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APPLICATION OF GENERALIZED WEAK CONTRACTION IN INTEGRAL EQUATION

Amrish Handa

Abstract. This manuscript is divided into three segments. In the first segment, we prove a unique common fixed point theorem satisfying generalized weak contraction on partially ordered metric spaces and also give an example to support our results presented here. In the second segment of the article, some common coupled fixed point results are derived from our main results. In the last segment, we investigate the solution of integral equation as an application. Our results generalize, extend and improve several well-known results of the existing literature.

1. Introduction and Preliminaries

Weak contraction was first studied in partially ordered metric spaces by Harjani and Sadarangani [15]. In [4], Choudhury and Kundu established some coincidence point results for generalized weak contractions with discontinuous control functions on a partially ordered metric spaces. Choudhury et al. [5] proved coincidence point results by assuming a weak contraction inequality with three control functions, two of which are not continuous. The results are obtained under two sets of additional conditions. Hussain et al. [17] introduced the notion of generalized compatibility of a pair $\{F, G\}$, of mappings $F, G : X \times X \to X$, then the authors employed this notion to obtained coupled coincidence point results for such pair of mappings involving $(\varphi, \psi)-$contractive condition without mixed $g-$monotone property of F. Erhan et al. [12], claimed that the results established in Hussain et al. [17] can be easily derived from the coincidence point results in the literature. For more details one can consult [1, 6, 7, 8, 9, 10, 11, 12, 14, 16, 21].

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partially ordered metric spaces and also give an example to support our results presented here. In the second segment of the article, some common coupled fixed point results are derived from our main results. In the last segment, we investigate the solution of integral equation to demonstrate the fruitfulness of our results. Our results generalize, extend and improve the results of Choudhury and Kundu [4], Choudhury et al. [5], Harjani and Sadarangani [15] and several well-known results.

2. PRELIMINARIES

Definition 2.1 ([13]). Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a *coupled fixed point* of F if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y.$$

Definition 2.2 ([2]). Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all x, $y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y), \\ y_1,\ y_2 &\in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

Definition 2.3 ([19]). Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of the mappings F and G if

$$F(x,\ y) = Gx \text{ and } F(y,\ x) = Gy.$$

Definition 2.4 ([19]). Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *common coupled fixed point* of the mappings F and G if

$$x = F(x,\ y) = Gx \text{ and } y = F(y,\ x) = Gy.$$

Definition 2.5 ([19]). Mappings $F : X^2 \to X$ and $G : X \to X$ are said to be *commutative* if

$$GF(x,\ y) = F(Gx,\ Gy),\ \text{for all } (x,\ y) \in X^2.$$

Definition 2.6 ([19]). Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $G : X \to X$ are given mappings. We say that F has the *mixed $G-$monotone* *property* if for all $x,\ y \in X$, we have

$$\begin{aligned} x_1,\ x_2 \ &\in\ X,\ Gx_1 \preceq Gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),\\ y_1,\ y_2 \ &\in\ X,\ Gy_1 \preceq Gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

If G is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.7 ([3])**.** Mappings $F : X^2 \to X$ and $G : X \to X$ are said to be *compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(GF(x_n,\ y_n),\ F(Gx_n,\ Gy_n)) &= 0,\\ \lim_{n\to\infty} d(GF(y_n,\ x_n),\ F(Gy_n,\ Gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} Gx_n = x \in X,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} Gy_n = y \in X. \end{aligned}$$

Definition 2.8 ([17])**.** Suppose that $F,\ G : X^2 \to X$ are two mappings. F is said to be *G-increasing* with respect to $\preceq$ if for all $x,\ y,\ u,\ v \in X$, with $G(x,\ y) \preceq G(u,\ v)$ we have $F(x,\ y) \preceq F(u,\ v)$.

Definition 2.9 ([17])**.** Let $F,\ G : X^2 \to X$ be two mappings. We say that the pair $\{F,\ G\}$ is *commuting* if $F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x))$, for all $x,\ y \in X$.

Definition 2.10 ([17])**.** Suppose that $F,\ G : X^2 \to X$ are two mappings. An element $(x,\ y) \in X^2$ is called a *coupled coincidence point* of mappings F and G if $F(x,\ y) = G(x,\ y)$ and $F(y,\ x) = G(y,\ x)$.

Definition 2.11 ([17])**.** Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is *g-increasing* with respect to $\preceq$ if for any $x,\ y \in X$,

$$\begin{aligned} gx_1 &\preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),\\ gy_1 &\preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.12 ([17])**.** Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is *increasing* with respect to $\preceq$ if for any $x,\ y \in X$,

$$\begin{aligned} x_1 &\preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),\\ y_1 &\preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.13 ([17])**.** Let $F,\ G : X^2 \to X$ be two mappings. We say that the pair $\{F,\ S\}$ is *generalized compatible* if

$$\begin{aligned}\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0,\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x \in X,\\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y \in X.\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Definition 2.14 ([2, 12])**.** A partially ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$. A partially ordered metric space $(X,\ d,\ \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. A partially ordered metric space $(X,\ d,\ \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular. Let $F,\ G : X \to X$ be two mappings. We say that F is $(G,\ \preceq)$-*non-decreasing* if $Fx \preceq Fy$ for all $x,\ y \in X$ such that $Gx \preceq Gy$. If G is the identity mapping on X, we say that F is $\preceq$-*non-decreasing.*

Definition 2.15 ([5])**.** Two self-mappings G and F of a non-empty set X are said to be *commutative* if $GFx = FGx$ for all $x \in X$.

Definition 2.16 ([12])**.** Let $(X,\ d,\ \preceq)$ be a partially ordered metric space. Two mappings $F,\ G : X \to X$ are said to be $O-$*compatible* if

$$\lim_{n\to\infty} d(FGx_n,\ GFx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that $\{Gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n \in X.$$

Definition 2.17 ([18])**.** Two self-mappings G and F of a non-empty set X are said to be *weakly compatible* if they commute at their coincidence points, that is, if $Gx = Fx$ for some $x \in X$, then $GFx = FGx$.

Definition 2.18 ([9])**.** Let X be a non-empty set. Two mappings $F, G : X \times X \to X$ are called *generalized weakly compatible* if $F(x,\ y) = G(x,\ y)$, $F(y,\ x) = G(y,\ x)$ implies that $G(F(x,\ y),\ F(y,\ x)) = F(G(x,\ y),\ G(y,\ x))$, $G(F(y,\ x),\ F(x,\ y)) = F(G(y,\ x),\ G(x,\ y))$, for all $x,\ y \in X$. Obviously, a generalized compatible pair is generalized weakly compatible but converse is not true in general.

3. Fixed Point Results

In this section, we prove a unique common fixed point theorem for mappings α, $\beta : X \to X$ in a partially ordered metric space $(X,\ d,\ \preceq)$, where X is a non-empty set. For simplicity, we denote $\beta(x)$ by βx where $x \in X$.

Choudhury et al. [5] used the following classes of functions.

The class Ψ of all functions $\psi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(x) = 0 \Leftrightarrow x = 0$.
and Θ is the class of all functions $\theta : [0,\ +\infty) \to [0,\ +\infty)$ satisfying
(i_θ) θ is bounded on any bounded interval in $[0,\ +\infty)$,
(ii_θ) θ is continuous at 0 and $\theta(0) = 0$.

Theorem 3.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\beta(X) \subseteq \alpha(X)$ for which there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ such that*

$$\psi(x) \le \varphi(y) \Rightarrow x \le y, \tag{3.1}$$

for any sequence $\{x_n\}$ in $[0,\ +\infty)$ with $x_n \to x > 0$,

$$\psi(x) - \overline{\lim}\varphi(x_n) + \underline{\lim}\theta(x_n) > 0, \tag{3.2}$$

and

$$\psi(d(\alpha x,\ \alpha y)) \le \varphi(d(\beta x,\ \beta y)) - \theta(d(\beta x,\ \beta y)), \tag{3.3}$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Furthermore assume that, at least, one of the following conditions holds.

(a) $(X,\ d)$ is complete, α and β are continuous and the pair $(\alpha,\ \beta)$ is $O-$compatible,

(b) $(\beta(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(c) $(X,\ d)$ is complete, β is continuous and monotone non-decreasing, the pair $(\alpha,\ \beta)$ is $O-$compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.

Then α and β have a coincidence point. Furthermore, suppose that for every x, $y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Since $\alpha(X) \subseteq \beta(X)$, there exists $x_1 \in X$ such that $\alpha x_0 = \beta x_1$. Then $\beta x_0 \preceq \alpha x_0 = \beta x_1$. As α is $(\beta,\ \preceq)$-non-decreasing and so $\alpha x_0 \preceq \alpha x_1$. Continuing in this manner, we get a sequence $\{x_n\}_{n\geq 0}$ such that $\{\alpha x_n\}$ is $\preceq$-non-decreasing, $\beta x_{n+1} = \alpha x_n \preceq \alpha x_{n+1} = \beta x_{n+2}$ and

$$\beta x_{n+1} = \alpha x_n \text{ for all } n \geq 0. \tag{3.4}$$

Let $d_n = d(\beta x_n,\ \beta x_{n+1})$ for all $n \geq 0$. Since $\beta x_n \preceq \beta x_{n+1}$, by using the contractive condition (3.3) and (3.4), we have

$$\begin{aligned} \psi(d(\beta x_{n+1},\ \beta x_{n+2})) &= \psi(d(\alpha x_n,\ \alpha x_{n+1})) \\ &\leq \varphi(d(\beta x_n,\ \beta x_{n+1})) - \theta(d(\beta x_n,\ \beta x_{n+1})). \end{aligned}$$

Thus

$$\psi(d_{n+1}) \leq \varphi(d_n) - \theta(d_n), \tag{3.5}$$

which, by the fact that $\theta \geq 0$, implies $\psi(d_{n+1}) \leq \varphi(d_n)$, follows from (3.1) that $d_{n+1} \leq d_n$ for all positive integers $n \geq 0$, that is, $\{d_n\}$ is a monotone non-increasing sequence. Hence there exists a $d \geq 0$ such that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = d. \tag{3.6}$$

Taking the limit supremum on both sides of (3.5), using (3.6), the property (i_θ) of φ, θ and the continuity of ψ, we obtain

$$\psi(d) \leq \overline{\lim}\varphi(d_n) - \overline{\lim}\theta(d_n), \text{ that is, } \psi(d) - \overline{\lim}\varphi(d_n) + \underline{\lim}\theta(d_n) \leq 0,$$

which is a contradiction and so $d = 0$. Thus

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = 0. \tag{3.7}$$

We now prove that $\{\beta x_n\}_{n\geq 0}$ is a Cauchy sequence in X. If possible, suppose that $\{\beta x_n\}_{n\geq 0}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can find two subsequences such that for all positive integers k

$$d(\beta x_{n(k)},\ \beta x_{m(k)}) \geq \varepsilon \text{ for } n(k) > m(k) > k.$$

Let $n(k)$ be the smallest such positive integer, we get

$$d(\beta x_{n(k)-1},\ \beta x_{m(k)}) < \varepsilon.$$

Now

$$\begin{aligned}\varepsilon &\leq \omega_k = d(\beta x_{n(k)},\ \beta x_{m(k)}) \\ &\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + d(\beta x_{n(k)-1},\ \beta x_{m(k)}) \\ &\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (3.7), we have

$$\lim_{k\to\infty} \omega_k = \lim_{k\to\infty} d(\beta x_{n(k)},\ \beta x_{m(k)}) = \varepsilon. \tag{3.8}$$

By applying triangle inequality, we have

$$\begin{aligned}d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) &\leq d(\beta x_{n(k)+1},\ \beta x_{n(k)}) + d(\beta x_{n(k)},\ \beta x_{m(k)}) \\ &\quad + d(\beta x_{m(k)},\ \beta x_{m(k)+1}).\end{aligned}$$

On taking $k \to \infty$ in the above inequality, using (3.7) and (3.8), we have

$$\lim_{k\to\infty} d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) = \varepsilon. \tag{3.9}$$

As $n(k) > m(k)$, $\beta x_{n(k)} \succeq \beta x_{m(k)}$ and so by using the contractive condition (3.3) and (3.4), we have

$$\begin{aligned}\psi(d(\beta x_{n(k)+1},\ \beta x_{m(k)+1})) &= \psi(d(\alpha x_{n(k)},\ \alpha x_{m(k)})) \\ &\leq \varphi(d(\beta x_{n(k)},\ \beta x_{m(k)})) - \theta(d(\beta x_{n(k)},\ \beta x_{m(k)})).\end{aligned}$$

Taking the limit supremum on both sides of the above inequality and by using (3.8), (3.9), the property (i_θ) of φ and θ and the continuity of ψ, we get

$$\psi(\varepsilon) \leq \overline{\lim}\varphi(\omega_k) - \underline{\lim}\theta(\omega_k), \text{ that is, } \psi(\varepsilon) - \overline{\lim}\varphi(\omega_k) + \underline{\lim}\theta(\omega_k) \leq 0,$$

which is a contradiction. Consequently $\{\beta x_n\}_{n\geq 0}$ is a Cauchy sequence in X. We now prove that α and β have a coincidence point distinguishing between cases $(a) - (c)$.

First suppose that (a) holds, that is, $(X,\ d)$ is complete, α and β are continuous and the pair $(\alpha,\ \beta)$ is $O-$compatible. Since $(X,\ d)$ is complete, there exists $x \in X$ such that $\{\beta x_n\} \to x$, which, by (3.4), implies $\{\alpha x_n\} \to x$. As α and β are continuous, $\{\alpha\beta x_n\} \to \alpha x$ and $\{\beta\beta x_n\} \to \beta x$. Also, the pair (α, β) is $O-$compatible, it follows that

$$d(\beta x,\ \alpha x) = \lim_{n\to\infty} d(\beta\beta x_{n+1},\ \alpha\beta x_n) = \lim_{n\to\infty} d(\beta\alpha x_n,\ \alpha\beta x_n) = 0,$$

that is, x is a coincidence point of α and β.

Secondly suppose that (b) holds, that is, $(\beta(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{\beta x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete space $(\beta(X),\ d)$ and so there exists $y \in \beta(X)$ such that $\{\beta x_n\} \to y$. Let $x \in X$ be any point such that $y = \beta x$, then $\{\beta x_n\} \to \beta x$. Also, since $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{\beta x_n\}$ is $\preceq$-non-decreasing and converging to βx, we have $\beta x_n \preceq \beta x$ for all $n \geq 0$. Using the contractive condition (3.3), we have

$$\begin{aligned} \psi(d(\beta x_{n+1},\, \alpha x)) &= \psi(d(\alpha x_n,\, \alpha x)) \\ &\leq \varphi(d(\beta x_n,\, \beta x)) - \theta(d(\beta x_n,\, \beta x)). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (ii_θ) of θ, φ and the fact that $\{\beta x_n\} \to \beta x$, we get $\psi(d(\beta x,\, \alpha x)) = 0$, which implies, by (ii_ψ), that $d(\beta x,\, \alpha x) = 0$, that is, x is a coincidence point of α and β.

Finally suppose that (c) holds, that is, $(X,\, d)$ is complete, β is continuous and monotone non-decreasing, the pair $(\alpha,\, \beta)$ is $O-$compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular. As $(X,\, d)$ is complete and so there exists $x \in X$ such that $\{\beta x_n\} \to x$, which, by (3.4), implies $\{\alpha x_n\} \to x$. As β is continuous, $\{\beta\beta x_n\} \to \beta x$ and since the pair $(\alpha,\, \beta)$ is $O-$compatible, therefore we have $\lim_{n\to\infty} d(\beta\beta x_{n+1}$, $\alpha\beta x_n) = \lim_{n\to\infty} d(\beta\alpha x_n,\, \alpha\beta x_n) = 0$, which suggest that $\{\alpha\beta x_n\} \to \beta x$.

Since $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{\beta x_n\}$ is $\preceq$-non-decreasing and converging to x, we have $\beta x_n \preceq x$, which, by the monotonicity of β, implies $\beta\beta x_n \preceq \beta x$. Using the contractive condition (3.3), we have

$$\psi(d(\alpha\beta x_n,\, \alpha x)) \leq \varphi(d(\beta\beta x_n,\, \beta x)) - \theta(d(\beta\beta x_n,\, \beta x)).$$

On taking limit $n \to \infty$ in the above inequality, by using (ii_θ) of θ, φ and the fact that $\{\beta\beta x_n\} \to \beta x$ and $\{\alpha\beta x_n\} \to \beta x$, we get $\psi(d(\beta x,\, \alpha x)) = 0$, which implies, by (ii_ψ), that $d(\beta x,\, \alpha x) = 0$, that is, x is a coincidence point of α and β.

Consequently the set of coincidence points of α and β is non-empty. Let x and y be two coincidence points of α and β, that is, $\alpha x = \beta x$ and $\alpha y = \beta y$. Now, we claim that $\beta x = \beta y$. By the assumption, there exists $z \in X$ such that αz is comparable with αx and αy. Put $z_0 = z$ and choose $z_1 \in X$ so that $\beta z_0 = \alpha z_1$. Then, we can inductively define the sequence $\{\beta z_n\}$ where $\beta z_{n+1} = \alpha z_n$ for all $n \geq 0$. Hence $\alpha x = \beta x$ and $\alpha z = \alpha z_0 = \beta z_1$ are comparable. One can easily get that $\beta z_n \preceq \beta x$ for each $n \geq 0$.

Let $e_n = d(\beta x,\, \beta z_n)$ for all $n \geq 0$. As $\beta z_n \preceq \beta x$ and so by using the contractive condition (3.3) and (3.4), we have

$$\psi(d(\beta x,\, \beta z_{n+1})) = \psi(d(\alpha x,\, \alpha z_n)) \leq \varphi(d(\beta x,\, \beta z_n)) - \theta(d(\beta x,\, \beta z_n)),$$

which implies that

$$\psi(e_{n+1}) \le \varphi(e_n) - \theta(e_n), \tag{3.10}$$

which, by the fact that $\theta \ge 0$, implies $\psi(e_{n+1}) \le \varphi(e_n)$ and so (3.1) follows that $e_{n+1} \le e_n$ for all $n \ge 0$. Thus $\{e_n\}$ is a monotone non-increasing sequence. Hence there exists an $e \ge 0$ such that

$$\lim_{n\to\infty} e_n = d(\beta x,\ \beta z_n) = e. \tag{3.11}$$

Taking the limit supremum on both sides of (3.10), using (3.11), the property of φ, θ and the continuity of ψ, we obtain

$$\psi(e) \le \overline{\lim}\varphi(e_n) - \underline{\lim}\theta(e_n),\ \text{that is,}\ \psi(e) - \overline{\lim}\varphi(e_n) + \underline{\lim}\theta(e_n) \le 0,$$

which contradicts (3.2). Hence $e = 0$ and so we must have

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} d(\beta x,\ \beta z_n) = 0. \tag{3.12}$$

Similarly, one can obtain that

$$\lim_{n\to\infty} d(\beta y,\ \beta z_n) = 0. \tag{3.13}$$

Hence, by (3.12) and (3.13), we get

$$\beta x = \beta y. \tag{3.14}$$

Since $\alpha x = \beta x$, by weak compatibility of α and β, we have $\beta\beta x = \beta\alpha x = \alpha\beta x$. Let $u = \beta x$, then $\beta u = \alpha u$, that is, u is a coincidence point of α and β. Then from (3.14) with $y = u$, it follows that $\beta x = \beta u$, that is, $u = \beta u = \alpha u$. Hence u is a common fixed point of α and β. To prove the uniqueness, assume that v is another common fixed point of α and β. Then by (3.14) we have $v = \beta v = \beta u = u$, that is, the common fixed point of α and β is unique. □

If we take $\psi = I$ (the identity mapping) and $\theta(x) = 0$ for all $x \ge 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ for which there exists some $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ +\infty)$ with $x_n \to x > 0$,*

$$\overline{\lim}\varphi(x_n) < x,$$

and

$$d(\alpha x,\ \alpha y) \le \varphi(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Furthermore, suppose that for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

If we take $\theta(x) = 0$ and $\varphi(x) = k\psi(x)$ with $0 \leq k < 1$ and for all $x \geq 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ for which there exist some $\psi \in \Psi$ and $k \in [0,\ 1)$ such that*

$$\psi(d(\alpha x,\ \alpha y)) \leq k\psi(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Furthermore, suppose that for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Taking $\varphi = \psi$ in Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ for which there exist some $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ +\infty)$ with $x_n \to x > 0$,*

$$\overline{\lim}\theta(x_n) > 0,$$

and

$$\psi(d(\alpha x,\ \alpha y)) \leq \psi(d(\beta x,\ \beta y)) - \theta(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Furthermore, suppose that for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

If we take $\psi = \varphi = I$ (the identity mappings) in Theorem 3.1, we have the following corollary.

Corollary 3.5. *Let $(X, d, \preceq)$ be a partially ordered metric space and $\alpha, \beta : X \to X$ be two mappings such that α is $(\beta, \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists some $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, +\infty)$ with $x_n \to x > 0$,*

$$\overline{\lim}\theta(x_n) > 0,$$

and

$$d(\alpha x,\ \alpha y) \leq d(\beta x,\ \beta y) - \theta(d(\beta x,\ \beta y)),$$

for all $x, y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Furthermore, suppose that for every $x, y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair (α, β) is weakly compatible. Then α and β have a unique common fixed point.

If we take $\psi = \varphi = I$ (the identity mapping) and $\theta(t) = (1-k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.6. *Let $(X, d, \preceq)$ be a partially ordered metric space and $\alpha, \beta : X \to X$ be two mappings such that α is $(\beta, \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists $k \in [0, 1)$ such that*

$$d(\alpha x,\ \alpha y) \leq kd(\beta x,\ \beta y),$$

for all $x, y \in X$ such that $\beta x \preceq \beta y$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Furthermore, suppose that for every $x, y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy, and also the pair (α, β) is weakly compatible. Then α and β have a unique common fixed point.

Example 3.1. Let $X = [0, 1]$ be equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$. Let $\alpha, \beta : X \to X$ be defined as

$$\alpha x = \frac{x^2}{3} \text{ and } \beta x = x^2, \text{ for all } x \in X.$$

Define $\varphi, \psi, \theta : [0, +\infty) \to [0, +\infty)$ as follows

$$\psi(t) = t^2,\ \varphi(t) = \begin{cases} \frac{1}{3}[t]^2, \text{ if } 3 < t < 4, \\ \frac{1}{9}t^2, \text{ otherwise,} \end{cases} \text{ and } \theta(t) = \begin{cases} \frac{1}{9}[t]^2, \text{ if } 3 < t < 4, \\ 0, \text{ otherwise.} \end{cases}$$

Then ψ, φ and θ have all the required properties. One can easily see that the contractive condition of Theorem 3.1 is satisfied for all $x, y \in X$. Furthermore, all the other conditions of Theorem 3.1 are also satisfied and $u = 0$ is a unique common fixed point of α and β.

4. Coupled Fixed Point Results

In this section, we derive two dimensional version of Theorem 3.1 for mappings $F, G : X^2 \to X^2$ in a partially ordered metric space $(X^2, \Delta_2, \sqsubseteq)$, where X is a non-empty set, with the help of the results established in the previous section. For given $n \in \mathbb{N}$ where $n \geq 2$, X^n denote the n^{th} Cartesian product $X \times X \times ... \times X$ (n times). For this, we shall consider the partially ordered metric space $(X^2, \Delta_2, \sqsubseteq)$, where $\sqsubseteq$ was introduced by

$$W \sqsubseteq V \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } W = (u, v), V = (x, y) \in X^2.$$

Let $F, G : X^2 \to X$ be two mappings. Define the mappings $\Phi_F, \Phi_G : X^2 \to X^2$, for all $V = (x, y) \in X^2$, by

$$\Phi_F(V) = (F(x, y), F(y, x)) \text{ and } \Phi_G(V) = (G(x, y), G(y, x)).$$

Definition 4.1 ([1]). Let (X, d) be a metric space. Define $\Delta_n : X^n \times X^n \to [0, +\infty)$, for $A = (a_1, a_2, ..., a_n)$, $B = (b_1, b_2, ..., b_n) \in X^n$, by

$$\Delta_n(A, B) = \frac{1}{n}\sum_{i=1}^{n} d(a_i, b_i).$$

Then Δ_n is metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) is complete.

Lemma 4.1 ([6, 8]). *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose F, $G : X^2 \to X$ and $\Phi_F, \Phi_G : X^2 \to X^2$ are mappings. Then*

(1) If $(X, d, \preceq)$ is regular, then $(X^2, \Delta_2, \sqsubseteq)$ is also regular.

(2) If F is d-continuous, then Φ_F is Δ_2-continuous.

(3) If F is G-increasing with respect to $\preceq$, then Φ_F is $(\Phi_G, \sqsubseteq)$-non-decreasing.

(4) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $V_0 = (x_0, y_0) \in X^2$ such that $\Phi_G(V_0) \sqsubseteq \Phi_F(V_0)$.

(5) For any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $\Phi_F(X^2) \subseteq \Phi_G(X^2)$.

(6) If the pair $\{F, G\}$ is generalized compatible, then the mappings Φ_F and Φ_G are $O-$compatible in $(X^2, \Delta_2, \sqsubseteq)$.

(7) A point $V = (x,\ y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of Φ_F and Φ_G.

Theorem 4.1. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings for which there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ satisfying (3.1), (3.2) and*

$$\begin{aligned} \psi&\left(\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}\right) \\ \leq\ \varphi&\left(\frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2}\right) \\ &-\theta\left(\frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2}\right), \end{aligned} \tag{4.1}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$ such that F is G-increasing with respect to $\preceq$, G is continuous and monotone non-decreasing and there exist two elements $x_0,\ y_0 \in X$ with

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u).$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point. Furthermore, suppose that for every $(x,\ y),\ (z,\ w) \in X^2$, there exists a point $(u,\ v) \in X^2$ such that $(F(u,\ v),\ F(v,\ u))$ is comparable to $(F(x,\ y),\ F(y,\ x))$ and $(F(z,\ w),\ F(w,\ z))$ and also the pair $(F,\ G)$ is weakly compatible. Then F and G have a unique common coupled fixed point.

Proof. One can easily obtain that the contractive condition (4.1) imply that,

$$\psi(\Delta_2(\Phi_F(V),\ \Phi_F(W))) \leq \varphi(\Delta_2(\Phi_G(V),\ \Phi_G(W))) - \theta(\Delta_2(\Phi_G(V),\ \Phi_G(W)),$$

for all $V = (x, y), W = (u, v) \in X^2$ with $\Phi_G(V) \sqsubseteq \Phi_G(W)$. Thus it is only necessary to utilize Theorem 3.1 to the mappings $\alpha = \Phi_F$ and $\beta = \Phi_G$ in the partially ordered metric space $(X^2,\ \Delta_2,\ \sqsubseteq)$ taking into account all items of Lemma 4.1. □

Now, we deduce result without mixed monotone property of F.

Corollary 4.2. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose $F : X^2 \to X$ is an increasing mapping with respect to $\preceq$ for which there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ satisfying* (3.1), (3.2) *and*

$$\begin{aligned}(4.2)\qquad & \psi\left(\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}\right) \\ \leq\ & \varphi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right) - \theta\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right),\end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

In a similar way, we may state the results analog of Corollary 3.2, Corollary 3.3, Corollary 3.4, Corollary 3.5 and Corollary 3.6 for Theorem 4.1, and Corollary 4.2.

5. Application to Integral Equations

In the last segment, we investigate the solution of a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$(5.1)\qquad x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))[f(q,\ x(q)) + g(q,\ x(q))]dq + h(p),$$

for all $p \in I = [a,\ b]$.

Let Ω denote the set of all functions $\zeta : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_ζ) ζ is non-decreasing,

(ii_ζ) $\zeta(p) \leq \frac{1}{2}p$.

Definition 4.1 ([20]). A pair $(\alpha,\ \beta) \in X^2$ with $X = C(I, \mathbb{R})$, where $C(I, \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a *coupled lower-upper solution* of equation (5.1) if, for all $p \in I$,

$$\begin{aligned}\alpha(p)\ \leq\ & \int_a^b K_1(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq + h(p)\end{aligned}$$

and

$$\begin{aligned}\beta(p) \quad \geq \quad & \int_a^b K_1(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq + h(p).\end{aligned}$$

Theorem 5.1. *Consider the integral equation* (5.1) *with* $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, f, $g \in C(I \times \mathbb{R},\ \mathbb{R})$ *and* $h \in C(I,\ \mathbb{R})$ *satisfying the following conditions:*

(*i*) $K_1(p,\ q) \geq 0$ *and* $K_2(p,\ q) \geq 0$ *for all* $p,\ q \in I$.

(*ii*) *There exist positive numbers* λ, μ *and* $\zeta \in \Omega$ *such that for all* $x,\ y \in \mathbb{R}$ *with* $x \succeq y$, *the following conditions hold:*

$$0 \quad \leq \quad f(q,\ x) - f(q,\ y) \leq \lambda\zeta(x - y), \tag{5.2}$$

$$0 \quad \leq \quad g(q,\ x) - g(q,\ y) \leq \mu\zeta(x - y). \tag{5.3}$$

(*iii*)

$$(\lambda + \mu) \sup_{p \in I} \int_a^b (K_1(p,\ q) + K_2(p,\ q))dq \leq \frac{1}{2}. \tag{5.4}$$

Suppose that there exists a coupled lower-upper solution $(\alpha,\ \beta)$ *of* (5.1). *Then the integral equation* (5.1) *has a solution in* $C(I,\ \mathbb{R})$.

Proof. Consider $X = C(I,\ \mathbb{R})$, the natural partial order relation, that is, for x, $y \in X$,

$$x \preceq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I.$$

Notice that X is a regular complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} |x(p) - y(p)|\,.$$

Now consider on X^2 the following partial order: for $(x,\ y)$, $(u,\ v) \in X^2$,

$$(x,\ y) \sqsubseteq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \text{for all } p \in I.$$

Define φ, ψ, $\theta : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$\psi(t) = t^2,\ \varphi(t) = \begin{cases} [t]^2, \text{ if } 3 < t < 4, \\ \frac{1}{4}t^2, \text{ otherwise,} \end{cases} \quad \text{and } \theta(t) = \begin{cases} \frac{1}{4}[t]^2, \text{ if } 3 < t < 4, \\ 0, \text{ otherwise,} \end{cases}$$

and the mapping $F : X^2 \to X$ by

$$\begin{aligned} F(x,\ y)(p) &= \int_a^b K_1(p,\ q)[f(q,\ x(q)) + g(q,\ y(q))]dq \\ &\quad + \int_a^b K_2(p,\ q)[f(q,\ y(q)) + g(q,\ x(q))]dq + h(p), \end{aligned}$$

for all $p \in I$. It is easy to prove, like in [17], that F is increasing. Now for x, y, u, $v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = & \int_a^b K_1(p,\ q)[(f(q,\ x(q)) - f(q,\ u(q))) + (g(q,\ y(q)) - g(q,\ v(q)))]dq \\ & + \int_a^b K_2(p,q)[(f(q,\ y(q)) - f(q,\ v(q))) + (g(q,\ x(q)) - g(q,\ u(q)))]dq. \end{aligned}$$

Thus, by using (5.2) and (5.3), we get

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\zeta\left(x(q) - u(q)\right) + \mu\zeta\left(y(q) - v(q)\right)\right] dq \\ & + \int_a^b K_2(p,\ q)\left[\lambda\zeta\left(y(q) - v(q)\right) + \mu\zeta\left(x(q) - u(q)\right)\right] dq. \end{aligned} \tag{5.5}$$

Now, by the monotonicity of ζ, we have

$$\begin{aligned} \zeta\left(x(q) - u(q)\right) &\leq \zeta(\sup_{q\in I} |x(q) - u(q)|) = \zeta(d(x,\ u)), \\ \zeta\left(y(q) - v(q)\right) &\leq \zeta(\sup_{q\in I} |y(q) - v(q)|) = \zeta(d(y,\ v)). \end{aligned}$$

Hence, by (5.5), we found that

$$\begin{aligned} & |F(x,\ y)(p) - F(u,\ v)(p)| \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\zeta(d(x,\ u)) + \mu\zeta(d(y,\ v))\right] dq \\ & + \int_a^b K_2(p,\ q)\left[\lambda\zeta(d(y,\ v)) + \mu\zeta(d(x,\ u))\right] dq, \end{aligned} \tag{5.6}$$

as all the quantities on the right hand side of (5.5) are non-negative. Similarly one can get that

$$
\begin{aligned}
(5.7) \qquad & |F(y,\ x)(p) - F(v,\ u)(p)| \\
\leq\ & \int_a^b K_1(p,\ q)[\lambda\zeta(d(x,\ u)) + \mu\zeta(d(y,\ v))]dq \\
& + \int_a^b K_2(p,\ q)[\lambda\zeta(d(y,\ v)) + \mu\zeta(d(x,\ u))]dq,
\end{aligned}
$$

By summing up (5.6) and (5.7), dividing by 2 and then taking supremum with respect to p, by using (5.4), we get

$$
\begin{aligned}
& \frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} \\
\leq\ & (\lambda+\mu)\sup_{p\in I}\int_a^b (K_1(p,\ q) + K_2(p,\ q))dq.\frac{\zeta(d(x,\ u)) + \zeta(d(y,\ v))}{2} \\
\leq\ & \frac{\zeta(d(x,\ u)) + \zeta(d(y,\ v))}{4}.
\end{aligned}
$$

By the monotonicity of ζ, we have

$$
\begin{aligned}
\zeta(d(x,\ u)) &\leq \zeta(d(x,\ u) + d(y,\ v)), \\
\zeta(d(y,\ v)) &\leq \zeta(d(x,\ u) + d(y,\ v)),
\end{aligned}
$$

which, by (ii_ζ), implies

$$
\begin{aligned}
\frac{\zeta(d(x,\ u)) + \zeta(d(y,\ v))}{4} &\leq \frac{1}{2}\zeta(d(x,\ u) + d(y,\ v)) \\
&\leq \frac{d(x,\ u) + d(y,\ v)}{4}.
\end{aligned}
$$

It follows that

$$
(5.8) \qquad \frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} \leq \frac{1}{2}\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right).
$$

Thus, by (5.8), we have

$$
\begin{aligned}
& \psi\left(\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}\right) \\
=\ & \left(\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}\right)^2 \\
\leq\ & \frac{1}{4}\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right)^2 \\
\leq\ & \varphi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right) - \theta\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right),
\end{aligned}
$$

which is the contractive condition of Corollary 4.2. Again, let $(\alpha, \beta) \in X^2$ be a coupled upper-lower solution of integral equation (5.1), then one can have $\alpha(p) \leq F(\alpha, \beta)(p)$ and $\beta(p) \geq F(\beta, \alpha)(p)$, for all $p \in I$. This demonstrate that all hypothesis of Corollary 4.2 are satisfied. Hence F has a coupled fixed point $(x, y) \in X^2$ which is the solution in $X = C(I, \mathbb{R})$ of the integral equation (5.1). □

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Professor: Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M. P.), India
Email address: amrishhanda83@gmail.com