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ON s-TOPOLOGICAL d-ALGEBRAS

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Abstract. The aim of this paper is to study the concept of s-topological d-algebras which is a d-algebra supplied with a certain type of topology that makes the binary operation defined on it d-topologically continuous. This concept is a generalization of the concept of topological d-algebra. We obtain several properties of s-topological d-algebras.

1. Introduction

Among the important subjects of pure mathematics are topology and algebra. Topology studies continuity, convergence, and so on, while algebra studies all types of operations and provides a basis for calculations and algorithms. The basic principle describing the relation between topology and algebraic operations is to make these operations topologically continuous, maybe in the first or second variable or in jointly continuous which is defined as Topological Algebra. In recent years, several researchers have contributed to the development of this subject. Algebras and topology, are two fundamental subjects of pure mathematics. From the beginning of twentieth century many mathematicians have contributed to the development of this subject. After Y. Imai and K. Iseki [1] gave an axiom system of propositional calculus in 1966 and in the same year K. Iseki [2] gave an algebraic formulation for the BCK-propositional calculus system, several mathematicians have been written on the concept of BCK-algebras and found many of the algebraic properties of the BCK-algebras. In 1999, J. Neggers and H. S. Kim introduced and studied the concept of d-algebrs in [5]. In [8], N. Nagamani and N. Kandara studied the topological aspects of the d-structure. In this paper, we study d-algebras equipped with certain

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topologies in which the operation of the structure satisfied a certain type of continuity, we name this d-algebra joined with such topologies by s-topological d-algebra. It is proved that every topological d-algebra is s-topological d-algebra. Further, many topological properties of a d-algebra were found.

2. PRELIMINARIES

For the development of this paper, we give necessary definitions and properties of a d-algebra and investigate the concept of a topological d-algebra.

Definition 2.1 ([5]). A d-algebra is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms: for every $x, y \in X$,

(1) $x * x = 0$,
(2) $0 * x = 0$,
(3) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Definition 2.2 ([2]). By a BCK-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in X$,

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$,
(5) $0 * x = 0$.

Remark 1 ([5]). Every BCK-algebra is a d-algebra.

Definition 2.3. A subset F of a d-algebra X is called a *filter* of X if it satisfy the following properties:

(1) the constant $0 \in F$, and
(2) for all $x, y \in X$, if $x * y \in F$, $x \in F$, then $y \in F$.

Definition 2.4 ([6]). A subset I of a d-algebra X is called an *ideal* of X if it satisfies the following properties:

(1) the constant $0 \in I$,
(2) for all $x, y, z \in X$. If $x \star (y \star z) \in I$, $y \in I$, then $x \star z \in I$.

From the definition of d-algebras we can get the following properties.

Definition 2.5 ([8]). A d-algebra X equipped with a topology τ is called a *topological d-algebra* (for short *td-algebra*) if $f : X \times X \to X$ defined by $f(x, y) = x * y$ is continuous for all $(x, y) \in X \times X$ where $X \times X$ has the product topology. Equivalently, if for each open set O containing $x * y$, there exist open sets U and V containing x and y respectively such that $U * V \subseteq O$.

Definition 2.6 ([8]). Let X be a d-algebra, and $a \in X$. A left map $L_a : X \to X$ defined by, $L_a(x) = a*x$, for all $x \in X$ and a right map $R_a : X \to X$ by $R_a(x) = x*a$ for all $x \in X$. We denote $L(X)$ to be the family of all L_a for all $a \in X$.

Definition 2.7 ([8]). A d-algebra X is called a *positive implicative d-algebra*, if $(y * x) * (z * x) = (y * z) * x$ for all $x, y, z \in X$.

Definition 2.8 ([4]). For a subset A of a topological space (X, τ), we say that A is *regular open* [9] if $A = \mathrm{Int}(\mathrm{Cl}(A))$ and it is *semi-open* [4] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called a *semi-closed*. The closure, interior, semi-closure and semi-interior of A are denoted, respectively by $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$, $s\,\mathrm{Cl}(A)$ and $s\,\mathrm{Int}(A)$.

Definition 2.9 ([3]). A BCK-algebra X equipped with a topology τ is called an *s-topological BCK-algebra* if the function $f : X \times X \to X$ defined by $f(x, y) = x * y$ has the property that for each open set O containing $x * y$, there exists an open set U containing x and a semi-open set V containing y such that $U * V \subseteq O$ for all $x, y \in X$.

Remark 2. For a topological space (X, τ), we set the following
$\tau(x) = \{U : x \in U \textit{ and } U \in \tau\}$
$SO(x) = \{U : x \in U \textit{ and } \mathrm{U} \textit{ is semiopen in } \mathrm{X}\}$

3. Properties of s-topological d-algebras

In this section, we introduce the concept of s-topological d-algebras and establish some of their properties.

Definition 3.1. A d-algebra X equipped with a topology τ is called an *s-topological d-algebra* (*sd-algebra*, for short) if the function $f : X \times X \to X$ defined by $f(x, y) = x*y$ has the property that for each $O \in \tau(x*y)$, there exist $U \in \tau(x)$ and $V \in SO(y)$ such that $U * V \subseteq O$ for all $x, y \in X$.

Lemma 3.2. *A d-algebra $(X, f, 0)$, (f stands for the operation $*$), equipped with a topology τ is a sd-algebra if $f^{-1}(O)$ is a semi-open set in $X \times X$ for each $O \in \tau(x*y)$ and all $x, y \in X$.*

Example 1. It is obvious that every *td*-algebra is *sd*-algebra but not conversely. Consider a d-algebra $X = \{0, a, b, c\}$ with the following following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Let $\tau = P(X\backslash\{0\}) \cup X$. Then X is *sd*-algebra which is not *td*-algebra.

Proposition 3.3. *For any subset A of an sd-algebra X and any element $x \in X$, the following statements are true:*

(1) $\mathrm{Cl}(A) * x \subset \mathrm{Cl}(A * x)$.
(2) *If* $\mathrm{Cl}(A) * x$ *is closed, then* $\mathrm{Cl}(A) * x = \mathrm{Cl}(A * x)$.

Proof. (1). Let $y \in \mathrm{Cl}(A) * x$ and $U \in \tau(y)$. So $y = a * x$ where $a \in \mathrm{Cl}(A)$. Since X is *sd*-algebra, there exist $V \in \tau(a)$ and $G \in SO(x)$ such that $V * G \subseteq U$. Also we have $a \in \mathrm{Cl}(A)$ implies that $A \cap V \neq \emptyset$. Suppose that $b \in A \cap V$, so $b * x \in A * x$ and $b * x \in V * x \subseteq V * G \subseteq U$. Hence $y \in \mathrm{Cl}(A * x)$.
(2). Suppose that $\mathrm{Cl}(A) * x$ is closed and $y \in \mathrm{Cl}(A * x)$. If $y \notin \mathrm{Cl}(A) * x$, then $y \in X\backslash(\mathrm{Cl}(A) * x)$, which is an open set. Hence $A * x \subseteq \mathrm{Cl}(A) * x$. Then we have $A * x \cap X\backslash(\mathrm{Cl}(A) * x) = \emptyset$, a contradiction. $\square$

The following example shows that the equality in (1) of Proposition 3.3 is not true and $\mathrm{Cl}(A) * x$ is not closed.

Example 2. Consider a d-algebra $X = \{0, a, b, c\}$ with the following following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Consider the topology τ on X defined as: $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Then X is *sd*-algebra. If $A = \{0, a\}$, then $\mathrm{Cl}(A)*b = \{0\}$, which is not closed and $\mathrm{Cl}(A*b) = \{0, a\}$, so $\mathrm{Cl}(A) * b \neq \mathrm{Cl}(A * b)$.

Proposition 3.4. *For any subset A of an sd-algebra X and any element $x \in X$, the following statements are true:*

(1) $x * s\,\mathrm{Cl}(A) \subset \mathrm{Cl}(x * A)$.
(2) *If* $x * s\,\mathrm{Cl}(A)$ *is closed, then* $x * s\,\mathrm{Cl}(A) = \mathrm{Cl}(x * A)$.
(3) *In general, the equality in (1) is not true and* $x * s\,\mathrm{Cl}(A)$ *is not closed.*

Proof. (1). Let $y \in x * s\,\mathrm{Cl}(A)$ and $U \in \tau(y)$. So $y = x * a$, where $a \in s\,\mathrm{Cl}(A)$. Since X is *sd*-algebra, there exist $V \in \tau(x)$ and $G \in SO(a)$ such that $V * G \subseteq U$. Also $a \in s\,\mathrm{Cl}(A)$ implies $A \cap G \neq \emptyset$. Suppose that $b \in A \cap G$, so $x * b \in x * A$ and $x * b \in x * G \subseteq V * G \subseteq U$. Hence $y \in \mathrm{Cl}(A * x)$.
(2). Suppose that $x * s\,\mathrm{Cl}(A)$ is closed and $y \in \mathrm{Cl}(x * A)$. If $y \notin \mathrm{Cl}(x * A)$, then $y \in X\backslash(x * s\,\mathrm{Cl}(A) \in \tau$. Then $x * A \subseteq x * s\,\mathrm{Cl}(A)$, so we get $(x * A) \cap X\backslash(x * s\,\mathrm{Cl}(A)) = \emptyset$, a contradiction.
(3). In Example 2, if $A = \{0, a\}$, then $b * s\,\mathrm{Cl}(A) = \{b\}$, which is not closed and $\mathrm{Cl}(b * A) = \{0, a, b\}$, so $b * s\,\mathrm{Cl}(A) \neq \mathrm{Cl}(b * A)$. □

Corollary 3.5. *For any subset A of an sd-algebra X and any element $x \in X$, the following statements are true:*

(1) *If* $A * x$ *is closed, then* $\mathrm{Cl}(A) * x = A * x$.
(2) *If* $x * A$ *is closed, then* $x * s\,\mathrm{Cl}(A) = x * A$.

Proof. Follows from Propositions 3.3 and 3.4. □

Proposition 3.6. *For any subsets A and B of an sd-algebra X, the following statements are true:*

(1) $\mathrm{Cl}(A) * s\,\mathrm{Cl}(B) \subseteq \mathrm{Cl}(A * B)$.
(2) *If* $\mathrm{Cl}(A) * s\,\mathrm{Cl}(B)$ *is closed, then* $\mathrm{Cl}(A) * s\,\mathrm{Cl}(B) = \mathrm{Cl}(A * B)$.

Proof. (1). Let $x \in \mathrm{Cl}(A) * s\,\mathrm{Cl}(B)$ and $U \in \tau(x)$. So $x = a * b$, where $a \in \mathrm{Cl}(A)$ and $b \in s\,\mathrm{Cl}(B)$. Since X is *sd*-algebra, there exist $V \in \tau(a)$ and $G \in SO(b)$ such that $V * G \subseteq U$. Also $a \in \mathrm{Cl}(A)$ and $b \in s\,\mathrm{Cl}(B)$ implies that $A \cap V \neq \emptyset$ and $B \cap G \neq \emptyset$. Suppose that $a_1 \in A \cap V$ and $b_1 \in B \cap G$, so $a_1 * b_1 \in A * B$ and $a_1 * b_1 \in V * G \subseteq U$. Hence $x \in \mathrm{Cl}(A * B)$.
(2). Suppose that $\mathrm{Cl}(A) * s\,\mathrm{Cl}(B)$ is closed and let $x \in \mathrm{Cl}(A * B)$. If $x \notin \mathrm{Cl}(A) *$

$s\,\mathrm{Cl}(B)$, then $x \in X \backslash (\mathrm{Cl}(A) * s\,\mathrm{Cl}(B)) \in \tau$. Then $A * B \subseteq \mathrm{Cl}(A) * s\,\mathrm{Cl}(B)$, so we get $(A * B) \cap X \backslash (\mathrm{Cl}(A) * s\,\mathrm{Cl}(B)) = \emptyset$, a contradiction. $\square$

Proposition 3.7. *In an sd-algebra X, if $\{0\}$ is open, then X is discrete.*

Proof. Suppose that $\{0\} \in \tau$ and let $x \in X$. Since $x * x = 0$ for all $x \in X$ and X is *sd*-algebra, there exist $U \in \tau(x)$ and $G \in SO(x)$ such that $U * G \subseteq \{0\}$. Hence $V = U \cap G \in SO(x)$. If V contains any other point y, then $x * y = 0$ and $y * x = 0$, a contradiction. Hence $V \in SO(x)$. Then $\{x\} \in \tau$. Hence X is discrete. $\square$

Definition 3.8 ([7])**.** A topological space (X, τ) is called

(1) *semi-T_1* if for each two distinct points $x, y \in X$ there two semi-open sets U and V such that U containing x but not y and V containing y but not x.

(2) *semi-T_2* if for each two distinct points $x, y \in X$ there two disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.

Proposition 3.9. *In an sd-algebra X, if $\{0\}$ is closed, then X is semi-T_2.*

Proof. Suppose that $\{0\}$ is closed and $x, y \in X$. Then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $y * x \neq 0$. Then there exist $V \in \tau(y)$ and $G \in SO(x)$ such that $V * G \subseteq X \backslash \{0\}$. Hence $V \in SO(x)$ and $G \in SO(y)$ such that $V \cap G = \emptyset$. Hence X is semi-T_2. $\square$

Proposition 3.10. *If the sd-algebra $(X, *, \tau)$ is T_0, then it is semi-T_1.*

Proof. Let $x, y \in X$ and $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $x * y \neq 0$. Since X is T_0, there exists an open set W containing one of them but not the other. Suppose that $W \in \tau(x*y)$ and $0 \notin W$. Since $(X, *, \tau)$ is *sd*-algebra, there exist $U \in \tau(x)$ and $V \in SO(y)$ such that $U * V \subseteq W$. Then $U \in SO(x), V \in SO(y)$ such that $U \cap V = \emptyset$. If $0 \in W$ and $x * y \notin W$. Then $x * x = 0 \in W$, so there s an open set U containing x and a semi-open set V containing x such that $U * V \in W$, and $y * y = 0 \subseteq W$, there s an open set U_1 containing y and a semi-open set V_1 containing y such that $U_1 * V_1 \subseteq W$. Therefore, $G = U \cap V$ and $H = U_1 \cap V_1$ are two semi-open sets containing x and y respectively. It is clear that $y \notin G$ and $x \notin H$. Hence $(X, *, \tau)$ is a semi-T_1 space. $\square$

Definition 3.11 ([6])**.** A non-empty subset A of a d-algebra X is called a *d-subalgebra* of X if $x * y \in A$ $\forall x, y \in A$.

Proposition 3.12. *If Y is an open d-subalgebra of an sd-algebra X, then Y is also an sd-algebra.*

Proof. Let $x, y \in Y$ and let U be any open set in the subspace Y containing $x * y$, then there exist $V \in \tau(x * y)$ such that $U = Y \cap V$. Since X is sd-algebra, there exist $W \in \tau(x)$ and $G \in SO(y)$ such that $W * G \subseteq V$. But $O = W \cap Y$ is an open set in Y containing x and $H = G \cap Y$ is a semi-open set in Y containing y, we have $(W \cap Y) * G \cap Y = (W * G) \cap Y \subseteq V \cap Y = U$. □

Proposition 3.13. *If A is an ideal in an sd-algebra X and $0 \in \mathrm{Int}(A)$, then A is open.*

Proof. Let $x \in A$. Since $0 \in \mathrm{Int}(A)$ and $x * x = 0$, there exists $U \in \tau(0)$ such that $0 \in U \subseteq A$. Since X is sd-algebra, there exist $V \in \tau(x)$ such that $V * x \subseteq U$. If $y \in V \cap (X \backslash A)$, the we have $y * x \in A$. Since $x \in A$ and A is ideal, $y \in A$, a contradiction. Hence $x \in V \subseteq A$ implies that A is open. □

Proposition 3.14. *If A is an open ideal in an sd-algebra X, then A is semi-closed.*

Proof. Let $x \notin A$. Then there exist $V \in \tau(x)$ and $U \in SO(x)$ such that $V * U \subseteq A$, since $x * x = 0$. Hence if $W = V \cap U$, then $W \in SO(x)$ and $W * W \subseteq A$. If $y \in W \cap A$ and since A is ideal, then $W \subseteq A$, a contradiction. Hence $W \subseteq X \backslash A$ and hence A is semi-closed. □

Definition 3.15. Let $(X, *, 0)$ be a sd-algebra and $F \subseteq X$. Then F is said to be a *filter* if

(1) $0 \in F$,
(2) If $0 \neq x \in F$ and $x * y \in F$, then $y \in F$.

Proposition 3.16. *Let $(X, *, \tau)$ be sd-algebra and F be a filter on X. If 0 is an interior point of F, then F is semi-open.*

Proof. Suppose that 0 is an interior point of F. Then there exists $U \in \tau(0)$ such that $U \subseteq F$. Let $x \in F$. Since $x * x = 0$, there exist $V \in \tau(x)$ and $W \in SO(x)$ such that $V * W \subseteq U \subseteq F$. Now, for each $W \in SO(y)$, we have $x * y \in F$. Since F is a filter and $x \in F$, $y \in F$. Hence $x \in W \subseteq F$ and so $F \in SO(X)$. □

Proposition 3.17. *Let $(X, *, \tau)$ be a sd-algebra and F a filter of X. If F is open, then it is closed.*

Proof. Let F be a filter of X, which is open in X. We show that $X\backslash F$ is open. Let $x \in X\backslash F$. Since F is open, 0 is an interior point of F. Since $x * x = 0$, there exist $V \in \tau(x)$ and $W \in SO(x)$ such that $V * W \subseteq F$. We claim that $V \subseteq X\backslash F$. If $V \nsubseteq X\backslash F$, then there exists $y \in V \cap F$. For each $z \in W$, we have $y * z \in V * W \subseteq F$, since $y \in F$ and F is a filter, $z \in F$. Hence $W \subseteq F$ and so $x \in F$, a contradiction. Then $x \in V \subseteq X\backslash F$, which implies that $X\backslash F$ is open and hence, F is closed. □

Definition 3.18. Let X be a d-algebra, U a non-empty subset of X and $a \in X$. The subsets U_a and ${}_aU$ are defined as follows: $U_a = \{x \in X : x * a \in U\}$ and ${}_aU = \{x \in X : a * x \in U\}$. Also if $K \subseteq X$ we put ${}_KU = \bigcup\limits_{a\in K} {}_aU$ and $U_K = \bigcup\limits_{a\in K} U_a$

Proposition 3.19. *Let X be a d-algebra and A, B, W, K any non-empty subsets of X. Then*

(1) *If $A \subseteq B$, then ${}_AW \subseteq_B W$.*
(2) *If $W \subseteq K$, then $A_W \subseteq A_K$.*
(3) *If $F \subseteq X$, then $X\backslash F_a = (X\backslash F)_a$ and $X\backslash_a F = {}_a(X\backslash F)$ for each $a \in X$.*

Proposition 3.20. *Let X be an sd-algebra, U and F any two non-empty subsets of X. Then*

(1) *If U is open, then U_a is open and ${}_aU$ is semi-open.*
(2) *If F is closed, then F_a is closed and ${}_aF$ is semi-closed.*

Proof. (1). Let $U \in \tau$, $a \in X$ and $x \in U_a$. Then $x * a \in U$. Since X is sd-algebra, there exist $G \in \tau(x)$ and $A \in SO(a)$ such that $G * A \subseteq U$, $x * a \in G_a \subseteq U$, thus $G * a \subseteq U$. Then $x \in G \subseteq U_a$. So $U_a \in \tau$. To prove that ${}_aU \in SO(X)$. Let $x \in {}_aU$ implies that $a * x \in U$. Since X is sd-algebra, there exist $A \in \tau(a)$ and $H \in SO(x)$ such that $A * H \subseteq U$. Then $a * x \in {}_aH \subseteq U$ and $a * H \subseteq U$. Hence $x \in H \subseteq {}_aU$. Therefore, ${}_aU \in SO(X)$.
(2). Let F be closed, then $X\backslash F$ is open. Hence by (1), $(X\backslash F)_a \in \tau$ and ${}_a(X\backslash F) \in SO(X)$. By Proposition 3.19, $X\backslash F_a = (X\backslash F)_a$ and $X\backslash_a F = {}_a(X\backslash F)$. Hence $X\backslash F_a$ is open and $X\backslash_a F$ is semi-open. Consequently, F_a is closed and ${}_aF$ is semi-closed. □

Definition 3.21. Let X be a d-algebra. The binary operation $\odot$ will be defined on $L(X)$ as $(L_a \odot L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.

Theorem 3.22. *Let X be a positive implicative d-algebra, then $(L(X), \odot, L_0)$ is a d-algebra.*

Proof. Let $L_a, L_b \in L(X)$. Then $(L_a \odot L_b)(x) = L_a(x) * L_b(x) = (a * x) * (b * x)$. Since X is positive implication d-algebra, $(a * x) * (b * x) = (a * b) * x$. Hence $(L_a \odot L_b)(x) = L_{*b}(x)$, implies that $L_a \odot L_b = L_{a*b}$ for all $a, b \in X$. Now (1). $((L_x \odot L_y) \odot (L_x \odot L_z)) \odot (L_z \odot L_y) = (L_{x*y} \odot L_{x*z}) \odot L_{z*y} = L_{((x*y)*(x*z))*(z*y)} = L_0$.
(2). $(L_x \odot (L_x \odot L_y)) \odot L_y = (L_x \odot L_{x*y}) \odot L_y = L_{(x*(x*y)} \odot L_y = L_{(x*(x*y))*y} = L_0$.
(3). $L_x \odot L_x = L_{x*x} = L_0$.
(4). $L_x \odot L_y = L_0$ and $L_y \odot L_x = L_0$, then $L_x * y = L_0$ and $L_y * x = L_0$, implies that $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ and hence, $L_x = L_y$.
(5). $L_0 \odot L_x = L_0 * x = L_0$. Hence, $L(X)$ is a d-algebra. □

Definition 3.23. Let X be a d-algebra, we define a map $\Psi : X \to L(X)$ by $\Psi(x) = L_x$ for all $x \in X$ and if A is any subset of X, then $L_A = \{L_a : a \in A\}$.

Remark 3. If X is a positive implicative d-algebra, then the following statements can be easily proved.

(1) If $A \subseteq B$, then $\Psi(A) \subseteq \Psi(A)$.
(2) If A and B are any two subsets of X, then $\Psi(A \cup B) = \Psi(A) \cup \Psi(B)$ and $\Psi(A \cap B) = \Psi(A) \cap \Psi(B)$.

Proposition 3.24. *Let X be a positive implicative d-algebra, then the map $\Psi : X \to L(X)$ is a d-isomorphism.*

Proof. It is clear that Ψ is a bijection. We have $\Psi(x * y) = L_{x*y}$ and $L_{x*y}(z) = (x * y) * z$. Since X is positive implicative, $(x * y) * z = (x * z) * (y * z)$. Then $L_x * y(z) = L_x(z) \odot L_y(z) = (L_x \odot L_y)(z)$. Hence $\Psi(x * y) = \Psi(x) \odot \Psi(y)$ for all x, $x \in X$, so Ψ is a d-isomorphism. □

Proposition 3.25. *Let X be a positive implicative d-algebra and τ be a topology on X, then the following statements are true:*

(1) *The family $\sigma = \{\Psi(G) \subseteq L(X) : G \in \tau\}$ is a topology on $L(X)$.*
(2) *For any subset A of X, $L_{\mathrm{Cl}(A)} = \mathrm{Cl}(L_A)$.*
(3) *If A is any semi-open set in (X, τ), then $\Psi(A)$ is a semi-open set in $(L(X), \sigma)$.*

Proof. (1). The proof of σ is a topology and hence it is obvious.
(2). For any $A \subseteq X$, we have $A \subseteq \mathrm{Cl}(A)$. Hence $L_A \subseteq L_{\mathrm{Cl}(A)}$ and $\mathrm{Cl}(A)$ is closed in X. Then by definition of $\sigma L_{\mathrm{Cl}(A)}$ is closed in $L(X)$. Hence we obtain $\mathrm{Cl}(L_A) \subseteq \mathrm{Cl}(L_{\mathrm{Cl}(A)}) = L_{\mathrm{Cl}(A)}$. To prove $L_{\mathrm{Cl}(A)} \subseteq \mathrm{Cl}(L_A)$, let $L_x \in L_{\mathrm{Cl}(A)}$, then $x \in \mathrm{Cl}(A)$. Let $L_G \in \tau(L_x)$. Hence $G \in \tau(x)$, hence $A \cap G \neq \emptyset$. Therefore, $L_A \cap L_G \neq \emptyset$, implies

that $L_x \in \mathrm{Cl}(L_A)$, so $L_{\mathrm{Cl}(A)} \subseteq \mathrm{Cl}(L_A)$ and hence $L_{\mathrm{Cl}(A)} = \mathrm{Cl}(L_A)$.
(3). Let $A \in SO(X)$. Then there exists $O \in \tau$ such that $O \subseteq A \subseteq \mathrm{Cl}(O)$. Hence $L_O \subseteq L_A \subseteq L_{\mathrm{Cl}(O)}$ and by (2), $L_O \subseteq L_A \subseteq \mathrm{Cl}(L_{\mathrm{Cl}(o)})$. Hence L_A is semi-open in $L(X)$. $\square$

Proposition 3.26. *Let X be a positive implicative sd-algebra. Then $(L(X), \odot, \sigma)$ is an sd-algebra.*

Proof. Let L_W be an open set containing $L_x \odot L_y = L_{x*y}$. Hence $W \in \tau(x * y)$. Since X is an *sd*-algebra, there exist $U \in \tau(x)$, $V \in SO(y)$ and $U * V \subseteq W$. Then $L_U * V \subseteq L_W$. Since X is positive implicative, $L_U * V = L_U \odot L_V \subseteq L_W$. By Proposition 3.25, L_V is semi-open in $L(X)$ containing L_y. $\square$

Recall that a function $f : X \to Y$ is semi-continuous [4] if the inverse image of each open set in Y is a semi-open set in X, and it is semi-open if the image of each open set is semi-open.

Proposition 3.27. *Let X be an sd-algebra, then every left map on X is semi-continuous.*

Proof. Let $a \in X$, define a left map $L_a : X \to X$ by $L_a(x) = a * x$, for all $x \in X$. Let $W \in \tau(L_a(x) = a * x)$. Since X is an *sd*-algebra, there exist $U \in \tau(x)$ and $V \in SO(x)$ such that $U * V \subseteq W$. Clearly, $a * V \subseteq U * V \subseteq W$. Hence $L_a(V) \subseteq W$ and hence L_a is semi-continuous. $\square$

Definition 3.28. A d-algebra X is called *s-transitive* (resp. *s-open*) if for each $a \in X \backslash \{0\}$, the left map L_a is semi-continuous (resp. semi-open) and it is transitive open if the right map R_a is both continuous and open.

Proposition 3.29. *Let X be an sd-algebra such that for each $a \in X \backslash \{0\}$, the left map L_a is semi-open. If $U \in \tau$, then the following statements are true:*

(1) $a * U \in SO(X)$.
(2) $L_a^{-1}(U) = \{x \in X : a * x \in U\} \in SO(X)$.
(3) $A * U \in SO(X)$ *for each* $A \subseteq X$.

Proof. Since L_a is semi-open and U is open, $L_a(U) = a*U \in SO(X)$. By Proposition 3.27, L_a is semi-continuous. Hence $L_a^{-1}(U) = \{x \in X : a*x \in U\} \in SO(X)$. Lastly, we have $A * U = \bigcup_{a \in A} (a * U) \in SO(X)$. $\square$

Proposition 3.30. *Let X be an sd-algebra, then every right map on X is continuous.*

Proof. Let $a \in X$, define a right map $R_a : X \to X$ by $R_a(x) = x * a$ for all $x \in X$. Let $W \in \tau(R_a(x) = x * a)$. Since X is an sd-algebra, there exist $U \in \tau(x)$ and $V \in SO(a)$ such that $U * V \subseteq W$. Clearly, $U * a \subseteq U * V \subseteq W$. Hence $R_a(U) \subseteq W$ and hence R_a is continuous. □

Proposition 3.31. *Let U be an open subset of a transitive open sd-algebra X and let $a \in X$. Then the following statements are true:*

(1) $U * a \in \tau$.
(2) $R_a^{-1}(U) = \{x \in X : x * a \in U\} \in \tau$.
(3) $U * A \in \tau$ *for each* $A \subseteq X$.

Proof. Since $R_a \in \tau$ and $U \in \tau$, $L_a(U) = U * a \in \tau$. By Proposition 3.30, R_a is continuous. So $R_a^{-1}(U) = \{x \in X : a * x \in U\} \in \tau$. Also $U * A = \bigcup_{a \in A} (U * a) \in \tau$. □

Definition 3.32. A d-algebra X is called an *edge d-algebra* if $x * X = \{0, x\}$ for each $x \in X$.

Proposition 3.33. *Let X be any s-transitive s-open edge d-algebra and τ be any topology on X, then there exists a topology σ on X which is sd-algebra.*

Proof. Let $x \in X \backslash \{0\}$, then L_x is s-open map. Since $X \in \tau$, by Proposition 3.3, $L_x(X) \in SO(X)$. Hence $L_x(X) = x * X = \{0, x\}$ as X is an edge d-algebra. Then $\{0, x\} \in SO(X)$ for all $x \in X$. Since $\{0, x\} \in SO(X)$ for all $x \in X$, $\mathrm{Int}(\{0, x\}) \neq \emptyset$ for all $x \in X$. Then we have the following cases: Either $\mathrm{Int}(\{0, x\}) = \{0\}$ or $\mathrm{Int}(\{0, x\}) = \{0, x\}$ or $\mathrm{Int}(\{0, x\}) = \{x\}$ for all $x \in X$. In the first two cases we obtain that $\{0\} \in \tau$, so σ is the discrete topology. The last case gives us $\{x\} \in \tau$ for all $x \in X \backslash \{0\}$. We claim that X equipped with the topology σ is an sd-algebra. For this, let $U \in \tau(x * y)$. If $x \neq 0$ and $y \neq 0$, then $\{x\} \in \tau(x)$ and $\{y\} \in \tau(y)$, so $\{x\} * \{y\} \subseteq U$. If $x = 0$, then $x * y = 0$ and hence if $U \in \tau(0)$, we have $U * \{y\} \subseteq U$. If $y = 0$, then $x * 0 = x$ and $U = \{x\} \in \tau$ and if $y \in X$ such that $x > y$, then $\{0, y\} \in SO(0)$ and $\{x\} * \{0, y\} \subseteq \{x\}$. Hence X is an sd-algebra. □

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References

1. Y. Imai & K. Iseki: On axiom systems of propositional calculi. Proc. Japan Acad. **42** (1966), 19-22. 10.3792/pja/1195522169
2. K. Iseki: An algebra related with a propositional calculus. Proc. Jpn. Acad. **42** (1966), 351-366. 10.3792/pja/1195522171
3. A.B. Khalaf & F.W. Ali: On s-topological BCK-algebras. Journal of University of Duhok (Pure and Eng. Sciences) **23** (2020), no. 1, 199-208. https://doi.org/10.26682/sjuod.2020.23.1.21
4. N. Levine: Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly **70** (1963), 36-41. https://doi.org/10.2307/2312781
5. J. Neggers & H.S. Kim: On d-algebras. Math. Slovaca **49** (1999) 19-26.
6. J. Neggers, Y.B. Jun & H.S. Kim: On d-ideals in d-algebras. Mathematica Slovaca **49** (1999), no. 3, 243-251.
7. S.N. Maheswari & R. Prasad: Some new separations axioms. Ann. Soc. Sci. Bruexles, Ser. I **89** (1975), 395-402.
8. N. Nagamani & N. Kandaraj: Topological Structures on d-Algebras. Journal of Physical Sciences **24**, 2019, 9-19.
9. M. Stone: Application of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc. **41** (1937), 374-481. https://doi.org/10.2307/1989788

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