psi_r + \psi_{r+1} \in \mathrm{E}^{\check{\lambda}}(D_{ab})$, $r \in \{0, 1, \ldots, m-1\}$, be a two-component eigenspinor of D_{ab} such that

$$D_0\psi_r = (-1)^{m+r}\nu\psi_r, \quad D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$$

hold for some real number $\nu \in \mathbb{R}$. Then, rewriting of (35) yields

$$\begin{aligned} &\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2} - b\right]^2 - \left[a\nu + \left(\frac{1}{a} - \frac{1}{2} + b\right)(m-1-2r)\right]^2 \\ &- \frac{m+1}{4m}(S+2m) - \left[\frac{1-2b}{a} - \frac{1}{a^2}\right](m-1-2r)^2 \geq 0. \end{aligned} \tag{42}$$

For $a = 1$ and $b = 0$, (42) becomes

$$\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2}\right]^2 - \left[\nu + \frac{1}{2}(m-1-2r)\right]^2 - \frac{m+1}{4m}(S+2m) \geq 0, \tag{43}$$

which implies

$$|\check{\lambda}| \geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}. \tag{44}$$

Comparing (44) with (38), one completes the proof of (39). In order to prove (40)-(41), we only consider the case $m \equiv 1 \bmod 4$, the other case $m \equiv 3 \bmod 4$ being similar. Let us denote $l := \frac{m-1}{2}$. Then, by (43), any negative eigenvalue $\check{\lambda}^- < 0$ of $\mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$ satisfies one of the inequalities

$$
\begin{aligned}
(-1)^{1+0}\check{\lambda}^- &\geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+2}\check{\lambda}^- &\geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\\
&\cdots\\
(-1)^{1+l}\check{\lambda}^- &\geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+1}\check{\lambda}^- &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+3}\check{\lambda}^- &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\\
&\cdots\\
(-1)^{1+l-1}\check{\lambda}^- &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\check{\lambda}^- &\leq \max\left\{\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\ -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)}\right\}\\
&= \frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)}. \qquad (45)
\end{aligned}
$$

Now comparing (45) with (38) leads to (40). Similarly, any positive eigenvalue $\check{\lambda}^+ > 0$ of $\mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$ satisfies one of the following inequalities:

$$
\begin{aligned}
(-1)^{1+0}\check{\lambda}^+ &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+2}\check{\lambda}^+ &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\\
&\cdots\\
(-1)^{1+l}\check{\lambda}^+ &\leq -\frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+1}\check{\lambda}^+ &\geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\\
(-1)^{1+3}\check{\lambda}^+ &\geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\\
&\cdots
\end{aligned}
$$

$$(-1)^{1+l-1}\check{\lambda}^+ \geq -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}.$$

Consequently, any positive eigenvalue $\lambda^+ > 0$ of $\mathrm{Spec}(D_{ab})$ satisfies

$$\begin{aligned}\lambda^+ &\geq \min\left\{ -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\ \frac{S+2m}{8m} + \frac{m}{2}\right\}\\ &= -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)},\end{aligned}$$

which proves (41). It remains to discuss the limiting case of (39)-(41). By (43) we see that the limiting case of (39) (resp. (40), (41)) occurs if and only if

$$\nu + \frac{1}{2}(m-1-2r) = 0.$$

In the case $r = 0$, we see by (37) that ψ_{r+1} must be a right Sasakian twistor spinor. Then, using (8), we conclude that $S + 2m = 4m(m+1)$ and so $(M^{2m+1}, \phi, \xi, \eta, g)$ have to be Einstein. By Lemma 2.5, (ψ_0, ψ_1) is a Sasakian duo and by Lemma 2.3, $\psi = \psi_0 + \psi_1$ turns out to be a Killing spinor. We should note here that any Killing spinor that is a section in either Σ_0 or Σ_m also satisfies the limiting case of (39)-(41). Suppose now that $r \neq 0$ and $(M^{2m+1}, \phi, \xi, \eta, g)$ is non-Einstein. In this case, making use of (36), Lemma 2.5 and (6) jointly, we conclude that $2r = m - 1$ and $\psi = \psi_r + \psi_{r+1}$ must be a Sasakian Killing spinor. □

Remark 4.5. In the case $m \equiv 1 \bmod 4$ (resp. $m \equiv 3 \bmod 4$), one easily verifies that the equality $\lambda_1^+ = -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}$ in (41) (resp. $\lambda_1^- = \frac{1}{2} - \sqrt{\frac{m+1}{4m}(S+2m)}$ in (40)) occurs if and only if $(M^{2m+1}, \phi, \xi, \eta, g)$ is Einstein and admits a Killing spinor.

As preparation for the proof of Theorem 4.9, we now need to establish several lemmas.

Lemma 4.6. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 3$. Assume that the scalar curvature S satisfies $S + 2m > 0$.*

(a) *Then, for any $r \in \{0, 1, \ldots, m-1\}$, the inequality*

$$\begin{aligned}&\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2 (m-1-2r)^2\\ &\geq \frac{m+1}{4m}(S+2m)\end{aligned}$$

holds.

(b) *If $m \equiv 0 \bmod 2$ and $r \in \left\{0, 1, \ldots, \frac{m}{2} - 1\right\}$, then*

$$\frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2 (m-1-2r)^2$$

$$\geq \frac{m+2}{4(m+1)}(S+2m)+\left[\frac{S+2m}{8m(m+1)}+\frac{1}{2}\right]^2$$
$$\geq \frac{m+1}{4m}(S+2m).$$

Proof. Rewrite the first two inequalities of the lemma to obtain

$$(m-1-2r)^2\left[S+2m-4m(m+1)\right]^2 \geq 0$$

and

$$(m-2r)(m-2-2r)\left[S+2m-4m(m+1)\right]^2 \geq 0,$$

respectively. □

Lemma 4.7. *Let $(M^{2m+1},\phi,\xi,\eta,g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 9$, $m \equiv 0 \mod 2$. Then, for any $\lambda \in \mathrm{Spec}(D_{ab},\mathrm{II}) \setminus \mathrm{Spec}(D_{ab},\mathrm{I})$ with $(a=1, b=0)$, there exist some real number $\nu \in \mathbb{R}$ and $r \in \{0,1,\ldots,\frac{m}{2}-1\}$ such that*

$$\left[(-1)^{m+r}\lambda+\frac{1}{2}\right]^2-\left[\nu+\frac{1}{2}(m-1-2r)\right]^2 \geq \frac{m+1}{4m}(S+2m) \tag{46}$$

holds for $S+2m>0$ and

$$\left[(-1)^{m+r}\lambda+\frac{1}{2}\right]^2-\left[\nu+\frac{3m}{2}+\frac{1}{2}-r\right]^2$$
$$\geq \frac{m+2}{4m}(S+2m)-(m+1)(m+2) \tag{47}$$

holds for $S+2m \geq 4m(m+1)$. Moreover, the inequality

$$\frac{m+1}{4m}(S+2m) \geq \frac{m+2}{4m}(S+2m)-(m+1)(m+2) \tag{48}$$

holds if

$$4m(m+1) \leq S+2m \leq 4m(m+1)(m+2).$$

Proof. Inequality (43) yields (46). On the other hand, it follows from (33) that

$$\left[(-1)^{m+r}\lambda+\frac{1}{2}-b\right]^2$$
$$-\left[a\nu-\left(\frac{1}{2}-b\right)(m-1-2r)+\frac{2}{a}(m-r)\right]^2$$
$$\geq \frac{m-r}{2m}(S+2m)-\frac{4}{a^2}(m-r)^2+\frac{2(1-2b)}{a}(m-r)(m-1-2r). \tag{49}$$

For $a=1$ and $b=0$, (49) becomes

$$\left[(-1)^{m+r}\lambda+\frac{1}{2}\right]^2-\left[\nu+\frac{3m}{2}+\frac{1}{2}-r\right]^2$$
$$\geq \frac{m-r}{2m}(S+2m)-2(m+1)(m-r).$$

If the scalar curvature satisfies $S+2m \geq 4m(m+1)$, then the function

$$\chi(r) := \frac{m-r}{2m}(S+2m) - 2(m+1)(m-r)$$

is decreasing and

$$\chi\Big(\frac{m}{2}-1\Big) = \frac{m+2}{4m}(S+2m) - (m+1)(m+2).$$

We thus proved (47). □

Explicit computations yield the inequalities in the following lemma.

Lemma 4.8. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 9$, $m \equiv 0 \bmod 2$. Then we have*

$$\frac{S+2m}{8m} + \frac{m}{2} \geq \max\Bigg\{ -\frac{1}{2} + \sqrt{\frac{m+2}{4(m+1)}(S+2m) + \Big[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\Big]^2}, \\ -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)}\ \Bigg\},$$

if $S+2m > 0$ and

$$\frac{S+2m}{8m} + \frac{m}{2} > -\frac{1}{2} + \sqrt{\frac{m+2}{4m}(S+2m) - (m+1)(m+2)}$$

if $S+2m \geq 4m(m+1)$.

Theorem 4.9. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed eta-Einstein Sasakian spin manifold of dimension $2m+1 \geq 9$, $m \equiv 0 \bmod 2$. Assume that the free parameters a, b of D_{ab} are chosen to be*

$$a = 1, \quad b = 0$$

and the scalar curvature S satisfies

$$S > -2m.$$

Then the first eigenvalue λ_1 of D_{ab} satisfies

$$|\lambda_1| \geq \begin{cases} -\frac{1}{2} + \sqrt{\frac{m+2}{4(m+1)}(S+2m) + \Big[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\Big]^2} \\ \quad \textit{for} \ \ 0 < S+2m \leq \frac{4m(m+1)(3m+1)}{m-1}, \\ -\frac{1}{2} + \sqrt{\frac{m+1}{4m}(S+2m)} \\ \quad \textit{for} \ \frac{4m(m+1)(3m+1)}{m-1} \leq S+2m \leq 4m(m+1)(m+2), \\ -\frac{1}{2} + \sqrt{\frac{m+2}{4m}(S+2m) - (m+1)(m+2)} \\ \quad \textit{for} \ 4m(m+1)(m+2) \leq S+2m. \end{cases} \tag{50}$$

The equality

$$|\lambda_1| = -\frac{1}{2} + \sqrt{\frac{m+2}{4(m+1)}(S+2m) + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2}$$

in (50) *occurs if and only if* $(M^{2m+1}, \phi, \xi, \eta, g)$ *admits a Sasakian duo* $\left(\psi_{\frac{m}{2}-1}, \psi_{\frac{m}{2}}\right)$ *with characteristic numbers*

$$(-1)^{\frac{m}{2}-1} c_\pm = \mp\left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right] + \sqrt{\frac{(m+2)(S+2m)}{4(m+1)} + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2}.$$

Proof. We first prove, under the condition $0 < S + 2m \leq \frac{4m(m+1)(3m+1)}{m-1}$, that any $\check{\lambda} \in \mathrm{Spec}(D_{ab}, \mathrm{II}) \setminus \mathrm{Spec}(D_{ab}, \mathrm{I})$ satisfies the inequality

$$\left\{(-1)^{m+r}\check{\lambda} + \frac{1}{2}\right\}^2$$
$$\geq \frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2 (m-1-2r)^2. \tag{51}$$

By (28) we see that

$$\begin{aligned}
&\left[(-1)^{m+r}\check{\lambda} + \frac{1}{2}\right]^2 \\
&\geq \nu^2 + \frac{8(r+1)(m-r) + (m-1-2r)^2}{8m(m+1)}(S+2m) + \frac{1}{4}(m-1-2r)^2 \\
&\geq \frac{(r+1)(m-r)}{m(m+1)}(S+2m) + \left[\frac{S+2m}{8m(m+1)} + \frac{1}{2}\right]^2 (m-1-2r)^2
\end{aligned}$$

holds if

$$\nu^2 \geq \left[\frac{m-1-2r}{8m(m+1)}(S+2m)\right]^2.$$

Next, we prove (51) under the condition

$$\nu^2 \leq \left[\frac{m-1-2r}{8m(m+1)}(S+2m)\right]^2.$$

Let $\psi = \psi_r + \psi_{r+1} \in \mathrm{E}^{\check{\lambda}}(D_{ab})$, $r \in \{0, 1, \ldots, m-1\}$, be a two-component eigenspinor of D_{ab} such that $D_0\psi_r = (-1)^{m+r}\nu\psi_r$ and $D_0\psi_{r+1} = (-1)^{m+r+1}\nu\psi_{r+1}$ hold for some real number $\nu \in \mathbb{R}$. Since the complex-antilinear mapping j_1 satisfies

$$D_{ab} \circ j_1 = (-1)^{m+1} j_1 \circ D_{ab},$$

we may assume without loss of generality that $r \in \left\{0, 1, \ldots, \frac{m}{2} - 1\right\}$. Note that rewriting of (33) yields

$$\left[(-1)^{m+r}\lambda + \frac{1}{2} - b\right]^2 - a^2\left[\nu + \frac{m-1-2r}{8m(m+1)}(S+2m)\right]^2$$

$$
\begin{aligned}
&+\left[2a\left(\frac{a(S+2m)}{8m(m+1)}+\frac{1}{2}-b\right)(m-1-2r)-4(m-r)\right]\\
&\times\left[\nu+\frac{m-1-2r}{8m(m+1)}(S+2m)\right]\\
&-\frac{(r+1)(m-r)}{m(m+1)}(S+2m)-\left[\frac{a(S+2m)}{8m(m+1)}+\frac{1}{2}-b\right]^2(m-1-2r)^2\geq 0. \qquad (52)
\end{aligned}
$$

For $a=1$ and $b=0$, (52) becomes

$$
\begin{aligned}
&\left[(-1)^{m+r}\lambda+\frac{1}{2}\right]^2-\left[\nu+\frac{m-1-2r}{8m(m+1)}(S+2m)\right]^2\\
\geq\ &-\left[2\left(\frac{S+2m}{8m(m+1)}+\frac{1}{2}\right)(m-1-2r)-4(m-r)\right]\\
&\quad\times\left[\nu+\frac{m-1-2r}{8m(m+1)}(S+2m)\right]\\
&\quad+\frac{(r+1)(m-r)}{m(m+1)}(S+2m)+\left[\frac{S+2m}{8m(m+1)}+\frac{1}{2}\right]^2(m-1-2r)^2. \qquad (53)
\end{aligned}
$$

Since

$$
\begin{aligned}
&S+2m\leq\frac{4m(m+1)(3m+1)}{m-1}\\
\Rightarrow\ &\frac{S+2m}{8m(m+1)}+\frac{1}{2}\leq 1+\frac{m+1}{m-1}\leq 1+\frac{m+1}{m-1-2r}\\
\Rightarrow\ &\left[\frac{S+2m}{8m(m+1)}+\frac{1}{2}\right](m-1-2r)\leq 2(m-r),
\end{aligned}
$$

(53) immediately becomes (51) under the condition $|\nu|\leq\frac{m-1-2r}{8m(m+1)}(S+2m)$. Finally, we take Lemma 4.6, Lemma 4.7, Lemma 4.8 and inequality (38) into account to obtain (50). □

Remark 4.10. In (50), the equality $|\lambda_1|=-\frac{1}{2}+\sqrt{\frac{m+1}{4m}(S+2m)}$ cannot be attained (see the proof of Theorem 4.4). It is an open question whether the equality

$$
|\lambda_1|=-\frac{1}{2}+\sqrt{\frac{m+2}{4m}(S+2m)-(m+1)(m+2)}
$$

can occur.

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