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CLASSIFYING PERMUTATIONS USING FULTON'S ESSENTIAL SET AND MID SET

Sook Min

Abstract. We study permutatons that satisfy a necessary and sufficient condition for the equality of Fulton's essential set and the maximal-inversion-descent(MID) set.

1. Introduction

Let S_n denote the symmetric group on $[n] = \{1, 2, ..., n\}$. We write a permutation of S_n in one-line notation. For example, $\pi = 213$ means $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 3$. The *permutation matrix* for $\pi \in S_n$ is an $n \times n$ matrix, considered as an n-by-n array of squares in the plane, where square $(i, \pi(i))$ has a dot for every $i = 1, 2, ..., n$ and all other squares are white. So there is exactly one dot in each row and column. The *inverse* π^{-1} of $\pi \in S_n$ is defined by $\pi^{-1}(a_i) = i \Leftrightarrow \pi(i) = a_i$ for every $i = 1, 2, ..., n$.

The essential set, together with a rank function, was introduced by Fulton [3]. The *essential set* $\mathcal{E}(\pi)$ of $\pi \in S_n$ is defined by

$$\begin{aligned}\mathcal{E}(\pi) \quad = \quad & \{(i,j) \in [1, n-1]^2 \mid \pi(i) > j,\ \pi^{-1}(j) > i, \\ & \pi(i+1) \leq j, \text{ and } \pi^{-1}(j+1) \leq i\}.\end{aligned}$$

For example, if $\pi = 462513 \in S_6$, then the essential set of π is $\mathcal{E}(\pi) = \{(2, 3), (2, 5), (4, 1), (4, 3)\}$.

We can also represent the essential set $\mathcal{E}(\pi)$ of $\pi \in S_n$ using a permutation matrix as follows. Create the permutation matrix using the permutation π, and in each square with a dot, shade in the direction east of the dot and shade in the direction south of the dot. Then a white (unshaded) square appears in the permutation matrix. We call a white square a *white corner* if the squares directly to the right and below of it

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are shaded. The *essential set* $\mathcal{E}(\pi)$ of $\pi \in S_n$ is defined to be the set of white corners of the permutation matrix for π.

Baxter permutations are named after a study by Baxter[1]. A *Baxter permutation* is exactly a permutation π in S_n that satisfies the following two conditions: for every $1 \leq i < j < k < l \leq n$,

(1) if $\pi(i) + 1 = \pi(l)$ and $\pi(l) < \pi(j)$ then $\pi(k) > \pi(l)$, and
(2) if $\pi(l) + 1 = \pi(i)$ and $\pi(i) < \pi(k)$ then $\pi(j) > \pi(i)$.

For example, 2413 and 3142 are the only permutations on four elements which are not Baxter permutations. We have the following properties that Baxter permutations can be represented as the essential sets [2].

THEOREM 1.1 ([2] Proposition 5.2). *A permutation $\pi \in S_n$ is a Baxter permutation if and only if its essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and column.*

DEFINITION 1.2 ([4] Definition 3, [5] Definition 2.1). Let $\pi \in S_n$. We say that the pair $(i, b_i), 1 \leq i < n$, is a *maximal-inversion* if b_i is the maximum of $\pi(k)$'s such that $\pi(k) < \pi(i)$ for every $k > i$. The *maximal-inversion set* of π, denoted by $MI(\pi)$, is the set of all maximal-inversions.

For example, the maximal-inversion set of $\pi = 462513 \in S_6$ is $MI(\pi) = \{(1, 3), (2, 5), (3, 1), (4, 3)\}$.

DEFINITION 1.3 ([5] Definition 2.3). A *maximal-inversion-descent* of a permutation π in S_n is an element (i, b_i) in $MI(\pi)$ with descent in position i. The *maximal-inversion-descent set* of π, denoted by $MID(\pi)$, is the set of all maximal-inversion-descents:

$$MID(\pi) = \{(i, b_i) \in MI(\pi) \mid \pi(i) > \pi(i+1)\}.$$

For example, the maximal-inversion-descent set of $\pi = 462513 \in S_6$ is $MID(\pi) = \{(2, 5), (4, 3)\}$. Note that the number of the elements in $MID(\pi)$ for every permutation $\pi \in S_n$ is equal to the number of elements in the descent set of π. We have the following properties of the maximal-inversion-descent (MID) sets and the essential sets [5].

PROPOSITION 1.4 ([5] Proposition 3.4). *For every permutation π, we have $MID(\pi) \subset \mathcal{E}(\pi)$.*

THEOREM 1.5 ([5] Theorem 3.5). *If a permutation π is a Baxter, then $MID(\pi) = \mathcal{E}(\pi)$.*

The purpose of this paper is to study permutations $\pi \in S_n$ that satisfy a necessary and sufficient condition in terms of $MID(\pi) = \mathcal{E}(\pi)$.

2. Main results

We introduce new definitions associated with Baxter permutations:

DEFINITION 2.1. A *c-pseudoBaxter permutation* is exactly a permutation $\pi \in S_n$ that satisfies the following two conditions:

(1) for every $1 \leq i < j < k < l \leq n$,

$$\text{if } \pi(i) + 1 = \pi(l) \text{ and } \pi(j) > \pi(l) \text{ then } \pi(k) > \pi(l), \text{ and}$$

(2) there exist indices $1 \leq i_c < j_c < k_c < l_c \leq n$ such that

$$\pi(l_c) + 1 = \pi(i_c), \pi(k_c) > \pi(i_c), \text{ and } \pi(j_c) < \pi(i_c).$$

DEFINITION 2.2. A *r-pseudoBaxter permutation* is exactly a permutation $\pi \in S_n$ that satisfies the following two conditions:

(1) for every $1 \leq i < j < k < l \leq n$,

$$\text{if } \pi(l) + 1 = \pi(i) \text{ and } \pi(k) > \pi(i) \text{ then } \pi(j) > \pi(i), \text{ and}$$

(2) there exist indices $1 \leq i_r < j_r < k_r < l_r \leq n$ such that

$$\pi(i_r) + 1 = \pi(l_r),\ \pi(j_r) > \pi(l_r), \text{ and } \pi(k_r) < \pi(l_r).$$

DEFINITION 2.3. A *rc-pseudoBaxter permutation* is exactly a permutation $\pi \in S_n$ that satisfies the following two conditions:

(1) there exist indices $1 \leq i_r < j_r < k_r < l_r \leq n$ such that

$$\pi(i_r) + 1 = \pi(l_r),\ \pi(j_r) > \pi(l_r), \text{ and } \pi(k_r) < \pi(l_r), \text{ and}$$

(2) there exist indices $1 \leq i_c < j_c < k_c < l_c \leq n$ such that

$$\pi(l_c) + 1 = \pi(i_c),\ \pi(k_c) > \pi(i_c), \text{ and } \pi(j_c) < \pi(i_c).$$

EXAMPLE 2.4. *The permutation $\pi = 3\,1\,5\,4\,2$ is a c-pseudoBaxter permutation because for indices $i_c = 1, j_c = 2, k_c = 3, l_c = 5$ it satisfies the second condition (2) of Definition 2.1, and because it satisfies the first condition (1) of Definition 2.1.*

EXAMPLE 2.5. *The permutation $\pi = 3\,5\,2\,4\,1$ is an r-pseudoBaxter permutation because for indices $i_r = 1, j_r = 2, k_r = 3, l_r = 4$ it satisfies the second condition (2) of Definition 2.2, and because it satisfies the first condition (1) of Definition 2.2.*

EXAMPLE 2.6. *The permutation $\pi = 5\,3\,1\,7\,2\,8\,4\,6$ is an rc-pseudoBaxter permutation.*

We summarize our main results. As mentioned in Introduction, Min and Park proved the next theorem.

Theorem 2.7 ([5] Theorem 3.5). *If a permutation $\pi \in S_n$ is a Baxter permutation, then $MID(\pi) = \mathcal{E}(\pi)$.*

Proposition 2.8 ([2] Proposition 5.2). *A permutation $\pi \in S_n$ is a Baxter permutation if and only if its essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and column.*

Theorem 2.9. *If a permutation $\pi \in S_n$ is a c-pseudoBaxter permutation, then $MID(\pi) = \mathcal{E}(\pi)$.*

Proposition 2.10. *A permutation $\pi \in S_n$ is a c-pseudoBaxter permutation if and only if its essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and at least two white corners in some columns.*

Note 2.11. The meaning of $MID(\pi) = \mathcal{E}(\pi)$ of $\pi \in S_n$ is as follows.

(1) The essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and column, or
(2) The essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and at least two white corners in some columns.

Then our main theorem:

Theorem 2.12. *A permutation $\pi \in S_n$ is a Baxter permutation or a c-pseudoBaxter permutation if and only if $MID(\pi) = \mathcal{E}(\pi)$.*

Proof. [Proof (necessity)] It is proved by Theorem 2.7 and Theorem 2.9.

[Proof (sufficiency)] If the essential set $\mathcal{E}(\pi)$ satisfies the first condition (1) of Note 2.11, the permutation π is Baxter permutation by Proposition 2.8. Also, if the essential set $\mathcal{E}(\pi)$ satisfies the second condition (2) of Note 2.11, the permutation π is c-pseudoBaxter permutation by Proposition 2.10. $\square$

3. Proof of Main results

In this section we prove Proposition 2.8, Theorem 2.9, and Proposition 2.10.

Lemma 3.1. *Let $\pi \in S_n$. If $(i, a),\ (i, b) \in \mathcal{E}(\pi)$ with $a < b$, then π satisfies the second condition (2) of Definition 2.2.*

Proof. Since $(i, a), (i, b) \in \mathcal{E}(\pi)$, by the definition of the essential set it can be written as

$$\pi(i) > a,\ \pi^{-1}(a) > i,\ \pi(i+1) \leq a,\ \text{and}\ \pi^{-1}(a+1) \leq i,\ \text{and}$$

$$\pi(i) > b, \pi^{-1}(b) > i, \pi(i+1) \leq b, \text{ and } \pi^{-1}(b+1) \leq i.$$

We first claim that the permutation π is represented by

$$\pi = \cdots (a+1) \cdots \pi(i)\pi(i+1) \cdots b \cdots ,$$

where $\pi(i) > b > a+1 > \pi(i+1)$.

(1) $\pi^{-1}(b) > i+1$, since $\pi^{-1}(b) > i$, moreover, if $\pi(i+1) = b$, then $a \geq \pi(i+1) = b$, however, this is a contradiction to the assumption $a < b$.

(2) $\pi^{-1}(a+1) < i$, since if $\pi^{-1}(a+1) = i$, then $a+1 = \pi(i) > b$, however, this is a contradiction to the assumption $a < b$.

(3) $a+1 < b$, since if $a+1 = b$, then $i < \pi^{-1}(b) = \pi^{-1}(a+1) \leq i$, however, this is a contradiction.

Let $b - a = l$, where $l \geq 2$. Second, we claim that the permutation π is represented by, for some $k \in \{1, 2, \ldots, l-1\}$

$$\pi = \cdots (a+k) \cdots \pi(i)\pi(i+1) \cdots (a+k+1) \cdots ,$$

where $\pi(i) > b \geq a+k+1$ and $\pi(i+1) < a+1 < a+k+1$.

(1) $\pi(i), \pi(i+1) \notin \{a+1, a+2, \ldots, a+l\}$, since $\pi(i) > b = a+l$ and $\pi(i+1) < a+1$, and

(2) there exists an integer $k \in \{1, 2, \ldots, l-1\}$ such that $\pi^{-1}(a+k) < i$ and $\pi^{-1}(a+k+1) > i$, since consecutive numbers $a+1, a+2, \ldots, a+l$ are placed into the positions $\pi(1), \ldots, \pi(i-1), \pi(i+2), \ldots, \pi(n)$, $\pi^{-1}(a+1) < i$, and $\pi^{-1}(a+l)(= \pi^{-1}(b)) > i$.

Thus π satisfies the second condition (2) of Definition 2.2. □

LEMMA 3.2. *Let $\pi \in S_n$. If $(i, b), (j, b) \in \mathcal{E}(\pi)$ for some indices $1 \leq i < j < n$, then π satisfies the second condition (2) of Definition 2.1.*

Proof. Since $(i, b), (j, b) \in \mathcal{E}(\pi)$, by the definition of the essential set and inverse we see that $(b, i), (b, j) \in \mathcal{E}(\pi^{-1})$ with $1 \leq i < j < n$. Let $j - i = l$, where $l \geq 2$. By the proof of Lemma 3.1, the inverse π^{-1} of π is represented by, for some $k \in \{1, 2, \ldots, l-1\}$

$$\pi^{-1} = \cdots (i+k) \cdots \pi^{-1}(b)\pi^{-1}(b+1) \cdots (i+k+1) \cdots$$

where $\pi^{-1}(b) > i+k+1$ and $\pi^{-1}(b+1) < i+k+1$ (i.e. $\pi^{-1}(b+1) \leq i+k-1$). So the permutation π can be represented by

$$\pi = \cdots (b+1) \cdots \pi(i+k)\pi(i+k+1) \cdots b \cdots ,$$

where $\pi(i+k+1) > b+1$ and $\pi(i+k) < b+1$. Thus, π satisfies the second condition (b) of Definition 2.1. □

LEMMA 3.3. *Let $\pi \in S_n$. If $\mathcal{E}(\pi)$ has at most one white corner in each row, then π satisfies the first condition (1) of Definition 2.1.*

Proof. Suppose not, that is to say, there exist indices $1 \leq i < j < k < l \leq n$ such that $\pi(i)+1 = \pi(l), \pi(j) > \pi(l)$, and $\pi(k) < \pi(l)$. Choose the largest j_0 $(i < j_0 < l-1)$ such that $\pi(j_0) > \pi(l)$ and choose the smallest $k_0(j_0 < k_0 < l)$ such that $\pi(k_0) < \pi(l)$. It is clear that there exist two indices j_0 and k_0 and that they satisfy $k_0 = j_0 + 1$. Then there exits $(j_0, b_{j_0}) \in MID(\pi)$ since $\pi(j_0) > \pi(j_0 + 1)$. Here, b_{j_0} is the maximum value of $\pi(m)$, where $m > j_0$ and $\pi(m) < \pi(j_0)$. So by Proposition 1.4, $(j_0, b_{j_0}) \in \mathcal{E}(\pi)$.

Now we claim that there exists $(j_0, a) \in \mathcal{E}(\pi)$ such that $a < b_{j_0}$. Choose the largest number a such that $a < \pi(i)$, $\pi^{-1}(a) > j_0$, and $\pi^{-1}(a+1) < j_0$. We know that such a number a exists: Since $\pi(j_0+1) = \pi(k_0) < \pi(l) = \pi(i) + 1$ and $j_0 + 1 \neq i$, it means that $\pi(j_0 + 1) < \pi(i)$. So there exists a $\pi(m)$ such that $\pi(m) < \pi(i)$ and $m > j_0$. Also, since $\pi(i)(> \pi(m))$ is placed into the positions $\pi(1), \ldots, \pi(j_0 - 1)$, it means that the index of $\pi(m) + 1$ is less than j_0, there exists $\pi(m)$. Finally, it is shown that a has the following properties.

(1) $a < b_{j_0}$, since
  (i) $a < \pi(i)$,
  (ii) $\pi(i) < b_{j_0}$, since $\pi(j_0) > \pi(l) = \pi(i) + 1$ and Definition 1.2.

(2) $(j_0, a) \in \mathcal{E}(\pi)$, since
  (i) $\pi(j_0) > a$, since $\pi(j_0) > \pi(i) + 1 > \pi(i) > a$,
  (ii) $\pi^{-1}(a) > j_0$,
  (iii) $\pi(j_0 + 1) \leq a$, since $\pi(j_0 + 1) = \pi(k_0) < \pi(i)$ and the maximality of a,
  (iv) $\pi^{-1}(a + 1) \leq j_0$.

Thus (j_0, b_{j_0}) and (j_0, a) are in $\mathcal{E}(\pi)$ with $a < b_{j_0}$, however, this is a contradiction to the hypothesis. □

Proof of Proposition 2.8. Suppose that π is a Baxter permutation. Then by Theorem 2.7, $\mathcal{E}(\pi) = MID(\pi)$, so let $(i, b_i), (j, b_j) \in \mathcal{E}(\pi)$, without loss of generality, suppose that $1 \leq i < j < n$ and that $b_i = b_j = b$. Then $(i, b), (j, b) \in MID(\pi) = \mathcal{E}(\pi)$. By Lemma 3.2, π satisfies the second condition (2) of Definition 2.1, however, this is a contradiction to the assumption that π is a Baxter permutation. Thus $i = j$ and so $\mathcal{E}(\pi)$ has at most noe white corner in each row and column.

Conversely, suppose that the essential set $\mathcal{E}(\pi)$ has at most one white corner in each row and column. First, if there is at most one white corner in each row, then by Lemma 3.3, permutations π satisfy the following condition: for all indices $1 \leq i < j < k < l \leq n$,

$$\text{if } \pi(i) + 1 = \pi(l) \text{ and } \pi(j) > \pi(l) \text{ then } \pi(k) > \pi(l).$$

Second, if there is at most one white corner in each column, then by considering the inverse of π, π satisfies the following condition: for all indices $1 \leq i < j < k < l \leq n$,

$$\text{if } \pi(l) + 1 = \pi(i) \text{ and } \pi(k) > \pi(i) \text{ then } \pi(j) > \pi(i).$$

Thus π is a Baxter permutation. □

Proof of Theorem 2.9. Suppose that π is a c-pseudoBaxter permutation. By Proposition 1.4, it suffices to show that $\mathcal{E}(\pi) \subseteq MID(\pi)$. Let $(i, a) \in \mathcal{E}(\pi)$. By the definition of the essential set it can be written as

$$\pi(i) > a,\ \pi^{-1}(a) > i,\ \pi(i+1) \leq a, \text{ and } \pi^{-1}(a+1) \leq i.$$

Since $\pi(i) > a \geq \pi(i+1)$(i.e. $\pi(i) > \pi(i+1)$), there exists the pair $(i, b_i) \in MID(\pi)(\subset \mathcal{E}(\pi))$. Here, b_i is the maximum value of $\pi(m)$ where $m > i$ and $\pi(m) < \pi(i)$, so $b_i \geq a$. If we assume that $b_i > a$, then by Lemma 3.1, π satisfies the second condition (2) of Definition 2.2. However this is a contradiction to the first condition (1) of Definition 2.1. Thus $a = b_i$ and so $(i, a) \in MID(\pi)$. □

Proof of Proposition 2.10. Suppose that π is a c-pseudoBaxter permutation. Then by Theorem 2.9, $\mathcal{E}(\pi) = MID(\pi)$, so its essential set has at most one white corner in each row. However, if its essential set has also at most one white corner in each column then π is a Baxter permutation which is impossible. So its essential set has at least two white corners in some columns.

Conversely, first, if its essential set has at most one white corner in each row, then π satisfies the first condition (1) of Definition 2.1, by Lemma 3.3. Second, if its essential set has at least two white corners in some columns, then $(i, b), (j, b) \in \mathcal{E}(\pi)$ with $1 \leq i < j \leq n$. By Lemma 3.2, π satisfies the second condition (2) of Definition 2.1. Thus π is a c-pseudoBaxter permutation. □

From the above results we deduce the following corollaries.

Corollary 3.4. *A permutation is an r-pseudoBaxter permutation if and only if its essential set has at most one white corner in each column and at least two white corners in some rows.*

Corollary 3.5. *A permutation is an rc-pseudoBaxter permutation if and only if its essential set has at least two white corners in some rows and some columns.*

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Sook Min
Division of Software
Yonsei University
Wonju, Republic of Korea
E-mail: `sookmin@yonsei.ac.kr`