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ALMOST UNIVERSAL SUMS OF TRIANGULAR NUMBERS WITH ONE EXCEPTION

Jangwon Ju

Abstract. For an arbitrary integer x, an integer of the form $T(x)=\frac{x^2+x}{2}$ is called a triangular number. Let $\alpha_1,\dots,\alpha_k$ be positive integers. A sum $\Delta_{\alpha_1,\dots,\alpha_k}(x_1,\dots,x_k)=\alpha_1T(x_1)+\cdots+\alpha_kT(x_k)$ of triangular numbers is said to be *almost universal with one exception* if the Diophantine equation $\Delta_{\alpha_1,\dots,\alpha_k}(x_1,\dots,x_k)=n$ has an integer solution $(x_1,\dots,x_k)\in\mathbb{Z}^k$ for any nonnegative integer n except a single one. In this article, we classify all almost universal sums of triangular numbers with one exception. Furthermore, we provide an effective criterion on almost universality with one exception of an arbitrary sum of triangular numbers, which is a generalization of "15-theorem" of Conway, Miller, and Schneeberger.

1. Introduction

In 1770, Lagrange proved that every nonnegative integer can be written as a sum of at most four squares of integers. Motivated by Lagrange's four-square theorem, Ramanujan provided a list of 55 candidates of diagonal quaternary integral quadratic forms that represent all nonnegative integers (for details, see [15]). Dickson pointed out that the diagonal quaternary quadratic form $x^2+2y^2+5z^2+5t^2$ in Ramanujan's list doesn't represent the integer 15, and confirmed that Ramanujan's assertion is correct for all the other 54 quadratic forms (for details, see [5]).

Ramanujan's assertion was generalized to find all *universal* quaternary quadratic forms, i.e., those representing all nonnegative integers. This was completely solved by Conway, Miller, and Schneeberger in 1993. They proved the so called "15-theorem", which states that a positive definite integral quadratic form is universal if and only if it represents the integers

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 14,\ \text{and } 15,$$

irrespective of its rank. Moreover, they provided a complete list of 204 quaternary quadratic forms with this property. Recently, Bhargava provided an elegant proof of the 15-theorem in [1].

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As a natural generalization of the 15-theorem, Bhargava and Hanke [2] proved the so-called "290-theorem", which states that every positive definite integer-valued quadratic form is universal if and only if it represents the integers

$$\begin{aligned}&1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 13,\ 14,\ 15,\ 17,\ 19,\ 21,\ 22,\ 23,\ 26,\ 29,\\ &30,\ 31,\ 34,\ 35,\ 37,\ 42,\ 58,\ 93,\ 110,\ 145,\ 203,\ \text{and } 290.\end{aligned}$$

Here a quadratic form $f(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i,j \leqslant n} a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ is called *integral* if $a_{ij} \in \mathbb{Z}$ for any i, j, and is called *integer-valued* if $a_{ii} \in \mathbb{Z}$ and $a_{ij} + a_{ji} \in \mathbb{Z}$ for any i, j. Moreover, they provided a complete list of 6436 such forms in four variables.

A next natural generalization of Ramanujan's assertion is to classify all quadratic forms representing all nonnegative integers with finitely many exceptions. A quadratic form with this property is said to be *almost universal*. At first, by using an escalation method, in [6] Halmos provided a list of 88 candidates of almost universal diagonal quaternary quadratic forms with one exception. It was pointed out that diagonal quaternary quadratic forms $x^2 + y^2 + 2z^2 + 22t^2$ and $x^2 + 2y^2 + 4z^2 + 22t^2$ in Halmos's list don't represent two integers 14 and 78. Halmos proved that 85 of those indeed represent all nonnegative integers except a single one. Moreover, he conjectured that the remaining form $x^2 + 2y^2 + 7z^2 + 13t^2$ represents all nonnegative integers except 5, and it was proved by Pall in [14].

In 2009, Bochnak and Oh [3] provided an effective characterization for deciding whether a positive definite integral quaternary quadratic form represents all nonnegative integers with finitely many exceptions. It can be considered as the final solution to the problem first addressed by Ramanujan in [15].

In this paper, we investigate representations by sums of triangular numbers. The n-th triangular number is the number of dots in the triangular arrangement with n dots on a side. More precisely, the n-th triangular number is defined by

$$T(n) = \frac{n^2 + n}{2}$$

for any nonnegative integer n. Note that $\{T(x) : x \in \mathbb{N} \cup \{0\}\} = \{T(x) : x \in \mathbb{Z}\}$.

For positive integers $\alpha_1, \ldots, \alpha_k$, we say a sum

$$\Delta_{\alpha_1,\ldots,\alpha_k}(x_1, \ldots, x_k) := \alpha_1 T(x_1) + \cdots + \alpha_k T(x_k)$$

of triangular numbers *represents* a nonnegative integer n if the Diophantine equation

$$\Delta_{\alpha_1,\ldots,\alpha_k}(x_1, \ldots, x_k) = n$$

has an integer solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$. Furthermore, a sum

$$\Delta_{\alpha_1,\ldots,\alpha_k}(x_1, \ldots, x_k) \text{ (simply, } \Delta_{\alpha_1,\ldots,\alpha_k})$$

of triangular numbers is called *universal* if it represents all nonnegative integers.

The famous Gauss' triangular theorem states that every positive integer can be expressed as a sum of three triangular numbers which was first asserted by Fermat in 1638. In 1862, Liouville proved that for positive integers a, b, and

c ($a \leqslant b \leqslant c$), a sum $\Delta_{a,b,c}$ of triangular numbers is universal if and only if (a,b,c) is one of the following triples:

$$(1,1,1), \quad (1,1,2), \quad (1,1,4), \quad (1,1,5), \quad (1,2,2), \quad (1,2,3), \quad \text{and} \quad (1,2,4),$$

which is a generalization of Gauss' triangular theorem.

In 2013, Bosma and Kane proved the triangular theorem of eight which states that for positive integers $\alpha_1, \ldots, \alpha_k$, an arbitrary sum $\Delta_{\alpha_1,\ldots,\alpha_k}$ of triangular numbers is universal if and only if it represents $1, 2, 4, 5$, and 8 (for details, see [4]). This might be considered as a natural generalization of the "15-theorem" of Conway, Miller, and Schneeberger.

For positive integers $\alpha_1, \ldots, \alpha_k$, a sum $\Delta_{\alpha_1,\ldots,\alpha_k}$ of triangular numbers is called *almost universal* if it represents all nonnegative integers with finitely many exceptions. Especially, if a sum of triangular numbers represents all nonnegative integers except a single one, then it is said to be *almost universal with one exception*. Furthermore, it is called *proper* if any proper partial sum of it doesn't represent at least two nonnegative integers.

We know that if a sum $\Delta_{\alpha_1,\ldots,\alpha_k}$ of triangular numbers is almost universal with one exception m, then m is inside $\{1, 2, 4, 5, 8\}$ by the triangular theorem of eight. By using an escalation method, we give a complete list of candidates of 490 proper almost universal sums of triangular numbers with one exception, actually, the numbers of ternary, quaternary, and quinary sums among them are 1, 235, and 254, respectively. We classify all almost universal sums of triangular numbers with one exception $1, 2, 4, 5$, and 8, respectively. Furthermore, we provide an effective criterion on almost universality with one exception of an arbitrary sum $\Delta_{\alpha_1,\ldots,\alpha_k}$ of triangular numbers. This might be considered as a natural generalization of the 15-theorem of Conway, Miller, and Schneeberger.

Theorem 1.1. *A sum of triangular numbers is almost universal with one exception* 1 *if and only if it represents the integers*

$$2,\ 3,\ 4,\ 8,\ 10,\ 16,\ \textit{and}\ 19$$

and doesn't represent 1*. There are exactly* 29 *proper almost universal sums of triangular numbers with one exception* 1*, actually, there are* 11 *quaternary and* 18 *quinary ones* (*see Table 1*)*.*

Theorem 1.2. *A sum of triangular numbers is almost universal with one exception* 4 *if and only if it represents the integers*

$$1,\ 2,\ 11,\ 14,\ 19,\ 25,\ 29,\ 46,\ \textit{and}\ 50$$

and doesn't represent 4*. There are exactly* 138 *proper almost universal sums of triangular numbers with one exception* 4*, actually, there are* 127 *quaternary and* 11 *quinary ones* (*see Table 2*)*.*

Theorem 1.3. *A sum of triangular numbers is almost universal with one exception* 5 *if and only if it represents the integers*

$$1,\ 2,\ 8,\ 14,\ 26,\ 40,\ 41,\ 47,\ 59,\ \textit{and}\ 71$$

and doesn't represent 5*. There are exactly* 171 *proper almost universal sums of triangular numbers with one exception* 5*, actually, there are* 56 *quaternary and* 115 *quinary ones* (*see Table 10*).

Theorem 1.4. *A sum of triangular numbers is almost universal with one exception* 8 *if and only if it represents the integers*

$$1,\ 2,\ 5,\ 17,\ \textit{and}\ 89$$

and doesn't represent 8*. There are exactly* 80 *proper almost universal sums of triangular numbers with one exception* 8*, actually, there are* 7 *quaternary and* 73 *quinary ones* (*see Table 13*).

Note that the sum $\Delta_{1,4,5}$ of triangular numbers is the unique candidate of ternary almost universal sums of triangular numbers with one exception (see Section 7). In [10], Kane proved that $\Delta_{1,4,5}$ represents all positive odd integers under the assuming GRH for L-functions of weight 2 newforms. We conjecture that it represents all nonnegative integers except 2. Actually, we checked that $\Delta_{1,4,5}$ represents all nonnegative integers up to 10^7 except 2.

Conjecture 1.5. *The ternary sum* $\Delta_{1,4,5}$ *of triangular numbers is almost universal with one exception* 2*.*

Assume that Conjecture 1.5 is true. Then we have the following theorem.

Theorem 1.6. *If Conjecture 1.5 is true, then a sum of triangular numbers is almost universal with one exception* 2 *if and only if it represents the integers*

$$1,\ 4,\ 5,\ 7,\ 8,\ 9,\ 11,\ 16,\ 17,\ 20,\ 29,\ \textit{and}\ 35$$

and doesn't represent 2*. There are exactly* 72 *proper almost universal sums of triangular numbers with one exception* 2*, actually, there are unique ternary,* 34 *quaternary, and* 37 *quinary ones* (*see Table 15*).

The complete list of proper almost universal sums of triangular numbers with one exception is given in Tables 1, 2, 10, 13, and 15. In the above tables, each sum of triangular numbers having a dagger mark with the last coefficient is almost universal with one exception that is not proper.

Let $f(x_1, x_2, \ldots, x_k) = \sum_{1 \leqslant i,j \leqslant k} a_{ij}x_ix_j$ ($a_{ij} = a_{ji} \in \mathbb{Z}$) be a positive definite integral quadratic form. The corresponding integral symmetric matrix of f is defined by $M_f = (a_{ij})$ and any matrix isometric to it is denoted by M_f also. For a diagonal quadratic form $f(x_1, x_2, \ldots, x_k) = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2$, we simply write

$$M_f = \langle a_1, a_2, \ldots, a_k \rangle.$$

For an integer n, we say n is *represented by* f if the equation $f(x_1, x_2, \ldots, x_k) = n$ has an integer solution $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k$, which is denoted by $n \longrightarrow f$. The genus of f, denoted by $\text{gen}(f)$, is the set of all quadratic forms that are isometric to f over the p-adic integer ring $\mathbb{Z}_p$ for any prime p. The number of isometry classes in $\text{gen}(f)$ is called the class number of f and denoted by $h(f)$.

A good introduction to the theory of quadratic forms may be found in [13], and we adopt the notations and terminologies from this book.

2. General tools

For positive integers $\alpha_1, \ldots, \alpha_k$, we define

$$\Delta_{\alpha_1,\ldots,\alpha_k}(x_1, \ldots, x_k) = \alpha_1 T(x_1) + \cdots + \alpha_k T(x_k).$$

Recall that a sum

$$\Delta_{\alpha_1,\ldots,\alpha_k}(x_1, \ldots, x_k) \text{ (simply, } \Delta_{\alpha_1,\ldots,\alpha_k})$$

of triangular numbers is called almost universal if it represents all nonnegative integers with finitely many exceptions. In particular, if the number of exceptions is one, then it is said to be almost universal with one exception, which is equivalent to the existence of an integer solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of

$$\alpha_1(2x_1 + 1)^2 + \cdots + \alpha_k(2x_k + 1)^2 = 8n + \alpha_1 + \cdots + \alpha_k$$

for any nonnegative integer n except a single one. Furthermore, this is equivalent to the existence of an integer solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of

$$\text{(1)} \qquad \alpha_1 x_1^2 + \cdots + \alpha_k x_k^2 = 8n + \alpha_1 + \cdots + \alpha_k \text{ with } x_1 \cdots x_k \equiv 1 \pmod 2$$

for any nonnegative integer n except a single one.

Now, we introduce our strategy to prove that a sum $\Delta_{\alpha_1,\ldots,\alpha_k}$ $(k \geqslant 4)$ of triangular numbers is almost universal with one exception. At first, take a suitable ternary section $\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ of $\Delta_{\alpha_1,\ldots,\alpha_k}$, where $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\} \subset \{\alpha_1, \ldots, \alpha_k\}$. Without loss of generality, we may assume that $\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}} = \Delta_{\alpha_1,\alpha_2,\alpha_3}$. We consider the equation

$$\text{(2)} \qquad \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 8n + \alpha_1 + \alpha_2 + \alpha_3 \text{ with } x_1 x_2 x_3 \equiv 1 \pmod 2.$$

Note that Equation (2) corresponds to the representations by a ternary quadratic form with congruence conditions. Since there are some methods for determining the existence of representations of integers by a ternary quadratic form, we try to find a suitable method on reducing Equation (2) to the representations of a ternary quadratic form, denoted by $f(x_1, x_2, x_3)$, without congruence conditions. To explain our method, for example, assume that $\alpha_1 \equiv \alpha_2 \equiv 0 \pmod 2$ and $\alpha_3 \equiv 1 \pmod 2$. Then Equation (2) has an integer solution if

$$\text{(3)} \quad f(x_1, x_2, x_3) = \alpha_1(x_3 - 2x_1)^2 + \alpha_2(x_3 - 2x_2)^2 + \alpha_3 x_3^2 = 8n + \alpha_1 + \alpha_2 + \alpha_3$$

has an integer solution. Hence, in this case, the problem can be reduced to the representations of a ternary quadratic form without congruence conditions.

After that for sufficiently large n, we find suitable $a_4, \ldots, a_k \in \mathbb{Z}$ such that

$$\text{(4)} \qquad \begin{array}{rl} \text{(i)} & a_4 \cdots a_k \equiv 1 \pmod 2; \\ \text{(ii)} & 8n + \alpha_1 + \cdots + \alpha_k - (\alpha_4 a_4^2 + \cdots + \alpha_k a_k^2) \geqslant 0; \\ \text{(iii)} & 8n + \alpha_1 + \cdots + \alpha_k - (\alpha_4 a_4^2 + \cdots + \alpha_k a_k^2) \longrightarrow f(x_1, x_2, x_3). \end{array}$$

Then we know that Equation (1) has an integer solution. Finally, we directly check that the sum $\Delta_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ of triangular numbers represents all remaining small integers except a single one.

The work of Oh [11,12], and work of Oh and the author [8] led to the development of a method that determines whether or not integers in an arithmetic progression are represented by some particular ternary quadratic form. We briefly introduce this method for those who are unfamiliar with it.

Let d be a positive integer and let a be a nonnegative integer ($a \leqslant d$). We define

$$S_{d,a} = \{dn + a \mid n \in \mathbb{N} \cup \{0\}\}.$$

For two positive definite integral ternary quadratic forms f, g, we define

$$R(g, d, a) = \{v \in (\mathbb{Z}/d\mathbb{Z})^3 \mid vM_g v^t \equiv a \pmod{d}\}$$

and

$$R(f, g, d) = \{T \in M_3(\mathbb{Z}) \mid T^t M_f T = d^2 M_g\}.$$

Since f and g are positive definite, the above two sets are always finite. A coset (or, a vector in the coset) $v \in R(g, d, a)$ is said to be *good* with respect to $f, g, d,$ and a if there is a $T \in R(f, g, d)$ such that $\frac{1}{d} \cdot vT^t \in \mathbb{Z}^3$. The set of all good vectors in $R(g, d, a)$ is denoted by $R_f(g, d, a)$. If $R(g, d, a) = R_f(g, d, a)$, we write

$$g \prec_{d,a} f.$$

Now, we introduce two theorems which play a crucial role in proving our results.

Theorem 2.1. *Under the same notations given above, if $g \prec_{d,a} f$, then*

$$S_{d,a} \cap Q(g) \subset Q(f).$$

Proof. The theorem follows directly from Lemma 2.2 of [11] (see also Theorem 2.1 in [7]). □

Theorem 2.2. *Assume that $T \in M_3(\mathbb{Z})$ satisfies the following conditions:*

(i) *$\frac{1}{d}T$ has an infinite order;*
(ii) *$T^t M_g T = d^2 M_g$;*
(iii) *for any vector $v \in \mathbb{Z}^3$ such that $v \pmod{d} \in B_f(g, d, a)$, $\frac{1}{d} \cdot vT^t \in \mathbb{Z}^3$.*

Then we have

$$S_{d,a} \cap Q(g) \backslash \{g(z) \cdot s^2 \mid s \in \mathbb{Z}\} \subset Q(f),$$

where the vector z is any integral primitive eigenvector of T.

Proof. See Theorem 2.1 of [8]. □

We define $B_f(g, d, a) = R(g, d, a) \backslash R_f(g, d, a)$ and its cardinality is denoted by $|B_f(g, d, a)|$(*simply*, $|B|$). In general, if d is large, then it is hard to compute the set $B_f(g, d, a)$ exactly by hand. A MAGMA computer program for computing this set is available upon request to the author.

3. Proof of Theorem 1.1 and outline of the proofs of the other main theorems

Let $\alpha_1, \dots, \alpha_k$ be positive integers ($k \geqslant 1$). For positive integer n, the n-th nonnegative integer that is not represented by $\Delta_{\alpha_1,\dots,\alpha_k}$ is called the n-th *truant* of $\Delta_{\alpha_1,\dots,\alpha_k}$ and denoted by $\mathfrak{T}_n(\Delta_{\alpha_1,\dots,\alpha_k})$ if it exists. If $\mathfrak{T}_n(\Delta_{\alpha_1,\dots,\alpha_k})$ doesn't exist, then we define $\mathfrak{T}_n(\Delta_{\alpha_1,\dots,\alpha_k}) = \infty$. We say every nonnegative integer is less than ∞ for convenience.

For a sum $\Delta_{\alpha_1,\dots,\alpha_k}$ of triangular numbers, without loss of generality, we may assume that $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. We say $\Delta_{\alpha_1,\dots,\alpha_k}$ is a *candidate* of almost universal sums of triangular numbers with one exception m if it satisfies the following conditions:

(i) $\Delta_{\alpha_1,\dots,\alpha_i}$ doesn't represent m;

(ii) $\alpha_i \leqslant \begin{cases} \mathfrak{T}_1(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) & \text{if } \mathfrak{T}_1(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) < m, \\ \mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) & \text{if } \mathfrak{T}_1(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) = m \end{cases}$

for all $1 \leqslant i \leqslant k$, where $\mathfrak{T}_1(\Delta_{\alpha_1,\dots,\alpha_{i-1}})$ and $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{i-1}})$ are defined above when $i \geqslant 2$ and we define $\mathfrak{T}_1(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) = 1$ and $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{i-1}}) = 2$ when $i = 1$. Note that every almost universal sum of triangular numbers with one exception m is contained in the set of all candidates of almost universal sums of triangular numbers with one exception m.

We say an almost universal sum $\Delta_{\alpha_1,\dots,\alpha_k}$ of triangular numbers with one exception is *proper* if for any proper subset $\{i_1, \dots, i_u\} \subset \{1, \dots, k\}$, the partial sum $\Delta_{\alpha_{i_l},\dots,\alpha_{i_u}}$ doesn't represent at least two nonnegative integers.

Proof of Theorem 1.1. Let $\alpha_1, \dots, \alpha_k$ be positive integers. Assume that a sum $\Delta_{\alpha_1,\dots,\alpha_k}$ of triangular numbers is almost universal with one exception 1. Without loss of generality, we may assume that $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. From the definition of the candidate of almost universal sums of triangular numbers with one exception, one may easily check that Δ_2 is the unique candidate of unary almost universal sums of triangular numbers with one exception 1. However, it doesn't represent 3. Since $\mathfrak{T}_2(\Delta_2) = 3$, we know that there are exactly two candidates $\Delta_{2,2}$ and $\Delta_{2,3}$ of binary almost universal sums of triangular numbers withe one exception 1. Note that

$$\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2}) = \begin{cases} 3 & \text{if } (\alpha_1, \alpha_2) = (2, 2), \\ 4 & \text{if } (\alpha_1, \alpha_2) = (2, 3). \end{cases}$$

Therefore, there are exactly four candidates

$$\Delta_{2,2,2}, \quad \Delta_{2,2,3}, \quad \Delta_{2,3,3}, \quad \text{and} \quad \Delta_{2,3,4}$$

of ternary almost universal sums of triangular numbers with one exception 1. One may easily check that there are no ternary almost universal sums of triangular numbers with one exception 1. Indeed, for each of the above four cases, the second truant is

$$\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3}) = \begin{cases} 3 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (2,2,2), \\ 10 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (2,2,3), \\ 4 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (2,3,3), \\ 8 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (2,3,4). \end{cases} \tag{5}$$

Therefore, if $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a candidate of almost universal sums of triangular numbers with one exception 1, then $\Delta_{\alpha_1,\alpha_2,\alpha_3}$ is a candidate of ternary almost universal sums of triangular numbers with one exception 1 and $\alpha_3 \leqslant \alpha_4 \leqslant \mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3})$ from (5). So there are 17 candidates of quaternary almost universal sums of triangular numbers with one exception 1. One may easily check that 6 sums of them don't represent at least two nonnegative integers. Actually, we know that

$$\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \begin{cases} \mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3}) & \text{if } (\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (2,2,2,2), (2,2,3,9), \\ & \qquad (2,3,3,3), (2,3,4,7), \\ 19 & \text{if } (\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (2,2,3,3), \\ 16 & \text{if } (\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (2,2,3,6). \end{cases} \tag{6}$$

We will prove that remaining 11 quaternary candidates represent all nonnegative integers except 1 (see Table 1).

Now, we classify all candidates of quinary almost universal sums of triangular numbers with one exception 1. Assume that $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$ is a candidate of quinary almost universal sums of triangular numbers with one exception 1. Then $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is one of the candidates of quaternary almost universal sums of triangular numbers with one exception 1. If $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, then it implies that $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ represents all nonnegative integers except 1. Therefore, α_5 can be any integer greater than or equal to α_4. In this case $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$ is almost universal with one exception 1 but not proper. If $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) \neq \infty$, then we have $\alpha_4 \leqslant \alpha_5 \leqslant \mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4})$ from (6). In this case, we have 36 candidates of quinary almost universal sums of triangular numbers with one exception 1. One may easily check that there are 12 sums among them that are almost universal with one exception 1 but not proper. Furthermore, if $\alpha_5 = \mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) - 1$ for each possible case, then $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$ is not almost universal with one exception since it doesn't represent 1 and $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4})$. We will prove that remaining 18 quinary candidates represent all nonnegative integers except 1 (see Table 1).

Finally, for $k \geqslant 6$, we classify all candidates of k-ary almost universal sums of triangular numbers with one exception 1. Assume that $\Delta_{\alpha_1,\dots,\alpha_k}$ is a candidate of k-ary almost universal sums of triangular numbers with one exception 1. Then $\Delta_{\alpha_1,\dots,\alpha_{k-1}}$ is one of the candidates of $k-1$-ary almost universal sums of triangular numbers with one exception 1. If $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) = \infty$, then it implies that $\Delta_{\alpha_1,\dots,\alpha_{k-1}}$ represents all nonnegative integers except 1. Therefore,

α_k can be any integer greater than or equal to α_{k-1}. In this case, $\Delta_{\alpha_1,\dots,\alpha_k}$ is almost universal with one exception 1 but not proper. If $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) \neq \infty$, then $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1}+1$ since $\mathfrak{T}_2(\Delta_{\alpha_1,\dots\alpha_{k-1}}) = \alpha_{k-1}+1$. Note that $\Delta_{\alpha_1,\dots,\alpha_{k-1},\alpha_{k-1}}$ is not almost universal with one exception since it doesn't represent 1 and $\alpha_{k-1}+1$. Furthermore, $\Delta_{\alpha_1,\dots,\alpha_{k-1},\alpha_{k-1}+1}$ is almost universal with one exception 1 but not proper since $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_{k-1}+1}$ or $\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{k-1}+1}$ is already almost universal with one exception 1 for each possible case (see Table 1). Therefore, there are no k-ary proper almost universal sums of triangular numbers with one exception 1 for any integer $k \geqslant 6$ (see Table 1).

Now, we prove that above 11 quaternary and 18 quinary sums of triangular numbers are proper almost universal with one exception 1. In all cases, it is enough to show that each sum represents all nonnegative integers except 1 since its properness is clear.

Table 1. Proper almost universal sums with one exception 1

Sums	Candidates	Conditions on α_k
Δ_{α_1}	$\alpha_1 = 2$	$\alpha_1 \neq 2$
Δ_{2,α_2}	$2 \leqslant \alpha_2 \leqslant 3$	$\alpha_2 \neq 2, 3$
$\Delta_{2,2,\alpha_3}$	$2 \leqslant \alpha_3 \leqslant 3$	$\alpha_3 \neq 2, 3$
$\Delta_{2,3,\alpha_3}$	$3 \leqslant \alpha_3 \leqslant 4$	$\alpha_3 \neq 3, 4$
$\Delta_{2,2,2,\alpha_4}$	$2 \leqslant \alpha_4 \leqslant 3$	$\alpha_4 \neq 2$
$\Delta_{2,2,3,\alpha_4}$	$3 \leqslant \alpha_4 \leqslant 10$	$\alpha_4 \neq 3, 6, 9$
$\Delta_{2,3,3,\alpha_4}$	$3 \leqslant \alpha_4 \leqslant 4$	$\alpha_4 \neq 3$
$\Delta_{2,3,4,\alpha_4}$	$4 \leqslant \alpha_4 \leqslant 8$	$\alpha_4 \neq 7$
$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$	$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, $\alpha_5 \geqslant \alpha_4$	$\alpha_5 \neq \alpha_4^{\dagger}, \alpha_4+1^{\dagger}, \dots$
$\Delta_{2,2,2,2,\alpha_5}$	$2 \leqslant \alpha_5 \leqslant 3$	$\alpha_5 \neq 2, 3^{\dagger}$
$\Delta_{2,2,3,3,\alpha_5}$	$3 \leqslant \alpha_5 \leqslant 19$	$\alpha_5 \neq 4^{\dagger}, 5^{\dagger}, 7^{\dagger}, 8^{\dagger}, 10^{\dagger}, 18$
$\Delta_{2,2,3,6,\alpha_5}$	$6 \leqslant \alpha_5 \leqslant 16$	$\alpha_5 \neq 7^{\dagger}, 8^{\dagger}, 10^{\dagger}, 15$
$\Delta_{2,2,3,9,\alpha_5}$	$9 \leqslant \alpha_5 \leqslant 10$	$\alpha_5 \neq 9, 10^{\dagger}$
$\Delta_{2,3,3,3,\alpha_5}$	$3 \leqslant \alpha_5 \leqslant 4$	$\alpha_5 \neq 3, 4^{\dagger}$
$\Delta_{2,3,4,7,\alpha_5}$	$7 \leqslant \alpha_5 \leqslant 8$	$\alpha_5 \neq 7, 8^{\dagger}$
$\Delta_{\alpha_1,\dots,\alpha_k}$ $(k \geqslant 6)$	$\Delta_{\alpha_1,\dots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) = \infty$, $\alpha_k \geqslant \alpha_{k-1}$	$\alpha_k \neq \alpha_{k-1}^{\dagger}, \alpha_{k-1}+1^{\dagger}, \dots$
$\Delta_{\alpha_1,\dots,\alpha_k}$ $(k \geqslant 6)$	$\Delta_{\alpha_1,\dots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) \neq \infty$, $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1}+1$	$\alpha_k \neq \alpha_{k-1}, \alpha_{k-1}+1^{\dagger}$

(i) Let $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 2)$. We show that $\Delta_{2,2,2,3}$ is an almost universal sum of triangular numbers with one exception 1. Since $\Delta_{1,1,1}$ is universal, $\Delta_{2,2,2}$ represents all nonnegative even integers. Let n be an odd integer greater than 1. Since $n-3$ is represented by $\Delta_{2,2,2}$, n is represented by $\Delta_{2,2,2,3}$. Therefore, $\Delta_{2,2,2,3}$ is an almost universal sum of triangular numbers with one exception 1.

(ii) Let $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 3)$. We show that $\Delta_{2,2,3,\alpha_4}$ $(3 \leqslant \alpha_4 \leqslant 10,\ \alpha_4 \neq 3, 6, 9)$ are almost universal sums of triangular numbers with one exception 1. Similarly in the proof of the case (i), since $\Delta_{1,1,2}$, $\Delta_{1,1,4}$, and $\Delta_{1,1,5}$ are universal, $\Delta_{2,2,3,4}$, $\Delta_{2,2,3,8}$, and $\Delta_{2,2,3,10}$ are almost universal with one exception 1, respectively.

Assume that α_5 is 5 or 7. Since the proofs are quite similar to each other, we only provide the proof of $\Delta_{2,2,3,5}$. By Equation (1), it suffices to show that the equation

$$2x^2 + 2y^2 + 3z^2 + 5t^2 = 8n + 12 \tag{7}$$

has an integer solution $(x, y, z, t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$ for any nonnegative integer n except 1. If $n = 0$ or $2 \leqslant n \leqslant 4$, then one may directly check that Equation (7) has a desired integer solution. Therefore, we may assume that $n \geqslant 5$. Note that the genus of $f(x, y, z) = 2(4x + y)^2 + 2y^2 + 3z^2$ consists of

$$M_f = \langle 3, 4, 16\rangle \quad \text{and} \quad M_2 = \langle 4\rangle \perp \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}.$$

For a nonnegative integer m, if $m \equiv 7 \pmod 8$ and $m \neq 3^{2u+1}(3v + 2)$ for any nonnegative integers u and v, then m is represented by M_f or M_2 by 102:5 of [13], for it is represented by M_f over $\mathbb{Z}_p$ for any prime p. One may easily check that

$$M_2 \prec_{8,7} M_f.$$

Note that $8n + 12 - 5d^2 \equiv 7 \pmod 8$ and $8n + 12 - 5d^2 \not\equiv 0 \pmod 3$, where

$$d = \begin{cases} 1 & \text{if } 8n + 12 \equiv 0 \pmod 3, \\ 3 & \text{if } 8n + 12 \not\equiv 0 \pmod 3. \end{cases}$$

Furthermore, since we are assuming $n \geqslant 5$, $8n+12-5d^2$ is positive. Therefore, the equation

$$2x^2 + 2y^2 + 3z^2 = 8n + 12 - 5d^2$$

has an integer solution $(x, y, z) \in \mathbb{Z}^3$ such that $x \equiv y \pmod 4$ by Theorem 2.1. This completes the proof.

(iii) Let $(\alpha_1, \alpha_2, \alpha_3) = (2, 3, 3)$. We show that $\Delta_{2,3,3,4}$ is an almost universal sum of triangular numbers with one exception 1. By Equation (1), it suffices to show that the equation

$$2x^2 + 3y^2 + 3z^2 + 4t^2 = 8n + 12 \tag{8}$$

has an integer solution $(x, y, z, t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$ for any nonnegative integer n except 1. The class number of $f(x, y, z) = 2x^2+3y^2+3z^2$

is one. For a nonnegative integer m, if $m \equiv 0 \pmod 8$ and $m \not\equiv 3^{2u}(3v+1)$ for any nonnegative integers u and v, then m is represented by f over $\mathbb{Z}_p$ for any prime p, in particular, it is primitively represented by f over $\mathbb{Z}_2$. Let $8n+12 = 3^{2\ell}(8k+4)$ for some nonnegative integers ℓ and k such that $8k+4 \not\equiv 0 \pmod 9$. If $k=0$, note that for any $\ell \geqslant 1$,

$$2(3^{\ell-1})^2 + 3(3^{\ell-1})^2 + 3(3^{\ell})^2 + 4(3^{\ell-1})^2 = 4 \cdot 3^{2\ell}.$$

If $k=2$, note that for any $\ell \geqslant 1$,

$$2(7 \cdot 3^{\ell-1})^2 + 3(5 \cdot 3^{\ell-1})^2 + 3(3^{\ell-1})^2 + 4(3^{\ell-1})^2 = 20 \cdot 3^{2\ell}.$$

One may directly check that if $0 \leqslant k \leqslant 12$, $k \neq 0, 2$, then the equation

$$2x^2 + 3y^2 + 3z^2 + 4t^2 = 8k + 4$$

has an integer solution $(x,y,z,t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$. Therefore we may assume that $k \geqslant 13$. One may easily check that $8k + 4 - 4d^2$ is represented by f over $\mathbb{Z}_p$ for any p, where

$$d = \begin{cases} 3 & \text{if } 8k+4 \equiv r \pmod 9 \text{ for any } r \in \{2,3,5,6,8\}, \\ 1 & \text{if } 8k+4 \equiv r \pmod 9 \text{ for any } r \in \{1,7\}, \\ 5 & \text{if } 8k+4 \equiv 4 \pmod 9), \end{cases}$$

in particular, it is primitively represented by f over $\mathbb{Z}_2$. Furthermore, since we are assuming $k \geqslant 12$, $8k + 4 - 4d^2$ is positive. By 102:5 of [13], the equation

$$2x^2 + 3y^2 + 3z^2 = 8k + 4 - 4d^2$$

has an integer solution $(x,y,z) \in \mathbb{Z}^3$ such that $xyz \equiv 1 \pmod 2$. This completes the proof.

(iv) Let $(\alpha_1, \alpha_2, \alpha_3) = (2,3,4)$. We show that $\Delta_{2,3,4,\alpha_4}$ $(4 \leqslant \alpha_4 \leqslant 8,\ \alpha_4 \neq 7)$ are almost universal sums of triangular numbers with one exception 1. Similarly in the proof of case (i), since $\Delta_{1,2,2}$, $\Delta_{1,2,3}$, and $\Delta_{1,2,4}$ are universal, $\Delta_{2,3,4,4}$, $\Delta_{2,3,4,6}$, and $\Delta_{2,3,4,8}$ are almost universal with one exception 1.

Assume $\alpha_4 = 5$. By Equation (1), It suffices to show that the equation

$$2x^2 + 3y^2 + 4z^2 + 5t^2 = 8n + 14 \tag{9}$$

has an integer solution $(x,y,z,t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$ for any nonnegative integer n except 1. If $n = 0$ or $2 \leqslant n \leqslant 218$, then one may directly check that Equation (9) has a desired integer solution. Therefore we may assume that $n \geqslant 219$. Note that the genus of $f(x,y,t) = 2(2x+y)^2 + 3y^2 + 5t^2$ consists of

$$M_f = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \perp \langle 5 \rangle \quad \text{and} \quad M_2 = \langle 1, 1, 120 \rangle.$$

For a nonnegative integer m, if $m \equiv 2 \pmod 8$ and $m \not\equiv 0 \pmod 3$, then m is represented by M_f or M_2 by 102:5 of [13], for it is represented by M_f over $\mathbb{Z}_p$ for any prime p. One may easily check that

$$M_2 \prec_{7,r} M_f$$

for any $r \in \{0,3,5,6\}$. Assume that $8n+14 \not\equiv 0 \pmod 3$. Note that $8n+14-4d^2 \equiv 2 \pmod 8$, $8n+14-4d^2 \not\equiv 0 \pmod 3$, and $8n+14-4d^2 \equiv r \pmod 7$ for some $r \in \{0,3,5,6\}$, where

$$d = \begin{cases} 21 & \text{if } 8n+14 \equiv r \pmod 7 \text{ for any } r \in \{0,3,5,6\}, \\ 3 & \text{if } 8n+14 \equiv r \pmod 7 \text{ for any } r \in \{1,4\}, \\ 9 & \text{if } 8n+14 \equiv 2 \pmod 7. \end{cases}$$

Assume that $8n+14 \equiv 0 \pmod 3$. Note that $8n+14-4d^2 \equiv 2 \pmod 8$, $8n+14-4d^2 \not\equiv 0 \pmod 3$, and $8n+14-4d^2 \equiv r \pmod 7$ for some $r \in \{0,3,5,6\}$, where

$$d = \begin{cases} 7 & \text{if } 8n+14 \equiv r \pmod 7 \text{ for any } r \in \{0,3,5,6\}, \\ 1 & \text{if } 8n+14 \equiv r \pmod 7 \text{ for any } r \in \{2,4\}, \\ 5 & \text{if } 8n+14 \equiv 1 \pmod 7. \end{cases}$$

Furthermore, since we are assuming $n \geqslant 219$, $8n+14-4d^2$ is positive. Therefore, the equation

$$2x^2+3y^2+5t^2 = 8n+14-4d^2$$

has an integer solution by Theorem 2.1. This completes the proof.

To explain our main method that determines whether or not integers in an arithmetic progression are represented by some particular ternary quadratic form, we give another proof for almost universality of $\Delta_{2,3,4,4}$ by using Theorem 2.2. By Equation (1), it suffices to show that the equation

$$2x^2+3y^2+4z^2+4t^2 = 8n+13 \tag{10}$$

has an integer solution $(x,y,z,t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$ for any nonnegative integer n except 1. If $n=0$ or $2 \leqslant n \leqslant 11$, then one may directly check that Equation (10) has a desired integer solution. Therefore, we may assume that $n \geqslant 12$. Note that the genus of $f(x,y,z) = 2x^2+3y^2+4z^2$ consists of

$$f(x,y,z) \quad \text{and} \quad g(x,y,z) = x^2+2y^2+12z^2.$$

For a nonnegative integer m, if $m \equiv 1 \pmod 8$, then m is represented by M_f or M_2 by 102:5 of [13], for it is represented by M_f over $\mathbb{Z}_p$ for any prime p. One may easily show that

$$B_f(g,5,1) = \{\pm(1,0,0)\} \quad \text{and} \quad B_f(g,5,4) = \{\pm(2,0,0)\}.$$

In each case, if we define

$$T = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -12 \\ 0 & 2 & 1 \end{pmatrix},$$

then one may easily show that it satisfies all conditions in Theorem 2.2. Note that $z = \pm(1,0,0)$ are the only integral primitive eigenvectors of T. Therefore, we have

$$S_{5,r} \cap Q(g) \backslash \{s^2 | s \in \mathbb{Z}\} \subset Q(f)$$

for any $r \in \{1,4\}$. Since g is contained in the spinor genus of f, every square t^2 of an integer that has a prime divisor greater than 3 is represented by f by Lemma 2.4 in [9]. If t is divisible by 2 (3), then t^2 is represented by f since 4 (9, respectively) is represented by f. Therefore, every integer greater than 1 that is congruent to 1 modulo 8 and congruent to 1 or 4 modulo 5 is represented by f. Note that $8n+13-4d^2$ is congruent to 1 modulo 8 and congruent to 1 or 4 modulo 5, where

$$d = \begin{cases} 1 & \text{if } 8n+13 \equiv r \pmod 5 \text{ for any } r \in \{0,3\}, \\ 5 & \text{if } 8n+13 \equiv r \pmod 5 \text{ for any } r \in \{1,4\}, \\ 3 & \text{if } 8n+13 \equiv 2 \pmod 5). \end{cases}$$

Furthermore, since we are assuming $n \geqslant 12$, $8n+13-4d^2 \geqslant 2$. Therefore, the equation

$$2x^2+3y^2+4z^2 = 8n+13-4d^2$$

has an integer solution. This completes the proof.

(v) Let $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (2,2,3,3)$. We show that $\Delta_{2,2,3,3,\alpha_5}$ $(3 \leqslant \alpha_5 \leqslant 19$, $\alpha_5 \neq 4,5,7,8,10,18)$ are almost universal sums of triangular numbers with one exception 1.

Assume $\alpha_5 = 3$. Since $\Delta_{1,1,1}$ is universal, $\Delta_{3,3,3}$ represents every nonnegative integer divisible by 3. If $2 \leqslant n \leqslant 4$, then one may easily check that n is represented by $\Delta_{2,2,3,3,3}$. Assume $n \geqslant 5$. Then one may easily check that $n-c-d$ is represented by $\Delta_{3,3,3}$, where

$$(c,d) = \begin{cases} (0,0) & \text{if } n \equiv 0 \pmod 3, \\ (2,2) & \text{if } n \equiv 1 \pmod 3, \\ (2,0) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Therefore, $\Delta_{2,2,3,3,3}$ is almost universal with one exception 1. Similarly, $\Delta_{2,2,3,3,6}$, $\Delta_{2,2,3,3,12}$, and $\Delta_{2,2,3,3,15}$ are almost universal with one exception 1.

If $\alpha_5 \not\equiv 0 \pmod 3$, then the proofs are quite similar to the proof of $\Delta_{2,2,3,5}$ in the case (ii).

Assume that $\alpha_5 = 9$. By Equation (1), it suffices to show that

$$2x^2+2y^2+3z^2+3t^2+9s^2 = 8n+19 \tag{11}$$

has an integer solution $(x,y,z,t,s) \in \mathbb{Z}^5$ such that $xyzts \equiv 1 \pmod 2$ for any nonnegative integer n except 1. Let $8n+19 = 3^{2\ell}(8k+3)$ for some nonnegative integers ℓ and k such that $8k+3 \not\equiv 0 \pmod 9$. For the case when $k=0$, note that for any $\ell \geqslant 2$,

$$2(3^\ell)^2+2(3^{\ell-1})^2+3(3^{\ell-1})^2+3(3^{\ell-1})^2+9(3^{\ell-2})^2 = 3\cdot 3^{2\ell}.$$

For the case when $k=1$, note that for any $\ell \geqslant 1$,

$$2(3^\ell)^2+2(3^\ell)^2+3(3^\ell)^2+3(3^\ell)^2+9(3^{\ell-1})^2 = 11\cdot 3^{2\ell}.$$

If $k = 2, 4$, then one may directly check that the equation

$$2x^2 + 2y^2 + 3z^2 + 3t^2 + 9s^2 = 8k + 3$$

has an integer solution $(x, y, z, t, s) \in \mathbb{Z}^5$ such that $xyzts \equiv 1 \pmod 2$. Therefore, we may assume that $k \geqslant 5$. Note that $8k + 3 - 3d^2 - 9e^2 \equiv 7 \pmod 8$ and $8k+3-3d^2-9e^2 \neq 3^{2u+1}(3v+2)$ for any nonnegative integers u and v, where

$$(d, e) = \begin{cases} (1, 1) & \text{if } 8k + 3 \not\equiv 0 \pmod 3, \\ (3, 1) & \text{if } 8k + 3 \equiv 3 \pmod 9, \\ (1, 1) & \text{if } 8k + 3 \equiv 6 \pmod 9. \end{cases}$$

Furthermore, since we are assuming that $k \geqslant 5$, $8k + 3 - 3d^2 - 9e^2$ is positive and it is represented by $2(4x + y)^2 + 2y^2 + 3z^2$ (see the proof of $\Delta_{2,2,3,5}$ in the case (ii)). Therefore, the equation

$$2x^2 + 2y^2 + 3z^2 = 8k + 3 - 3d^2 - 9e^2$$

has an integer solution such that $x \equiv y \pmod 4$. This completes the proof.
(vi) Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 2, 3, 6)$. We show that $\Delta_{2,2,3,6,\alpha_5}$ ($6 \leqslant \alpha_5 \leqslant 16$, $\alpha_5 \neq 7, 8, 10, 15$) are almost universal sums of triangular numbers with one exception 1. Similarly as in the proof of case (v), since $\Delta_{3,6,6}$, $\Delta_{3,6,9}$, and $\Delta_{3,6,12}$ represent all nonnegative integers divisible by 3, $\Delta_{2,2,3,3,6}$, $\Delta_{2,2,3,6,9}$, and $\Delta_{2,2,3,6,12}$ are almost universal with one exception 1.

If $\alpha_5 \not\equiv 0 \pmod 3$, then the proofs are quite similar to the proof of $\Delta_{2,2,3,5}$ in the case (ii). This completes the proof.

Now, we give a proof of the first statement of Theorem 1.1. For positive integers $\alpha_1, \ldots, \alpha_k$, assume that a sum $\Delta_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ of triangular numbers represents the integers

$$2,\ 3,\ 4,\ 8,\ 10,\ 16,\ \text{and } 19$$

and doesn't represent 1. By using the same escalation method to the above, we know that there is a subset $\{i_1, i_2, i_3, i_4\}$ of $\{\alpha_1, \ldots, \alpha_k\}$ such that $\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4}}$ is contained in the above 11 quaternary proper almost universal sums of triangular numbers with one exception 1, or a subset $\{j_1, j_2, j_3, j_4, j_5\}$ of $\{\alpha_1, \ldots, \alpha_k\}$ such that $\Delta_{\alpha_{j_1},\alpha_{j_2},\alpha_{j_3},\alpha_{j_4},\alpha_{j_5}}$ is contained in the above 18 quinary ones. Therefore, $\Delta_{\alpha_1,\ldots,\alpha_k}$ represents all nonnegative integers except 1. This completes the proof. $\square$

The proofs of Theorems 1.2, 1.3, 1.4, and 1.6 are quite similar to the proof of Theorem 1.1. In each proof, by using an escalation method, we find all candidates of proper almost universal sums of triangular numbers with one exception 4, 5, 8, and 2, respectively. Furthermore, we show that each candidate $\Delta_{\alpha_1,\ldots,\alpha_k}$ ($k = 4, 5$) represents all nonnegative integers except a single one. To show this, we take a suitable ternary quadratic form $f(x_1, x_2, x_3)$ related with ternary section $\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ of $\Delta_{\alpha_1,\ldots,\alpha_k}$ like in Equation (3). After that for sufficiently large integer n, we find integers $a_4, \ldots, a_k \in \mathbb{Z}$ satisfying condition

(4). Finally, we directly check that $\Delta_{\alpha_1,\dots,\alpha_k}$ represents all remaining small integers except a single one.

Note that in most cases, the class number of f is less than or equal to 2. The methods for computations for representations of f are categorized into the following three cases:

(i) if $h(f)=1$, then one may easily compute the representations of f by the local-global principle similarly to the proof of $\Delta_{2,3,3,4}$ in the case (iii) of Theorem 1.1;

(ii) if $h(f)=2$ and $|B|=|B_f(g,d,a)|=0$ for some integers d and a, where g is the genus mate of f, then one may compute the representations of f by Theorem 2.1 similarly to the proof of $\Delta_{2,2,3,5}$ in the case (ii) of Theorem 1.1 (see also the proof of $\Delta_{2,3,4,5}$ in the case (iv) of Theorem 1.1);

(iii) if $h(f)=2$ and $|B|\neq 0$, then one may compute the representations of f by Theorem 2.2 similarly to the second proof of $\Delta_{2,3,4,4}$ in the case (iv) of Theorem 1.1.

In the remaining sections, since most of proofs require laborious computation, we only provide all parameters for the computations for the representations of the ternary quadratic form f (see Sections 4, 5, 6, and 7). One may easily apply the given parameters to the local-global principle, Theorem 2.1 or Theorem 2.2 to compute the representations of the ternary quadratic form f. We leave to the readers to find suitable integers $\alpha_4,\dots,\alpha_k$ stated above and to check the representations of remaining small integers. For the complete list of proper almost universal sums of triangular numbers with one exception, see Tables 1, 2, 10, 13, and 15.

4. Proof of Theorem 1.2

We give a proof of Theorem 1.2. From a similar escalation method in the proof of Theorem 1.1, we find all candidates of 127 quaternary and 11 quinary proper almost universal sums of triangular numbers with one exception 4 (see Table 2).

Since the proof of almost universality of each candidate is quite similar to the proof of Theorem 1.1, we only provide all parameters for the computations for representations of the ternary quadratic form f (see Tables 3, 4, 5, 6, 7, 8, and 9).

5. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. From a similar escalation method in the proof of Theorem 1.1, we find all candidates of 56 quaternary and 115 quinary proper almost universal sums of triangular numbers with one exception 5 (see Table 10). The proof of almost universality of each candidate is quite similar to the proof of Theorem 1.1 except for the cases when $(\alpha_1,\alpha_2,\alpha_3)=(1,1,8)$.

Table 2. Proper almost universal sums with one exception 4

Sums	Candidates	Conditions on α_k
Δ_{α_1}	$\alpha_1 = 1$	$\alpha_1 \neq 1$
Δ_{1,α_2}	$\alpha_2 = 2$	$\alpha_2 \neq 2$
$\Delta_{1,2,\alpha_3}$	$5 \leqslant \alpha_3 \leqslant 11$	$\alpha_3 \neq 5, 6, 7, 8, 9, 10, 11$
$\Delta_{1,2,5,\alpha_4}$	$5 \leqslant \alpha_4 \leqslant 19$	$\alpha_4 \neq 10, 15$
$\Delta_{1,2,6,\alpha_4}$	$6 \leqslant \alpha_4 \leqslant 50$	$\alpha_4 \neq 46$
$\Delta_{1,2,7,\alpha_4}$	$7 \leqslant \alpha_4 \leqslant 11$	$\alpha_4 \neq 7$
$\Delta_{1,2,8,\alpha_4}$	$8 \leqslant \alpha_4 \leqslant 19$	$\alpha_4 \neq 15$
$\Delta_{1,2,9,\alpha_4}$	$9 \leqslant \alpha_4 \leqslant 46$	$\alpha_4 \neq 42$
$\Delta_{1,2,10,\alpha_4}$	$10 \leqslant \alpha_4 \leqslant 14$	$\alpha_4 \neq 10$
$\Delta_{1,2,11,\alpha_4}$	$11 \leqslant \alpha_4 \leqslant 25$	$\alpha_4 \neq 21$
$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$	$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4})$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, $\alpha_5 \geqslant \alpha_4$	$\alpha_5 \neq \alpha_4^{\dagger}, \alpha_4 + 1^{\dagger}, \ldots$
$\Delta_{1,2,5,10,\alpha_5}$	$10 \leqslant \alpha_5 \leqslant 29$	$\alpha_5 \neq 11^{\dagger}, 12^{\dagger}, 13^{\dagger}, 14^{\dagger}, 16^{\dagger}, 17^{\dagger}, 18^{\dagger}, 19^{\dagger}, 25$
$\Delta_{1,2,5,15,\alpha_5}$	$15 \leqslant \alpha_5 \leqslant 19$	$\alpha_5 \neq 15, 16^{\dagger}, 17^{\dagger}, 18^{\dagger}, 19^{\dagger}$
$\Delta_{1,2,6,46,\alpha_5}$	$46 \leqslant \alpha_5 \leqslant 50$	$\alpha_5 \neq 46, 47^{\dagger}, 48^{\dagger}, 49^{\dagger}, 50^{\dagger}$
$\Delta_{1,2,7,7,\alpha_5}$	$7 \leqslant \alpha_5 \leqslant 11$	$\alpha_5 \neq 7, 8^{\dagger}, 9^{\dagger}, 10^{\dagger}, 11^{\dagger}$
$\Delta_{1,2,8,15,\alpha_5}$	$15 \leqslant \alpha_5 \leqslant 19$	$\alpha_5 \neq 15, 16^{\dagger}, 17^{\dagger}, 18^{\dagger}, 19^{\dagger}$
$\Delta_{1,2,9,42,\alpha_5}$	$42 \leqslant \alpha_5 \leqslant 46$	$\alpha_5 \neq 42, 43^{\dagger}, 44^{\dagger}, 45^{\dagger}, 46^{\dagger}$
$\Delta_{1,2,10,10,\alpha_5}$	$10 \leqslant \alpha_5 \leqslant 14$	$\alpha_5 \neq 10, 11^{\dagger}, 12^{\dagger}, 13^{\dagger}, 14^{\dagger}$
$\Delta_{1,2,11,21,\alpha_5}$	$21 \leqslant \alpha_5 \leqslant 25$	$\alpha_5 \neq 21, 22^{\dagger}, 23^{\dagger}, 24^{\dagger}, 25^{\dagger}$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) = \infty$, $\alpha_k \geqslant \alpha_{k-1}$	$\alpha_k \neq \alpha_{k-1}^{\dagger}, \alpha_{k-1} + 1^{\dagger}, \ldots$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) \neq \infty$, $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1} + 4$	$\alpha_k \neq \alpha_{k-1} + \ell$ $(\ell = 0, 1^{\dagger}, 2^{\dagger}, 3^{\dagger}, 4^{\dagger})$

Table 3. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 5)$

$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$		Sufficient conditions for $m \to f$
f	$h(f)$	
$\Delta_{1,2,5}$		$m \equiv 0 \pmod 8$, $m \neq 5^{2u+1}(5v \pm 2)$
$x^2 + 2y^2 + 5z^2$	1	

Assume $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 8)$. We show that $\Delta_{1,1,8,\alpha_4}$ $(8 \leqslant \alpha_4 \leqslant 41,\ \alpha_4 \neq 30, 36)$ are almost universal sums of triangular numbers with one exception 5. Since the proofs are quite similar to each other, we only provide the proof of $\Delta_{1,1,8,8}$. By Equation (1), it suffices to show that the equation

$$x^2 + y^2 + 8z^2 + 8t^2 = 8n + 18 \tag{12}$$

has an integer solution $(x, y, z, t) \in \mathbb{Z}^4$ such that $xyzt \equiv 1 \pmod 2$. If $0 \leqslant n \leqslant 19$, then one may directly check that Equation (12) has a desired integer solution. Therefore, we may assume that $n \geqslant 20$. Note that the genus of

TABLE 4. Data for the proof of the candidates when $(\alpha_1,\alpha_2,\alpha_3)=(1,2,6)$

α_4	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$	d	a	$\lvert B\rvert$	T		Sufficient conditions for $m \to f$
f	$h(f)$						
M_f	M_2				z	$Q(z)$	
$\alpha_4 \not\equiv 0 \pmod 5$	$\Delta_{1,2,6}$	20	$5,13,17$	0			$m\equiv 1 \pmod 8$, $m\equiv 0,2,3 \pmod 5$
$(2x+y)^2+2y^2+6z^2$	2						
$\begin{pmatrix}3&1\\1&3\end{pmatrix}\perp\langle 6\rangle$	$\langle 1,2,24\rangle$						
$\alpha_4 = 10,20,40,45$	$\Delta_{1,2,6}$	28	$1,9,25$	16	$\begin{pmatrix}28&0&0\\0&26&-36\\0&3&26\end{pmatrix}$		$m>1$, $m\equiv 1 \pmod 8$, $m\equiv 1,2,4 \pmod 7$
$(2x+y)^2+2y^2+6z^2$	2						
$\begin{pmatrix}3&1\\1&3\end{pmatrix}\perp\langle 6\rangle$	$\langle 1,2,24\rangle$				$\pm(1,0,0)$	1	
$\alpha_4 = 25,30,35$	$\Delta_{1,2,6}$	52	$1,9,17,$ $25,29,49$	16	$\begin{pmatrix}52&0&0\\0&46&-84\\0&7&46\end{pmatrix}$		$m>1$, $m\equiv 1 \pmod 8$, $m\equiv 1,3,4,9,10,$ $12 \pmod{13}$
$(2x+y)^2+2y^2+6z^2$	2						
$\begin{pmatrix}3&1\\1&3\end{pmatrix}\perp\langle 6\rangle$	$\langle 1,2,24\rangle$				$\pm(1,0,0)$	1	
$\alpha_4 = 15,50$	$\Delta_{1,2,6}$	76	$1,5,9,$ $17,25,45,$ $49,61,73$	16	$\begin{pmatrix}76&0&0\\0&74&-60\\0&5&74\end{pmatrix}$		$m>1$, $m\equiv 1 \pmod 8$, $m\equiv 1,4,5,6,7,9,11,$ $16,17 \pmod{19}$
$(2x+y)^2+2y^2+6z^2$	2						
$\begin{pmatrix}3&1\\1&3\end{pmatrix}\perp\langle 6\rangle$	$\langle 1,2,24\rangle$				$\pm(1,0,0)$	1	

TABLE 5. Data for the proof of the candidates when $(\alpha_1,\alpha_2,\alpha_3)=(1,2,7)$

α_4	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$	d	a	$\lvert B\rvert$	Sufficient conditions for $m \to f$
f	$h(f)$				
M_f	M_2				
$\alpha_4 = 10$	$\Delta_{1,2,10}$				$m\equiv 5 \pmod 8$, $m\not\equiv 0 \pmod 5$
$x^2+2y^2+10t^2$	1				
$\langle 1,2,10\rangle$					
$\alpha_4 = 8,9,11$	$\Delta_{1,2,7}$	80	$10,26,34,$ $50,66,74$	0	$m\equiv 2 \pmod 8$, $m\equiv 0,1,4 \pmod 5$, $m\not\equiv 0 \pmod 7$
$(2x+y)^2+2y^2+7z^2$	2				
$\begin{pmatrix}3&1\\1&3\end{pmatrix}\perp\langle 7\rangle$	$\begin{pmatrix}2&0&1\\0&2&1\\1&1&15\end{pmatrix}$				

$f(x,y,z)=x^2+(2y+z)^2+8z^2$ consists of

$$M_f=\begin{pmatrix}1&0&0\\0&4&2\\0&2&9\end{pmatrix},\quad M_2=\langle 1,1,32\rangle,\quad\text{and}\quad M_3=\begin{pmatrix}2&0&1\\0&2&1\\1&1&9\end{pmatrix}.$$

TABLE 6. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 8)$

α_4 / f / M_f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ / $h(f)$ / M_2	d	a	$\lvert B\rvert$	Sufficient conditions for $m \to f$
$\alpha_4 = 10$	$\Delta_{1,2,10}$				$m \equiv 5 \pmod 8$, $m \not\equiv 0 \pmod 5$
$x^2 + 2y^2 + 10t^2$	1				
$\langle 1, 2, 10\rangle$					
$\alpha_4 \not\equiv 0 \pmod 5$	$\Delta_{1,2,8}$	40	$11, 19, 35$	0	$m \equiv 3 \pmod 8$, $m \equiv 0, 1, 4 \pmod 5$
$x^2 + 2(2y+z)^2 + 8z^2$	2				
$\langle 1\rangle \perp \begin{pmatrix} 8 & 4 \\ 4 & 10 \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 9 \end{pmatrix}$				

TABLE 7. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 9)$

α_4 / f / M_f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ / $h(f)$ / M_2	d	a	$\lvert B\rvert$	T: z	T: $Q(z)$	Sufficient conditions for $m \to f$
$\alpha_4 = 21$	$\Delta_{1,9,21}$						$m \equiv 7 \pmod 8$, $m = 3^u(3v+1), (u \leqslant 1)$
$x^2 + 9z^2 + 21t^2$	1						
$\langle 1, 9, 21\rangle$							
$\alpha_4 \not\equiv 0 \pmod 7$	$\Delta_{1,2,9}$	7	$0, 3, 5, 6$	0			$m \equiv 4 \pmod 8$, $m \equiv 0, 3, 5, 6 \pmod 7$
$(4x+z)^2 + 2y^2 + 9z^2$	2						
$\langle 2\rangle \perp \begin{pmatrix} 10 & 4 \\ 4 & 16 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 8 & 0 \\ 2 & 0 & 10 \end{pmatrix}$						
$\alpha_4 = 14, 28, 35$	$\Delta_{1,2,9}$	3	0	0			
$(4x+z)^2 + 2y^2 + 9z^2$	2	9	$1, 4, 7$	6	$\begin{pmatrix} 9 & 4 & 3 \\ 0 & 3 & 9 \\ 0 & -8 & 3 \end{pmatrix}$		$m > 4$, $m \equiv 4 \pmod 8$, $m \not\equiv 2 \pmod 3$
$\langle 2\rangle \perp \begin{pmatrix} 10 & 4 \\ 4 & 16 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 8 & 0 \\ 2 & 0 & 10 \end{pmatrix}$				$\pm(1, 0, 0)$	4	

For an integer m, if $m \equiv 2 \pmod 8$, then m is represented by M_f, M_2, or M_3 by 102:5 of [13], for it is represented by M_f over $\mathbb{Z}_p$ for any prime p. Furthermore, note that the spinor genus of f consists of unique class f, itself. One may easily

TABLE 8. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 10)$

<table>
<tr><td colspan="2">$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$</td><td rowspan="2">Sufficient conditions for $m \to f$</td></tr>
<tr><td>f</td><td>$h(f)$</td></tr>
<tr><td colspan="2">$\Delta_{1,2,10}$</td><td rowspan="2">$m \equiv 5 \pmod 8, m \not\equiv 0 \pmod 5)$</td></tr>
<tr><td>$x^2 + 2y^2 + 10z^2$</td><td>1</td></tr>
</table>

TABLE 9. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 11)$

<table>
<tr><td>α_4</td><td>$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$</td><td rowspan="3">$d$</td><td rowspan="3">$a$</td><td rowspan="3">$|B|$</td><td colspan="2" rowspan="2">T</td><td rowspan="3">Sufficient conditions for $m \to f$</td></tr>
<tr><td>f</td><td>$h(f)$</td></tr>
<tr><td>M_f</td><td>M_2</td><td>z</td><td>$Q(z)$</td></tr>
<tr><td>$\alpha_4 \not\equiv 0 \pmod 5$</td><td>$\Delta_{1,2,11}$</td><td rowspan="3">20</td><td rowspan="3">$2, 10, 18$</td><td rowspan="3">0</td><td colspan="2" rowspan="3"></td><td rowspan="3">$m \equiv 6 \pmod 8$,
$m \equiv 0, 2, 3 \pmod 5$</td></tr>
<tr><td>$(2x+y)^2 + 2y^2 + 11z^2$</td><td>2</td></tr>
<tr><td>$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle 11 \rangle$</td><td>$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 2 & 6 \end{pmatrix}$</td></tr>
<tr><td>$\alpha_4 = 15$</td><td>$\Delta_{1,2,15}$</td><td>3</td><td>0</td><td>0</td><td colspan="2"></td><td rowspan="4">$m > 2$,
$m \equiv 2 \pmod 8$,
$m \not\equiv 10, 26 \pmod{48}$,
$m \not\equiv 0 \pmod 5$</td></tr>
<tr><td>$(2x+y)^2 + 2y^2 + 15t^2$</td><td>2</td><td rowspan="3">48</td><td rowspan="3">$2, 34$</td><td rowspan="3">384</td><td colspan="2" rowspan="2">$\begin{pmatrix} 48 & 30 & 72 \\ 0 & -12 & -144 \\ 0 & 15 & -12 \end{pmatrix}$</td></tr>
<tr><td rowspan="2">$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle 15 \rangle$</td><td rowspan="2">$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle 24 \rangle$</td></tr>
<tr><td>$\pm(1, 0, 0)$</td><td>2</td></tr>
<tr><td>$\alpha_4 = 20$</td><td>$\Delta_{1,2,20}$</td><td rowspan="3">3</td><td rowspan="3">2</td><td rowspan="3">2</td><td colspan="2" rowspan="2">$\begin{pmatrix} 1 & -4 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$</td><td rowspan="6">$m \equiv 7 \pmod 8$,
$m \not\equiv 0 \pmod 5$,
$m \not\equiv 3, 6, 7, 10, 12,$
$13, 19 \pmod{21}$</td></tr>
<tr><td>$x^2 + 2y^2 + 20t^2$</td><td>2</td></tr>
<tr><td rowspan="4">$\langle 1, 2, 20 \rangle$</td><td rowspan="4">$\langle 2, 4, 5 \rangle$</td><td>$\pm(0, 0, 1)$</td><td>5</td></tr>
<tr><td rowspan="2">7</td><td rowspan="2">$1, 2, 4$</td><td rowspan="2">2</td><td colspan="2">$\begin{pmatrix} 3 & 0 & -10 \\ 0 & 7 & 0 \\ 4 & 0 & 3 \end{pmatrix}$</td></tr>
<tr><td>$\pm(0, 1, 0)$</td><td>4</td></tr>
<tr><td>21</td><td>0</td><td>0</td><td colspan="2"></td></tr>
<tr><td>$\alpha_4 = 25$</td><td>$\Delta_{1,2,11}$</td><td rowspan="3">52</td><td rowspan="3">$10, 14, 22, 30, 38, 42$</td><td rowspan="3">16</td><td colspan="2" rowspan="2">$\begin{pmatrix} 36 & -32 & 32 \\ 24 & 48 & 4 \\ -20 & 12 & 40 \end{pmatrix}$</td><td rowspan="3">$m \equiv 6 \pmod 8$,
$m \equiv 1, 3, 4, 9, 10,$
$12 \pmod{13}$</td></tr>
<tr><td>$(2x+y)^2 + 2y^2 + 11z^2$</td><td>2</td></tr>
<tr><td>$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle 11 \rangle$</td><td>$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 2 & 6 \end{pmatrix}$</td><td>$\pm(0, 1, 1)$</td><td>16</td></tr>
</table>

show that a positive integer a is a spinor exception of the genus of f only if $a = 2m^2$ for some $m \in \mathbb{Z}$ (for details, see [16]). Assume that

$$8n + 18 - 8 = 2m_1^2 \quad \text{and} \quad 8n + 18 - 8 \cdot 3^2 = 2m_2^2$$

for some $m_1, m_2 \in \mathbb{Z}$. Then $2m_1^2 - 2m_2^2 = 64$. So $(m_1, m_2) \in \{(9, 7), (6, 2)\}$. However this is impossible since we are assuming $n \geqslant 20$. Therefore, one of the integers $8n + 18 - 8$ or $8n + 18 - 8 \cdot 3^2$ is not spinor exception of the genus of f, in fact, it is represented by f. This implies that the equation

$$x^2 + y^2 + 8z^2 = 8n + 18 - 8d^2$$

Table 10. Proper almost universal sums with one exception 5

Sums	Candidates	Conditions on α_k
Δ_{α_1}	$\alpha_1 = 1$	$\alpha_1 \neq 1$
Δ_{1,α_2}	$\alpha_2 = 1$	$\alpha_2 \neq 1$
$\Delta_{1,1,\alpha_3}$	$6 \leqslant \alpha_3 \leqslant 8$	$\alpha_3 \neq 6, 7, 8$
$\Delta_{1,1,6,\alpha_4}$	$6 \leqslant \alpha_4 \leqslant 14$	$\alpha_4 \neq 6, 9$
$\Delta_{1,1,7,\alpha_4}$	$7 \leqslant \alpha_4 \leqslant 26$	$\alpha_4 \neq 7, 14, 21$
$\Delta_{1,1,8,\alpha_4}$	$8 \leqslant \alpha_4 \leqslant 41$	$\alpha_4 \neq 30, 36$
$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$	$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4})$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, $\alpha_5 \geqslant \alpha_4$	$\alpha_5 \neq \alpha_4^\dagger, \alpha_4 + 1^\dagger, \ldots$
$\Delta_{1,1,6,6,\alpha_5}$	$6 \leqslant \alpha_5 \leqslant 59$	$\alpha_5 \neq 7^\dagger, 8^\dagger, 10^\dagger, 11^\dagger, 12^\dagger, 13^\dagger, 14^\dagger, 54$
$\Delta_{1,1,6,9,\alpha_5}$	$9 \leqslant \alpha_5 \leqslant 14$	$\alpha_5 \neq 9, 10^\dagger, 11^\dagger, 12^\dagger, 13^\dagger, 14^\dagger$
$\Delta_{1,1,7,7,\alpha_5}$	$7 \leqslant \alpha_5 \leqslant 47$	$\alpha_5 \neq 8^\dagger, 9^\dagger, 10^\dagger, 11^\dagger, 12^\dagger, 13^\dagger, 15^\dagger, 16^\dagger, 17^\dagger, 18^\dagger, 19^\dagger, 20^\dagger, 22^\dagger, 23^\dagger, 24^\dagger, 25^\dagger, 26^\dagger, 42$
$\Delta_{1,1,7,14,\alpha_5}$	$14 \leqslant \alpha_5 \leqslant 40$	$\alpha_5 \neq 15^\dagger, 16^\dagger, 17^\dagger, 18^\dagger, 19^\dagger, 20^\dagger, 22^\dagger, 23^\dagger, 24^\dagger, 25^\dagger, 26^\dagger, 35$
$\Delta_{1,1,7,21,\alpha_5}$	$21 \leqslant \alpha_5 \leqslant 26$	$\alpha_5 \neq 21, 22^\dagger, 23^\dagger, 24^\dagger, 25^\dagger, 26^\dagger$
$\Delta_{1,1,8,30,\alpha_5}$	$30 \leqslant \alpha_5 \leqslant 71$	$\alpha_5 \neq 31^\dagger, 32^\dagger, 33^\dagger, 34^\dagger, 35^\dagger, 37^\dagger, 38^\dagger, 39^\dagger, 40^\dagger, 41^\dagger, 66$
$\Delta_{1,1,8,36,\alpha_5}$	$36 \leqslant \alpha_5 \leqslant 41$	$\alpha_5 \neq 36, 37^\dagger, 38^\dagger, 39^\dagger, 40^\dagger, 41^\dagger$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}})$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) = \infty$, $\alpha_k \geqslant \alpha_{k-1}$	$\alpha_k \neq \alpha_{k-1}^\dagger, \alpha_{k-1} + 1^\dagger, \ldots$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) \neq \infty$, $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1} + 5$	$\alpha_k \neq \alpha_{k-1} + \ell$ $(\ell = 0, 1^\dagger, 2^\dagger, 3^\dagger, 4^\dagger, 5^\dagger)$

has an integer solution $(x, y, z) \in \mathbb{Z}^3$ for some $d \in \{1, 3\}$ such that $y \equiv z \pmod 2$. This completes the proof.

Now, we show that $\Delta_{1,1,8,30,\alpha_5}$ ($30 \leqslant \alpha_5 \leqslant 71$, $\alpha_5 \neq 31$, 32, 33, 34, 35, 37, 38, 39, 40, 41, 66) are almost universal sums of triangular numbers with one exception 5. Similarly as above, one may easily show that $\Delta_{1,1,8,30}$ represents all nonnegative integers except 5 and 71. Therefore, every $\Delta_{1,1,8,30,\alpha_5}$ is an almost universal sum of triangular numbers with one exception 5. This completes the proof.

In the remaining cases, since the proof of the almost universality of each candidate is quite similar to the proof of Theorem 1.1, we only provide all parameters for the computations for representations of the ternary quadratic form f (see Tables 11 and 12).

6. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. From a similar escalation method in the proof of Theorem 1.1, we find all candidates of 7 quaternary

TABLE 11. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 6)$

α_4	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$	Sufficient conditions for $m \to f$
f	$h(f)$	
every case	$\Delta_{1,1,6}$	$m \equiv 0 \pmod 8, m \neq 3^{2u+1}(3v+1)$
$x^2+y^2+6z^2$	1	
$\alpha_4 = 12$	$\Delta_{1,1,12}$	$m \equiv 6 \pmod 8,\ m \neq 3^{2u+1}(3v+2)$
$x^2+y^2+12t^2$	1	

TABLE 12. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 7)$

$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$		d	a	$\lvert B\rvert$	T		Sufficient conditions for $m \to f$
f	$h(f)$						
M_f	M_2				z	$Q(z)$	
$\Delta_{1,1,7}$		8	1	8	$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 6 & -14 \\ 0 & 2 & 6 \end{pmatrix}$		$m > 1$, $m \not\equiv 0 \pmod{49}$, $m \equiv 1 \pmod 8$, $m \neq 7^{2u+1}(7v+r)$ for any $r \in \{3, 5, 6\}$
$(2x+y)^2+2y^2+7z^2$	2						
$\langle 2, 2, 7\rangle$	$\langle 1, 2, 14\rangle$				$\pm(1,0,0)$	1	

and 73 quinary proper almost universal sums of triangular numbers with one exception 8 (see Table 13).

TABLE 13. Proper almost universal sums with one exception 8

Sums	Candidates	Conditions on α_k
Δ_{α_1}	$\alpha_1 = 1$	$\alpha_1 \neq 1$
Δ_{1,α_2}	$\alpha_2 = 1$	$\alpha_2 \neq 1$
$\Delta_{1,1,\alpha_3}$	$\alpha_3 = 3$	$\alpha_3 \neq 3$
$\Delta_{1,1,3,\alpha_4}$	$9 \leqslant \alpha_4 \leqslant 17$	$\alpha_4 \neq 9, 12$
$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$	$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, $\alpha_5 \geqslant \alpha_4$	$\alpha_5 \neq \alpha_4^{\dagger}, \alpha_4 + 1^{\dagger}, \dots$
$\Delta_{1,1,3,9,\alpha_5}$	$9 \leqslant \alpha_5 \leqslant 17$	$\alpha_5 \neq 9, 10^{\dagger}, 11^{\dagger}, 13^{\dagger}, 14^{\dagger}, 15^{\dagger}, 16^{\dagger}, 17^{\dagger}$
$\Delta_{1,1,3,12,\alpha_5}$	$12 \leqslant \alpha_5 \leqslant 89$	$\alpha_5 \neq 13^{\dagger}, 14^{\dagger}, 15^{\dagger}, 16^{\dagger}, 17^{\dagger}, 81$
$\Delta_{\alpha_1,\dots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\dots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) = \infty$, $\alpha_k \geqslant \alpha_{k-1}$	$\alpha_k \neq \alpha_{k-1}^{\dagger}, \alpha_{k-1} + 1^{\dagger}, \dots$
$\Delta_{\alpha_1,\dots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\dots,\alpha_{k-1}}8$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\dots,\alpha_{k-1}}) \neq \infty$, $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1} + 8$	$\alpha_k \neq \alpha_{k-1} + \ell$ $(\ell = 0, 1^{\dagger}, 2^{\dagger}, 3^{\dagger}, 4^{\dagger}, 5^{\dagger}, 6^{\dagger}, 7^{\dagger}, 8^{\dagger})$

Since the proof of almost universality of each candidate is quite similar to the proof of Theorem 1.1, we only provide all parameters for the computations for representations of the ternary quadratic form f (see Table 14).

TABLE 14. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 3)$

α_4 / f / M_f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ / $h(f)$ / M_2	d	a	$\vert B\vert$	T / z	$Q(z)$	Sufficient conditions for $m \to f$
every case	$\Delta_{1,1,3}$						$m \equiv 5 \pmod 8$, $m \neq 3^{2u+1}(3v+2)$
$(2x+y)^2+y^2+3z^2$	1						
$\langle 2,2,3\rangle$							
$\alpha_4 = 6$	$\Delta_{1,1,6}$						$m \equiv 0 \pmod 8$, $m \neq 3^{2u+1}(3v+1)$
$x^2+y^2+6t^2$	1						
$\langle 1,1,6\rangle$							
$\alpha_4 = 15$	$\Delta_{1,1,15}$	8	1	8	$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & 3 & 7 \end{pmatrix}$		$m > 9$, $m \equiv 1 \pmod 8$, $m \neq 3^{2u+1}(3v+1)$
$(2x+y)^2+y^2+15t^2$	2						
$\langle 2,2,15\rangle$	$\langle 1,6,10\rangle$				$\pm(1,0,0)$	1	

7. Proof of Theorem 1.6

We give a proof of Theorem 1.6. For positive integers $\alpha_1, \dots, \alpha_k$ ($k \geqslant 1$), assume that a sum $\Delta_{\alpha_1,\dots,\alpha_k}$ of triangular numbers is almost universal with one exception 2. Without loss of generality, we may assume that $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. From the definition of the candidate of almost universal sums of triangular numbers with one exception, one may easily check that Δ_1 is the unique candidate of unary almost universal sums of triangular numbers with one exception 2. Note that $\mathfrak{T}_1(\Delta_1) = 2$ and $\mathfrak{T}_2(\Delta_1) = 4$. Since $\Delta_{1,1}$ and $\Delta_{1,2}$ represent 2, there are exactly two candidates

$$\Delta_{1,3} \quad \text{and} \quad \Delta_{1,4}$$

of binary almost universal sums of triangular numbers with one exception 2. Note that

$$\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2}) = \begin{cases} 5 & \text{if } (\alpha_1,\alpha_2) = (1,3), \\ 8 & \text{if } (\alpha_1,\alpha_2) = (1,4). \end{cases}$$

Therefore, there are exactly 8 candidates

$$\Delta_{1,3,3},\ \Delta_{1,3,4},\ \Delta_{1,3,5},\ \Delta_{1,4,4},\ \Delta_{1,4,5},\ \Delta_{1,4,6},\ \Delta_{1,4,7},\ \text{and } \Delta_{1,4,8}$$

of ternary almost universal sums of triangular numbers with one exception 2. If $\Delta_{\alpha_1,\alpha_2,\alpha_3} \neq \Delta_{1,4,5}$, then the second truants of the above candidates are

$$\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3}) = \begin{cases} 5 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,3,3), \\ 11 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,3,4), \\ 7 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,3,5), \\ 20 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,4,4), \\ 8 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,4,6), \\ 9 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,4,7), \\ 16 & \text{if } (\alpha_1,\alpha_2,\alpha_3) = (1,4,8). \end{cases}$$

On the other hand, we checked that $\Delta_{1,4,5}$ represents all nonnegative integers up to 10^7 except 2. We conjectured that the sum $\Delta_{1,4,5}$ of triangular numbers is almost universal with one exception 2 (see Conjecture 1.5). Note that there are no ternary almost universal sums of triangular numbers with one exception $1, 4, 5$, and 8 (see Theorems 1.1, 1.2, 1.3, and 1.4). Therefore, $\Delta_{1,4,5}$ is the unique candidate of ternary almost universal sums of triangular numbers with one exception. In this section we assume that Conjecture 1.5 is true.

Continuing on with a similar escalation method in the proof of Theorem 1.1, we find all candidates of 34 quaternary and 37 quinary proper almost universal sums of triangular numbers with one exception 2 (see Table 15).

TABLE 15. Proper almost universal sums with one exception 2

Sums	Candidates	Conditions on α_k
Δ_{α_1}	$\alpha_1 = 1$	$\alpha_1 \neq 1$
Δ_{1,α_2}	$\alpha_2 = 3, 4$	$\alpha_2 \neq 3, 4$
$\Delta_{1,3,\alpha_3}$	$3 \leqslant \alpha_3 \leqslant 5$	$\alpha_4 \neq 3, 4, 5$
$\Delta_{1,4,\alpha_3}$	$4 \leqslant \alpha_3 \leqslant 8$	$\alpha_4 \neq 4, 6, 7, 8$
$\Delta_{1,4,5,\alpha_4}$	$\alpha_4 \geqslant 5$	$\alpha_4 \neq 5^{\dagger}, 6^{\dagger}, \ldots$
$\Delta_{1,3,3,\alpha_4}$	$3 \leqslant \alpha_4 \leqslant 5$	$\alpha_4 \neq 3, 4$
$\Delta_{1,3,4,\alpha_4}$	$4 \leqslant \alpha_4 \leqslant 11$	$\alpha_4 \neq 5^{\dagger}, 9$
$\Delta_{1,3,5,\alpha_4}$	$5 \leqslant \alpha_4 \leqslant 7$	$\alpha_4 \neq 5$
$\Delta_{1,4,4,\alpha_4}$	$4 \leqslant \alpha_4 \leqslant 20$	$\alpha_4 \neq 5^{\dagger}, 15, 18$
$\Delta_{1,4,6,\alpha_4}$	$6 \leqslant \alpha_4 \leqslant 8$	$\alpha_4 \neq 6$
$\Delta_{1,4,7,\alpha_4}$	$7 \leqslant \alpha_4 \leqslant 9$	$\alpha_4 \neq 7$
$\Delta_{1,4,8,\alpha_4}$	$8 \leqslant \alpha_4 \leqslant 16$	$\alpha_4 \neq 14, 15$
$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$	$\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}) = \infty$, $\alpha_5 \geqslant \alpha_4$	$\alpha_5 \neq \alpha_4^{\dagger}, \alpha_4 + 1^{\dagger}, \ldots$
$\Delta_{1,3,3,3,\alpha_5}$	$3 \leqslant \alpha_5 \leqslant 5$	$\alpha_5 \neq 3, 5^{\dagger}$
$\Delta_{1,3,3,4,\alpha_5}$	$4 \leqslant \alpha_5 \leqslant 29$	$\alpha_5 \neq 4^{\dagger}, 5^{\dagger}, 6^{\dagger}, 7^{\dagger}, 8^{\dagger}, 10^{\dagger}, 11^{\dagger}, 27$
$\Delta_{1,3,4,9,\alpha_5}$	$9 \leqslant \alpha_5 \leqslant 11$	$\alpha_5 \neq 9, 10^{\dagger}, 11^{\dagger}$
$\Delta_{1,3,5,5,\alpha_5}$	$5 \leqslant \alpha_5 \leqslant 7$	$\alpha_5 \neq 5, 6^{\dagger}, 7^{\dagger}$
$\Delta_{1,4,4,15,\alpha_5}$	$15 \leqslant \alpha_5 \leqslant 35$	$\alpha_5 \neq 16^{\dagger}, 17^{\dagger}, 19^{\dagger}, 20^{\dagger}, 33$
$\Delta_{1,4,4,18,\alpha_5}$	$18 \leqslant \alpha_5 \leqslant 20$	$\alpha_5 \neq 18, 19^{\dagger}, 20^{\dagger}$
$\Delta_{1,4,6,6,\alpha_5}$	$6 \leqslant \alpha_5 \leqslant 8$	$\alpha_5 \neq 6, 7^{\dagger}, 8^{\dagger}$
$\Delta_{1,4,7,7,\alpha_5}$	$7 \leqslant \alpha_5 \leqslant 9$	$\alpha_5 \neq 7, 8^{\dagger}, 9^{\dagger}$
$\Delta_{1,4,8,14,\alpha_5}$	$14 \leqslant \alpha_5 \leqslant 16$	$\alpha_5 \neq 14, 16^{\dagger}$
$\Delta_{1,4,8,15,\alpha_5}$	$15 \leqslant \alpha_5 \leqslant 17$	$\alpha_5 \neq 15, 16^{\dagger}$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}}2$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) = \infty$, $\alpha_k \geqslant \alpha_{k-1}$	$\alpha_k \neq \alpha_{k-1}^{\dagger}, \alpha_{k-1} + 1^{\dagger}, \ldots$
$\Delta_{\alpha_1,\ldots,\alpha_k} (k \geqslant 6)$	$\Delta_{\alpha_1,\ldots,\alpha_{k-1}}$ is a candidate, $\mathfrak{T}_2(\Delta_{\alpha_1,\ldots,\alpha_{k-1}}) \neq \infty$, $\alpha_{k-1} \leqslant \alpha_k \leqslant \alpha_{k-1} + 2$	$\alpha_k \neq \alpha_{k-1} + \ell$ $(\ell = 0, 1^{\dagger}, 2^{\dagger})$

TABLE 16. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 3, 3)$

$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ f	 $h(f)$	Sufficient conditions for $m \to f$
$\Delta_{1,3,3}$ $(2x+y)^2+3y^2+3z^2$	 1	$m \equiv 7 \pmod 8,\ m \neq 3^{2u}(3v+2)$

TABLE 17. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 3, 4)$

α_4 f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ $h(f)$	Sufficient conditions for $m \to f$
$\alpha_4 \not\equiv 0 \pmod 3$ $(4x+y)^2+3y^2+4z^2$	$\Delta_{1,3,4}$ 1	$m \equiv 0 \pmod 8, m \not\equiv 0 \pmod 3$
$\alpha_4 = 6$ $(4x+y)^2+3y^2+6t^2$	$\Delta_{1,3,6}$ 1	$m \equiv 2 \pmod 8, m \neq 3^{2u}(3v+2)$

TABLE 18. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 3, 5)$

$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ f M_f	 $h(f)$ M_2	d	a	$\lvert B\rvert$	T z	 $Q(z)$	Sufficient conditions for $m \to f$
$\Delta_{1,3,5}$ $(2x+y)^2+3y^2+5z^2$ $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \perp \langle 5\rangle$	 2 $\langle 1\rangle \perp \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}$	8	1	4	$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & -4 & 6 \end{pmatrix}$ $\pm(1,0,0)$	 1	$m > 1$, $m \equiv 1 \pmod 8$, $m \not\equiv 0 \pmod 5$

TABLE 19. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 4, 4)$

α_4 f M_f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ $h(f)$ M_2	d	a	$\lvert B\rvert$	Sufficient conditions for $m \to f$
$\alpha_4 \not\equiv 0 \pmod 3$ $(2x+y)^2+4y^2+4z^2$ $\begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \perp \langle 4\rangle$	$\Delta_{1,4,4}$ 2 $\langle 1,4,16\rangle$	3	$0, 2$	0	$m \equiv 1 \pmod 8$, $m \equiv 0, 2 \pmod 3$
$\alpha_4 \equiv 0 \pmod 3$ $(2x+y)^2+4y^2+4z^2$ $\begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \perp \langle 4\rangle$	$\Delta_{1,4,4}$ 2 $\langle 1,4,16\rangle$	7	$0, 3, 5, 6$	0	$m \equiv 1 \pmod 8$, $m \equiv 0, 3, 5, 6 \pmod 7$

TABLE 20. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 4, 6)$

<table>
<tr><td colspan="2">$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$</td><td rowspan="2">Sufficient conditions for $m \to f$</td></tr>
<tr><td>f</td><td>$h(f)$</td></tr>
<tr><td colspan="2">$\Delta_{1,4,6}$</td><td rowspan="2">$m \equiv 3 \pmod 8,\ m \not\equiv 0 \pmod 3$</td></tr>
<tr><td>$x^2 + 4y^2 + 6z^2$</td><td>1</td></tr>
</table>

TABLE 21. Data for the proof of the candidates when $(\alpha_1, \alpha_2, \alpha_3) = (1, 4, 7)$

<table>
<tr><td>α_4</td><td>$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$</td><td rowspan="3">$d$</td><td rowspan="3">$a$</td><td rowspan="3">$\lvert B\rvert$</td><td colspan="2" rowspan="2">T</td><td rowspan="3">Sufficient conditions for $m \to f$</td></tr>
<tr><td>f</td><td>$h(f)$</td></tr>
<tr><td>M_f</td><td>M_2</td><td>z</td><td>$Q(z)$</td></tr>
<tr><td>$\alpha_4 = 8$</td><td>$\Delta_{1,4,8}$</td><td rowspan="3">3</td><td rowspan="3">1</td><td rowspan="3">2</td><td colspan="2" rowspan="2">$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -8 \\ 0 & 1 & 1 \end{pmatrix}$</td><td rowspan="6">$m > 1$,
$m \equiv 5 \pmod 8$,
$m \equiv 1 \pmod 3$ or
$m \equiv 2, 3 \pmod 5$ or
$m \equiv 0 \pmod{15}$</td></tr>
<tr><td>$(2x + t)^2 + 4y^2 + 8t^2$</td><td>2</td></tr>
<tr><td rowspan="4">$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 9 \end{pmatrix}$</td><td rowspan="4">$\langle 1, 4, 32\rangle$</td><td>$\pm(1, 0, 0)$</td><td>1</td></tr>
<tr><td rowspan="2">5</td><td rowspan="2">2, 3</td><td rowspan="2">2</td><td colspan="2">$\begin{pmatrix} 3 & -8 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$</td></tr>
<tr><td>$\pm(0, 0, 1)$</td><td>32</td></tr>
<tr><td>15</td><td>0</td><td>0</td><td colspan="2"></td></tr>
<tr><td>$\alpha_4 = 9$</td><td>$\Delta_{1,4,9}$</td><td>3</td><td>1</td><td>0</td><td colspan="2"></td><td rowspan="5">$m \equiv 6 \pmod 8$,
$m \equiv 1 \pmod 3$ or
$m \equiv 1, 4 \pmod 5$ or
$m \equiv 0, 10, 30, 35,$
$40, 50 \pmod{55}$</td></tr>
<tr><td>$x^2 + 4y^2 + 9t^2$</td><td>2</td><td rowspan="3">5</td><td rowspan="3">1, 4</td><td rowspan="3">2</td><td colspan="2" rowspan="2">$\begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$</td></tr>
<tr><td rowspan="3">$\langle 1, 4, 9\rangle$</td><td rowspan="3">$\langle 1, 1, 36\rangle$</td></tr>
<tr><td>$\pm(0, 0, 1)$</td><td>36</td></tr>
<tr><td>55</td><td>0, 10, 30
35, 40, 50</td><td>0</td><td colspan="2"></td></tr>
</table>

Since the proof of almost universality of each candidate is quite similar to the proof of Theorem 1.1, we only provide all parameters for the computations for representations of the ternary form f (see Tables 16, 17, 18, 19, 20, 21 and 22).

TABLE 22. Data for the proof of the candidates when $(\alpha_1,\alpha_2,\alpha_3)=(1,4,8)$

α_4 / f / M_f	$\Delta_{\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}}$ / $h(f)$ / M_2	d	a	$\lvert B\rvert$	T	z	$Q(z)$	Sufficient conditions for $m\to f$
$\alpha_4\neq 9,10,12$ $(2x+z)^2+4y^2+8z^2$ $\begin{pmatrix}4&0&2\\0&4&0\\2&0&9\end{pmatrix}$	$\Delta_{1,4,8}$ 2 $\langle 1,4,32\rangle$	3	1	2	$\begin{pmatrix}3&0&0\\0&1&-8\\0&1&1\end{pmatrix}$	$\pm(1,0,0)$	1	$m>1$, $m\equiv 5 \pmod 8$, $m\equiv 1 \pmod 3$ or $m\equiv 2,3 \pmod 5$ or $m\equiv 0 \pmod{15}$
		5	2, 3	2	$\begin{pmatrix}3&-8&0\\2&3&0\\0&0&5\end{pmatrix}$	$\pm(0,0,1)$	32	
		15	0	0				
$\alpha_4=9$ $x^2+4y^2+9t^2$ $\langle 1,4,9\rangle$	$\Delta_{1,4,9}$ 2 $\langle 1,1,36\rangle$	3	1	0				$m\equiv 6 \pmod 8$, $m\equiv 1 \pmod 3$ or $m\equiv 1,4 \pmod 5$ or $m\equiv 0,10,30,35,40,50 \pmod{55}$
		5	1, 4	2	$\begin{pmatrix}4&-3&0\\3&4&0\\0&0&5\end{pmatrix}$	$\pm(0,0,1)$	36	
		55	0, 10, 30 35, 40, 50	0				
$\alpha_4=10$ $4(2y+z)^2+8z^2+10t^2$ $\langle 10\rangle\perp\begin{pmatrix}12&4\\4&12\end{pmatrix}$	$\Delta_{4,8,10}$ 2 $\begin{pmatrix}6&-2&2\\-2&6&2\\2&2&42\end{pmatrix}$	3	1	2	$\begin{pmatrix}2&-3&1\\3&0&3\\0&0&-3\end{pmatrix}$	$\pm(1,1,-2)$	160	$m\equiv 6 \pmod 8$, $m\not\equiv 0 \pmod 5$, $m\equiv 1 \pmod 3$ or $m\equiv 0 \pmod{21}$ or $m\equiv 1,2,4 \pmod 7$
		7	1, 2, 4	2	$\begin{pmatrix}6&1&12\\3&4&-8\\-2&2&3\end{pmatrix}$	$\pm(1,1,0)$	8	
		21	0	0				
$\alpha_4=12$ $(2x+y)^2+4y^2+12t^2$ $\begin{pmatrix}4&2\\2&5\end{pmatrix}\perp\langle 12\rangle$	$\Delta_{1,4,12}$ 2 $\langle 1,12,16\rangle$	5	2, 3	2	$\begin{pmatrix}3&0&-16\\0&5&0\\1&0&3\end{pmatrix}$	$\pm(0,1,0)$	12	$m\equiv 1 \pmod 8$, $m\not\equiv 0 \pmod 3$, $m\equiv 2,3 \pmod 5$

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Jangwon Ju
Department of Mathematics Education
Korea National University of Education
Chungbuk 28173, Korea
Email address: jangwonju@knue.ac.kr