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The critical slab problem with the Anlı-Güngör scattering function

R.G. Türeci

Kırıkkale University, Kırıkkale Vocational School, Kırıkkale, Turkey

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ABSTRACT

The criticality problem in this study is studied with the recently investigated the Anlı-Güngör scattering function. The scattering function depends on the Legendre polynomials as the Mika scattering function, but it includes only one scattering parameter, *t*, and its orders. Both Mika and Anlı-Güngör scattering are the same for only linear anisotropic scattering. The difference appears for the quadratic scattering and further. The analytical calculations are performed with the H_N method, and the numerical results are calculated with Wolfram Mathematica. Interpolation technique in Mathematica is also used to approximate the isotropic scattering results when *t* parameter goes to zero. Thus, the calculated results could be compared with the literature data for isotropic scattering.

1. Introduction

The relation between the secondary neutron number, *c*, and the slab thickness, τ, is investigated in the criticality problem. The secondary neutron number defines the ratio between the reaction cross section and the total cross section. The value of *c* will be taken as a constant number since we are not deal with the reaction types here; besides that, it is a constant number for homogeneous medium in one-speed neutron transport theory.

The critical slab problem has been investigated with different geometries, with different scattering functions, different numerical, and semi-analytical methods. The aim is to find the critical equation which defines the relation between the secondary neutron number, *c*, and the critical size [1]. The problem was investigated by Mitsis [2] by the Case method [3,4], and studied by Carlvik [5] to find the decay constants for homogeneous spheres and homogeneous thin slabs.

The problem was studied to find the criticality time eigenvalues for isotropic scattering by Sahni and Sjöstrand [6] and Sahni [7]. It was studied to calculate the criticality factor by Sahni and Sjöstrand [8], and it was investigated to calculate the criticality and time eigenvalues by Sahni and Sjöstrand [9] for İnönü scattering function [10]. The scattering is the function of the forward and backward scatterings. Dahl and Sjöstrand [11] investigated the eigenvalue spectrum of multiplying slabs and spheres. The problem was investigated for reflecting boundary conditions by Garis [12], and Garis and Sjöstrand [13]. Siewert and Williams [14] searched the effect of the anisotropy on the critical slab problem.

The problem was investigated with different methods such as C_N method [15], F_N method [16] and the Case method for reflecting boundary conditions [17]. The reflecting boundary condition for the problem was also studied with H_N method [18] (which is also known as Modified FN method) by Türeci et al. [19]. The solution of the problem with linear anisotropic and quadratic anisotropic scatterings was investigated by Güleçyüz et al. [20], Türeci and Güleçyüz [21].

The terms of the scattering as linear or quadratic is based on the order of the scattering terms in Legendre expansion of the anisotropic scattering [22]. Mika proved that the Case method can be extended to Legendre expansion. The Legendre expansion function is

$$f(\mu,\mu')=\frac{1}{2}\sum_{n=0}^{N}(2n+1)f_nP_n(\mu)P_n(\mu'). \quad (1)$$

Here f_n is the scattering coefficient, $P_n(\mu)$ is the Legendre polynomial with nth order. If the scattering function is linear anisotropic scattering, then $f_0=1$ which corresponds to the isotropic scattering, $f_1\neq0$, and $f_{n\geq2}=0$. f_1, which corresponds to the linear anisotropic scattering coefficient, is defined in $f_1\in[-1/3,1/3]$. If the scattering function is pure-quadratic anisotropic scattering then $f_0=1, f_1=0, f_2\neq0$ and $f_{n\geq3}=0$. f_2, which corresponds to the quadratic anisotropic scattering coefficient, is defined in $f_2\in[-0.2,0.4]$.

E-mail address: tureci@kku.edu.tr.

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The scattering function defines the scattering probabilities of the neutrons. Therefore, it cannot be smaller than zero, and bigger than unity. As the scattering function becomes more complex, the relationship between the scattering coefficients should be well analysed. Gülderen et al. [23] performed such analysis for the linear-triplet anisotropic scattering. The positivity condition for the linear-triplet anisotropic scattering is investigated in this study and applied to the Milne problem. The criticality problem with the analysis of the scattering coefficients has been studied for tetra anisotropic scattering by Köklü and Özer [24].

Anlı-Güngör (AG) scattering function [25] is

$$f(\mu,\mu')=\frac{1}{2}\sum_{n=0}^{N}t^{n}P_n(\mu)P_n(\mu'), \tag{2}$$

contains the scattering parameter, t which is defined in $|t|\leq 1$ in the theory of AG scattering function. Although, the scattering parameter, t, is defined in the interval $t\in[-1,1]$, it must be defined in $t\in[-0.54,0.54]$ in terms of the physical meaning of the scattering parameter. Because, the scattering function defines the scattering probabilities of the neutrons.

If we think an analogue of the linear anisotropic scattering for both scattering functions, then we can take that $t\equiv 3f_1$. Therefore, t^2 for the quadratic scattering will be $(3f_1)^2$ and t^3 for triplet scattering will be $(3f_1)^3$. Therefore, we can think that AG scattering function is the scattering function with the moment of the linear anisotropic scattering of Mika scattering function.

Another important point is that we cannot select a specific scattering in the AG scattering function such as pure-quadratic scattering as in Legendre expansion. If any researcher wants to investigate the quadratic scattering as like in this study, the linear scattering must be automatically added to the problem. This is the other difference of the AG scattering from the Legendre expansion.

The scattering function has been studied with P_N method [26], Diffusion approximation [27] and U_N method [28]. It was also studied with Monte Carlo method for albedo problem by Maleki [29]. The slab albedo problem was also investigated with the quadratic AG scattering by Türeci [30]. The criticality problem is investigated in this study with quadratic AG scattering function. To get the numerical results H_N method is used. Since the H_N method is based on the usage of the Case method, the relations of the Case method must be investigated for AG scattering function. The analysis of that was performed by Türeci and Bülbül [31].

Time eigenvalues with dynamic mode decomposition was investigated by McClarren [32] in recently performed study. The dominant time-eigenvalues were also investigated by Gupta and Modak [33], was also investigated by Kornreich and Parsons [34] in multi-region cartesian geometry. Sanchez et al. [35] investigated alpha modes in multigroup diffusion. Unfortunately, we cannot calculate the time eigenvalues in this study. The main problem for this analysis is that we have two independent variables in our formalism. The time eigenvalues term will be in the secondary neutron number definition. Therefore, if we don't know the value of the secondary neutron number, then the discrete eigenvalues, which are determined by the numerical solving of Eq. (21), cannot be found for given critical slab thickness.

2. The Case method for Anlı-Güngör scattering

The one-speed, time-independent, homogeneous medium and plane geometry neutron transport equation for source-free is

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)=c\int_{-1}^{1}f(\mu,\mu')\psi(x,\mu')d\mu' \tag{3}$$

where x is the spatial variable in mfp unit, μ is the direction cosine, c is the secondary neutron number, and $f(\mu,\mu')$ is the scattering function. In this study the scattering function is the AG scattering. The Case eigenfunction for the AG scattering is given in Ref. [30] as following

$$\phi(\nu,\mu)=\frac{c\nu}{2}\frac{K_n(\nu,\mu)}{\nu-\mu} \tag{4}$$

where

$$K_n(\nu,\mu)=\sum_{n=0}^{N}t^{n}P_n(\mu)J_n(\nu) \tag{5}$$

and

$$J_n(\nu)=\int_{-1}^{1}P_n(\mu)\phi(\nu,\mu)d\mu. \tag{6}$$

$J_n(\nu)$ obeys the recursion relation, and it is given by

$$J_{k+1}(\nu)=\frac{\nu}{k+1}\left[(2k+1)-ct^{k}\right]J_k(\nu)-\frac{k}{k+1}J_{k-1}(\nu). \tag{7}$$

$J_0(\nu)$ corresponds to the normalization of the Case eigenfunction. The normalization condition is

$$\int_{-1}^{1}\phi(\nu,\mu)d\mu=1. \tag{8}$$

If $\nu\notin[-1,1]$, then the numerical solution of Eq. (8) gives one pair discrete eigenvalues, $\pm\nu_0$. If $\nu\in[-1,1]$, then we have a singular point at $\nu=\mu$. Finally, the discrete and the continuum eigenfunctions are

$$\phi(\pm\nu_0,\mu)=\frac{\pm c\nu_0}{2}\frac{K_n(\pm\nu_0,\mu)}{\nu_0\mp\mu}, \tag{9}$$

$$\phi(\nu,\mu)=\frac{c\nu_0}{2}P\frac{K_n(\nu,\mu)}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu), \tag{10}$$

where P corresponds to the Cauchy-principal value, and $\lambda(\nu)$, which corresponds to the Cauchy-principal value at $\nu=\mu$, is defined as

$$\lambda(\nu)=1-\frac{c\nu}{2}P\int_{-1}^{1}\frac{K_n(\nu,\mu)}{\nu-\mu}d\mu. \tag{11}$$

These eigenfunctions obey the orthogonality relations:

$$\int_{-1}^{1}\mu\phi(\pm\nu_0,\mu)\phi(\pm\nu_0,\mu)d\mu=M(\pm\nu_0),M(\pm\nu_0)=-M(\mp\nu_0), \tag{12}$$

$$\int_{-1}^{1} \mu\phi(\pm\nu_0,\mu)\phi(\mp\nu_0,\mu)d\mu = 0, \tag{13}$$

$$\int_{-1}^{1} \mu\phi(\nu,\mu)\phi(\nu,\mu)d\mu = M(\nu) \tag{14}$$

Finally, the solution of Eq. (3) for the AG scattering function is

$$\psi(x,\mu) = A(\nu_0)\phi(\nu_0,\mu)e^{-x/\nu_0} + A(-\nu_0)\phi(-\nu_0,\mu)e^{x/\nu_0} + \int_{-1}^{1} A(\nu)\phi(\nu,\mu)e^{-x/\nu}d\nu \tag{15}$$

where $A(\pm\nu_0)$ and $A(\nu)$ are the arbitrary expansion coefficients. We are interested in the quadratic AG scattering here. Therefore, the upper limit of the scattering is $N=2$ in the expansion of Eq. (2), and the fourth and beyond terms, $N>2$, are assumed to be small enough to be neglected. Thus, the scattering function studied in this study is

$$f(\mu,\mu') = \frac{1}{2}\left[1 + tP_1(\mu)P_1(\mu') + t^2P_2(\mu)P_2(\mu')\right]. \tag{16}$$

$K_2(\xi,\mu)$ where $\xi = \pm\nu_0, \nu$, is

$$K_2(\xi,\mu) = 1 + tJ_1(\xi)P_1(\mu) + t^2J_2(\xi)P_2(\mu) \tag{17}$$

where

$$J_0(\xi) = 1, \tag{18a}$$

$$J_1(\xi) = \xi(1-c), \tag{18b}$$

$$J_2(\xi) = \frac{1}{2}\left(\xi^2(1-c)(3-ct) - 1\right). \tag{18c}$$

Thus, Eq. (17) is given in the explicit form as

$$K_2(\xi,\mu) = 1 + t\xi(1-c)\mu + \frac{t^2}{4}\left(3\mu^2 - 1\right)\left(\xi^2(1-c)(3-ct) - 1\right), \tag{19}$$

and the Case eigenfunction for the quadratic AG scattering function is

$$\phi(\nu,\mu) = \frac{c\nu}{2}\frac{1 + t\nu(1-c)\mu + \frac{t^2}{4}\left(\nu^2(1-c)(3-ct) - 1\right)\left(3\mu^2 - 1\right)}{\nu - \mu}. \tag{20}$$

The normalization condition by using Eq.(8) is

$$\ln\left(\frac{1+\nu_0}{1-\nu_0}\right) = \frac{2}{c\nu_0}\frac{1 + c\nu_0 tJ_1(\nu_0) + \frac{3c\nu_0^2}{2}t^2J_2(\nu_0)}{1 + \nu_0 tJ_1(\nu_0) + t^2J_2(\nu_0)\frac{3\nu_0^2 - 1}{2}}. \tag{21}$$

The discrete eigenvalues, $\pm\nu_0$, are the numerical solutions of Eq. (21). Since Eq. (21) is a transcendental equation, it can be solved as numerical methods such as Newton-Raphson method or Muller's method. Both methods can give real and complex roots of any function. Thus, the discrete and the continuum eigenfunctions are

$$\phi(\pm\nu_0,\mu) = \frac{c\nu_0}{2}\frac{1 \pm t\nu_0(1-c)\mu + \frac{t^2}{4}\left(\nu_0^2(1-c)(3-ct) - 1\right)\left(3\mu^2 - 1\right)}{\nu_0 \mp \mu}, \tag{22}$$

$$\phi(\nu,\mu) = \frac{c\nu}{2}P\frac{1 + t\nu(1-c)\mu + \frac{t^2}{4}\left(\nu^2(1-c)(3-ct) - 1\right)\left(3\mu^2 - 1\right)}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu) \tag{23}$$

where

$$\lambda(\nu) = 1 + c\nu tJ_1(\nu) + \frac{3c\nu^2}{2}t^2J_2(\nu) - c\nu\left[1 + t\nu J_1(\nu) + t^2J_2(\nu)\frac{3\nu^2 - 1}{2}\right]\tanh^{-1}(\nu). \tag{24}$$

The explicit form of Eq. (12) for the quadratic AG scattering is

$$M(\nu_0) = \left(\frac{c\nu_0}{2}\right)^2\frac{2}{\nu_0^2 - 1}\left[(\alpha_0 + \nu_0\beta_0)\left(\alpha_0\nu_0 - 2\beta_0 + 3\nu_0^2\beta_0\right) + \frac{2}{3}\gamma_0\left(-6\nu_0\alpha_0 + 9\nu_0^3\alpha_0 - \beta_0\left(1 - 8\nu_0^2\right) + 12\beta_0\nu_0^4\right) + \frac{\nu_0\gamma_0^2}{3}\left(-2 - 10\nu_0^2 + 15\nu_0^4\right)\right] - \left(\frac{c\nu_0}{2}\right)^2\left(\alpha_0 + \beta_0\nu_0 + \gamma_0\nu_0^2\right)\left(\alpha_0 + 3\beta_0\nu_0 + 5\gamma_0\nu_0^2\right)\ln\left(\frac{\nu_0 + 1}{\nu_0 - 1}\right) \tag{25}$$

where

$$\alpha_0 \equiv \alpha(\nu_0) = 1 - \frac{1}{2}t^2J_2(\nu_0), \tag{26a}$$

$$\beta_0 \equiv \beta(\nu_0) = \nu_0^2t(1-c), \tag{26b}$$

$$\gamma_0 \equiv \gamma(\nu_0) = \frac{3}{2}t^2J_2(\nu_0). \tag{26c}$$

and Eq. (14) is

$$M(\nu) = \nu\lambda^2(\nu) + \frac{c^2\pi^2\nu^3}{4}\left[1 + tJ_1(\nu)P_1(\nu) + t^2J_2(\nu)P_2(\nu)\right]^2 \tag{27}$$

where $P_1(\nu)$ and $P_2(\nu)$ are the Legendre polynomials with the continuum eigenvalue, ν.

3. The criticality equation with H_N method

Here a slab reactor is taken into account, and it is defined in $x\in[-a,a]$, *i.e.* $\tau=2a$. The aim is to find the criticality equation which defines the mathematical relation between the secondary neutron number and the thickness of a slab reactor. The boundary condition of the criticality means that there are outgoing neutrons from reactor, but there are no incoming neutrons to reactor.

The neutron flux has a symmetry condition over the walls of the reactor. This means that the neutron fluxes are the same behaviour at the left and the right boundaries of the medium. The outgoing neutron fluxes are defined as a power series expansion:

$$\Psi(-a,-\mu)=\Psi(a,\mu)=\sum_{\ell=0}^{G}a_\ell\mu^\ell,\mu\in[0,1]. \tag{28}$$

Since there is no the incoming neutron to the medium, the incoming neutron fluxes are

$$\Psi(-a,\mu)=\Psi(-a,-\mu)=0,\mu\in[0,1]. \tag{29}$$

If the symmetry condition, which is given in Eq. (28), is used in the solution given in Eq. (15); then, the arbitrary expansion coefficients become

$$A(\nu_0)=A(-\nu_0)\text{ and }\quad A(\nu)=A(-\nu). \tag{30}$$

Thus, the solution of Eq. (3) turns into

$$\Psi(x,\mu)=A(\nu_0)\left[\phi(\nu_0,\mu)e^{-x/\nu_0}+\phi(-\nu_0,\mu)e^{x/\nu_0}\right]+\int_0^1A(\nu)\left[\phi(\nu,\mu)e^{-x/\nu}+\phi(-\nu,\mu)e^{x/\nu}\right]d\nu,\mu\in[-1,1]. \tag{31}$$

Now we can rewrite Eq.(31) by taking $x=a$ (or $x=-a$):

$$\Psi(a,\mu)=A(\nu_0)\left[\phi(\nu_0,\mu)e^{-a/\nu_0}+\phi(-\nu_0,\mu)e^{a/\nu_0}\right]+\int_0^1A(\nu)\left[\phi(\nu,\mu)e^{-a/\nu}+\phi(-\nu,\mu)e^{a/\nu}\right]d\nu,\mu\in[-1,1]. \tag{32}$$

Now we can apply the orthogonality relations to Eq. (32) by using Eqs. (28) and (29). But we must take into account that the neutron flux must go to zero when x goes to infinity. Therefore, we should apply the orthogonality relations by eliminating the negative exponential terms. Thus, Eq. (32) is multiplied to $\mu\phi(-\nu_0,\mu)$ and integrated over $\mu\in[-1,1]$. After some algebraic manipulations we find

$$A(\nu_0)M(-\nu_0)e^{a/\nu_0}=\int_{-1}^{1}\mu\phi(-\nu_0,\mu)\Psi(a,\mu)d\mu \tag{33}$$

$$A(\nu_0)=-\frac{e^{-a/\nu_0}}{M(\nu_0)}\frac{c\nu_0}{2}\sum_\ell a_\ell A_\ell(\nu_0) \tag{34}$$

Similarly, if Eq. (32) is multiplied to $\mu\phi(-\nu,\mu)$ and integrated over $\mu\in[-1,1]$, then we find the arbitrary coefficient for the continuum part.

$$A(\nu)=-\frac{e^{-a/\nu}}{M(\nu)}\frac{c\nu}{2}\sum_\ell a_\ell A_\ell(\nu). \tag{35}$$

where $A_\ell(\xi)$ and $B_\ell(\xi)$, $\xi=\nu,\nu_0$ are the moments of the Case eigenfunctions in interval $\mu\in[0,1]$. These functions are defined by

$$A_l(\xi)=\frac{2}{c\xi}\int_0^1\mu^{l+1}\phi(\xi,-\mu)d\mu, \tag{36}$$

$$B_l(\xi)=\frac{2}{c\xi}\int_0^1\mu^{l+1}\phi(\xi,\mu)d\mu, \tag{37}$$

$$A_0(\xi)=\alpha_\xi-\frac{\beta_\xi}{2}+\frac{\gamma_\xi}{3}-\xi\left[\alpha_\xi+\xi\beta_\xi+\xi^2\gamma_\xi\right]\ln\left(1+\frac{1}{\xi}\right)+\xi\beta_\xi-\frac{\xi\gamma_\xi}{2}+\xi^2\gamma_\xi, \tag{38}$$

$$\int_{-1}^{1}\mu\phi(\xi,\mu)d\mu=J_1(\xi)=\xi(1-c)\Rightarrow B_0(\xi)=\frac{2}{c\xi}J_1(\xi)+A_0(\xi), \tag{39}$$

where

$$\alpha_\xi\equiv\alpha(\xi)=1-\frac{1}{2}t^2J_2(\xi),\xi=\pm\nu_0,\nu, \tag{40a}$$

$$\beta_\xi\equiv\beta(\xi)=\xi^2t(1-c),\xi=\pm\nu_0,\nu, \tag{40b}$$

$$\gamma_\xi\equiv\gamma(\xi)=\frac{3}{2}t^2J_2(\xi),\xi=\pm\nu_0,\nu. \tag{40c}$$

These functions satisfy own recursion relations:

$$A_l(\xi)=\frac{\alpha_\xi}{l+1}-\frac{\beta_\xi}{l+2}+\frac{\gamma_\xi}{l+3}-\xi A_{l-1}(\xi) \tag{41}$$

$$B_l(\xi)=\xi A_{l-1}(\xi)-\frac{\alpha_\xi}{l+1}-\frac{\beta_\xi}{l+2}-\frac{\gamma_\xi}{l+3} \tag{42}$$

Now we want to get a linear equation system. This process is the difference of the H_N method from F_N method. In order to get a linear equation system, we apply the following steps:

- Equation (32) is rewritten in interval $\mu\in[0,1]$. Thus, it will correspond to the outgoing neutron flux from $x=a$,
- Equations (34 and 35) which corresponds the arbitrary expansion coefficients are written in the equation, which is defined in the first step,
- The equation which defined in the second step above is multiplied to μ^{m+1},
- The equation defined in the third step is integrated over $\mu\in[0,1]$ where m is an integer. The integer m takes its value from zero to G which correspond to the upper limit of the flux definition in Eq. (28).

Finally, we get the following equation:

Table 1
The discrete eigenvalues, $\pm i\nu_0$, for varying c and varying t.

$\pm i\nu_0$						
$c\backslash t$	−0.3	−0.2	−0.1	0.1	0.2	0.3
1.1	1.666597560	1.695358647	1.725334450	1.789447553	1.823869092	1.860078190
1.2	1.131838679	1.153130975	1.175240069	1.222309178	1.247481602	1.273897787
1.3	0.890066342	0.908039923	0.926656153	0.966159161	0.987222292	1.009281108
14	0.744231332	0.760176167	0.776663354	0.811566518	0.830133407	0.849543041
1.5	0.644105348	0.658614260	0.673600141	0.705271646	0.722087701	0.739638402
1.6	0.570052310	0.583456828	0.597292402	0.626497578	0.641980144	0.658114788
1.7	0.512560052	0.525069704	0.537976441	0.565197665	0.579609815	0.594607797
1.8	0.466369810	0.478128623	0.490258353	0.515825579	0.529347160	0.543399625
1.9	0.428299716	0.439412896	0.450876159	0.475028898	0.487790319	0.501035969
2.0	0.396293768	0.406841728	0.417722715	0.440642772	0.452742895	0.465286937

* *All of values are pure imaginer since* $c > 1$.

Table 2
The critical thickness values, $2a$, from $c = 1.1$ to $c = 1.5$.

c	N\ t	−0.5	−0.3	−0.1	0.1	0.3	0.5
1.1	0	3.9901372	4.0911202	4.1973173	4.3098855	4.4299367	4.5585423
	1	3.9693732	4.0682695	4.1720629	4.2818020	4.3984599	4.5229201
	2	3.9698265	4.0687225	4.1725305	4.2823005	4.3990090	4.5235462
	3	3.9698403	4.0687351	4.1725422	4.2823118	4.3990204	4.5235580
	4	3.9698402	4.0687350	4.1725422	4.2823119	4.3990205	4.5235581
	5	3.9698403	4.0687351	4.1725423	4.2823120	4.3990207	4.5235583
	6	3.9698403	4.0687351	4.1725423	4.2823120	4.3990207	4.5235583
	7	3.9698403	4.0687351	4.1725423	4.2823121	4.3990207	4.5235584
	8	3.9698403	4.0687351	4.1725423	4.2823121	4.3990207	4.5235584
1.2	0	2.4456060	2.5058318	2.5668771	2.6292762	2.6932587	2.7586578
	1	2.4329757	2.4906401	2.5491940	2.6090754	2.6704658	2.7332131
	2	2.4325708	2.4903003	2.5489035	2.6088243	2.6702481	2.7330251
	3	2.4326599	2.4903828	2.5489822	2.6089015	2.6703261	2.7331065
	4	2.4326599	2.4903827	2.5489820	2.6089012	2.6703259	2.7331062
	5	2.4326598	2.4903826	2.5489819	2.6089012	2.6703258	2.7331062
	6	2.4326598	2.4903826	2.5489819	2.6089012	2.6703259	2.7331063
	7	2.4326598	2.4903826	2.5489820	2.6089013	2.6703259	2.7331063
	8	2.4326598	2.4903826	2.5489820	2.6089013	2.6703259	2.7331063
1.3	0	1.7733849	1.8166223	1.8586921	1.8998328	1.9398406	1.9779185
	1	1.7757175	1.8165680	1.8568359	1.8967300	1.9361533	1.9746089
	2	1.7740097	1.8149767	1.8553008	1.8951984	1.9345711	1.9729110
	3	1.7741583	1.8151209	1.8554445	1.8953452	1.9347253	1.9730778
	4	1.7741668	1.8151279	1.8554504	1.8953503	1.9347299	1.9730821
	5	1.7741664	1.8151276	1.8554501	1.8953501	1.9347297	1.9730819
	6	1.7741664	1.8151276	1.8554501	1.8953501	1.9347297	1.9730819
	7	1.7741664	1.8151276	1.8554501	1.8953501	1.9347297	1.9730819
	8	1.7741664	1.8151276	1.8554501	1.8953501	1.9347297	1.9730819
1.4	0	1.3833442	1.4168427	1.4480989	1.4771690	1.5036054	1.5262813
	1	1.3989015	1.4304840	1.4607817	1.4898929	1.5176000	1.5432750
	2	1.3963201	1.4280244	1.4583562	1.4874181	1.5149845	1.5404042
	3	1.3964280	1.4281418	1.4584842	1.4875593	1.5151428	1.5405855
	4	1.3964567	1.4281669	1.4585067	1.4875800	1.5151624	1.5406046
	5	1.3964567	1.4281667	1.4585065	1.4875797	1.5151621	1.5406044
	6	1.3964566	1.4281667	1.4585065	1.4875797	1.5151621	1.5406044
	7	1.3964566	1.4281667	1.4585065	1.4875797	1.5151621	1.5406044
	8	1.3964566	1.4281667	1.4585065	1.4875797	1.5151621	1.5406044
1.5	0	1.1266752	1.1537300	1.1779767	1.1993539	1.2172884	1.2305194
	1	1.1515842	1.1772948	1.2013431	1.2237683	1.2442982	1.2622608
	2	1.1487977	1.1745986	1.1986449	1.2209752	1.2413041	1.2589297
	3	1.1487673	1.1745968	1.1986680	1.2210221	1.2413762	1.2590309
	4	1.1488190	1.1746439	1.1987119	1.2210641	1.2414175	1.2590726
	5	1.1488215	1.1746457	1.1987132	1.2210651	1.2414182	1.2590733
	6	1.1488213	1.1746456	1.1987131	1.2210651	1.2414182	1.2590733
	7	1.1488214	1.1746456	1.1987132	1.2210651	1.2414182	1.2590733
	8	1.1488214	1.1746456	1.1987132	1.2210651	1.2414182	1.2590733

Table 3
The critical thickness values, 2*a*, from $c = 1.6$ to $c = 2.0$.

c	N\ t	−0.5	−0.3	−0.1	0.1	0.3	0.5
1.6	0	0.9452723	0.9677094	0.9870824	1.0032532	1.0155960	1.0228329
	1	0.9757796	0.9974466	1.0172347	1.0351493	1.0508947	1.0637937
	2	0.9733718	0.9950785	1.0148279	1.0326198	1.0481432	1.0606901
	3	0.9731538	0.9949063	1.0146925	1.0325159	1.0480682	1.0606440
	4	0.9732190	0.9949686	1.0147531	1.0325761	1.0481293	1.0607078
	5	0.9732272	0.9949751	1.0147585	1.0325806	1.0481333	1.0607114
	6	0.9732271	0.9949750	1.0147583	1.0325805	1.0481332	1.0607114
	7	0.9732271	0.9949750	1.0147584	1.0325806	1.0481332	1.0607114
	8	0.9732271	0.9949750	1.0147584	1.0325806	1.0481332	1.0607114
1.7	0	0.8108849	0.8298416	0.8456671	0.8581728	0.8667213	0.8700794
	1	0.8440080	0.8627265	0.8794383	0.8941286	0.9064938	0.9158688
	2	0.8423805	0.8610841	0.8777280	0.8922879	0.9044450	0.9135082
	3	0.8419774	0.8607377	0.8774251	0.8920194	0.9042045	0.9132914
	4	0.8420393	0.8608010	0.8774901	0.8920870	0.9042760	0.9133688
	5	0.8420555	0.8608147	0.8775020	0.8920975	0.9042856	0.9133778
	6	0.8420560	0.8608149	0.8775021	0.8920975	0.9042856	0.9133778
	7	0.8420559	0.8608149	0.8775021	0.8920976	0.9042856	0.9133779
	8	0.8420560	0.8608149	0.8775021	0.8920976	0.9042856	0.9133779
1.8	0	0.7078557	0.7240962	0.7372505	0.7470962	0.7530064	0.7538209
	1	0.7414384	0.7579131	0.7723069	0.7845921	0.7944654	0.8012811
	2	0.7408152	0.7572263	0.7715337	0.7836985	0.7934041	0.7999876
	3	0.7402662	0.7567370	0.7710889	0.7832864	0.7930155	0.7996144
	4	0.7403069	0.7567851	0.7711435	0.7833476	0.7930842	0.7996926
	5	0.7403315	0.7568067	0.7711629	0.7833655	0.7931010	0.7997088
	6	0.7403336	0.7568082	0.7711640	0.7833662	0.7931015	0.7997093
	7	0.7403335	0.7568081	0.7711639	0.7833662	0.7931016	0.7997093
	8	0.7403335	0.7568081	0.7711640	0.7833662	0.7931016	0.7997093
1.9	0	0.6267328	0.6407984	0.6518868	0.6597521	0.6637893	0.6629223
	1	0.6593125	0.6740212	0.6866096	0.6970412	0.7050166	0.7099134
	2	0.6597772	0.6743846	0.6868799	0.6972149	0.7050811	0.7098487
	3	0.6591410	0.6738038	0.6863392	0.6967017	0.7045845	0.7093586
	4	0.6591463	0.6738230	0.6863701	0.6967434	0.7046375	0.7094243
	5	0.6591774	0.6738515	0.6863966	0.6967685	0.7046618	0.7094484
	6	0.6591820	0.6738550	0.6863993	0.6967708	0.7046636	0.7094499
	7	0.6591820	0.6738550	0.6863993	0.6967707	0.7046636	0.7094499
	8	0.6591821	0.6738550	0.6863993	0.6967708	0.7046637	0.7094500
2.0	0	0.5614597	0.5737495	0.5832044	0.5895627	0.5922461	0.5902654
	1	0.5921009	0.6053812	0.6165256	0.6254910	0.6319840	0.6354061
	2	0.5936388	0.6067931	0.6178484	0.6267510	0.6332036	0.6366112
	3	0.5929796	0.6061791	0.6172662	0.6261888	0.6326506	0.6360564
	4	0.5929409	0.6061603	0.6172640	0.6262014	0.6326775	0.6360986
	5	0.5929752	0.6061931	0.6172955	0.6262322	0.6327082	0.6361300
	6	0.5929830	0.6061994	0.6173008	0.6262366	0.6327120	0.6361333
	7	0.5929834	0.6061996	0.6173009	0.6262367	0.6327120	0.6361333
	8	0.5929834	0.6061996	0.6173009	0.6262367	0.6327121	0.6361334

Table 4
The interpolated data when $t \to 0$.

c	Interpolated results 2*a* for $t \to 0$	Ref. [2]	Ref. [16,17]	Ref. [18,21]	Ref. [19,28]	Ref. [14,15]	Ref. [24,29]	
1.1	4.226619250	4.24	4.22668	4.2776	4.22661933	4.22662	4.22728	4.23161
1.2	2.578758425	–	2.57963	–	–	2.57876	2.57947	2.58436
1.3	1.875450935	1.780	1.87764	1.8810	1.87545111	1.87546	–	–
1.4	1.473206920	–	1.47684	–	–	1.47321	–	–
1.5	1.210112838	–	1.21523	1.1911	1.21011304	1.21012	–	–
1.6	1.023925757	–	1.03034	–	–	1.02394	1.02528	1.03301
1.7	0.885073341	–	0.89278	0.8520	0.88507355	0.88509	–	–
1.8	0.777546656	–	0.78629	–	–	0.77756	0.77999	0.78919
1.9	0.691869009	1.022	0.70162	0.6507	0.69186921	0.69188	–	–
2.0	0.622051513	0.621	0.63257	–	0.62205180	0.62205	0.62589	0.63634

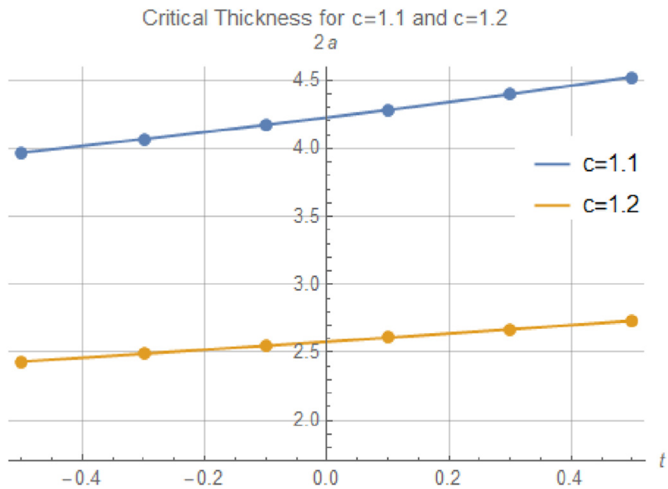

Fig. 1. Critical thickness for c = 1.1 and 1.2.

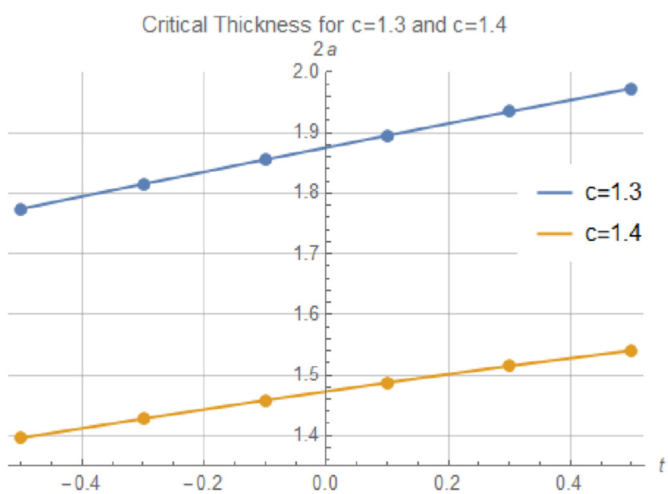

Fig. 2. Critical thickness for c = 1.3 and 1.4.

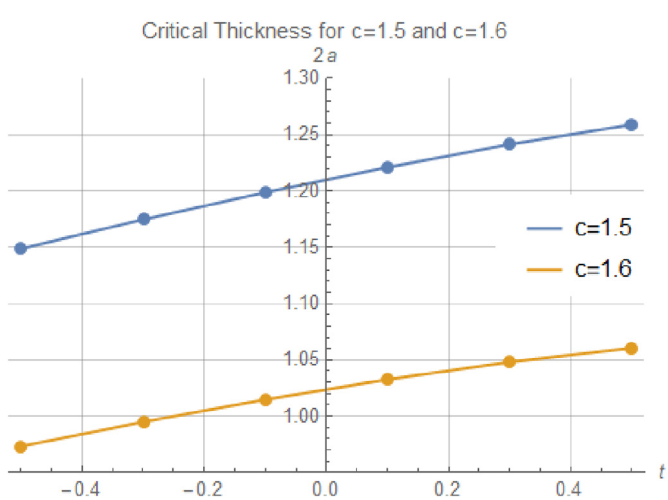

Fig. 3. Critical thickness for c = 1.5 and 1.6.

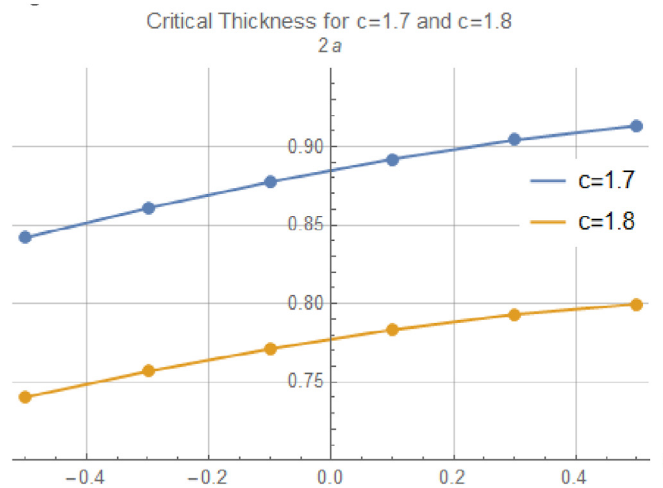

Fig. 4. Critical thickness for c = 1.7 and 1.8.

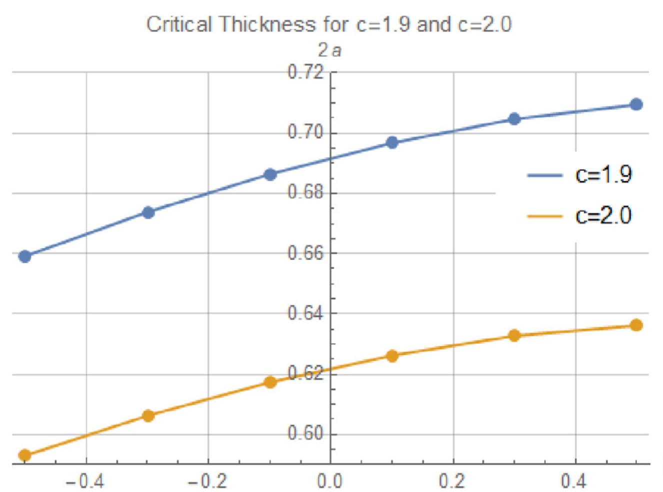

Fig. 5. Critical thickness for c = 1.9 and 2.0.

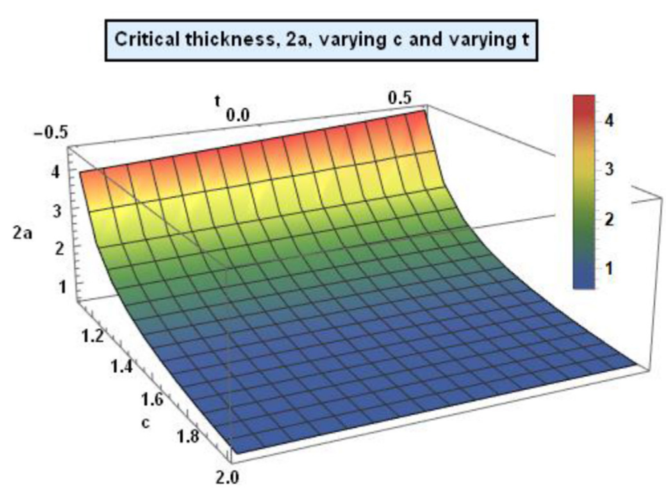

Fig. 6. The collective behaviour of the critical thickness for varying *c* and *t*.

$$\sum_{\ell=0}^{G} a_\ell \left\{ \frac{1}{m+\ell+2} + \left(\frac{c\nu_0}{2}\right)^2 \frac{A_\ell(\nu_0)B_m(\nu_0)}{M(\nu_0)} e^{-2a/\nu_0} + \left(\frac{c\nu_0}{2}\right)^2 \frac{A_\ell(\nu_0)A_m(\nu_0)}{M(\nu_0)} + \int_0^1 \left(\frac{c\nu}{2}\right)^2 \frac{A_\ell(\nu)B_m(\nu)}{M(\nu)} e^{-2a/\nu} d\nu + \int_0^1 \left(\frac{c\nu}{2}\right)^2 \frac{A_\ell(\nu)A_m(\nu)}{M(\nu)} d\nu \right\} = 0. \tag{43}$$

This last equation can be written as

$$\sum_{\ell=0}^{G} a_\ell T_{\ell m} = 0 \tag{44}$$

where

$$T_{\ell m} = \left\{ \frac{1}{m+\ell+2} + \left(\frac{c\nu_0}{2}\right)^2 \frac{A_\ell(\nu_0)B_m(\nu_0)}{M(\nu_0)} e^{-2a/\nu_0} + \left(\frac{c\nu_0}{2}\right)^2 \frac{A_\ell(\nu_0)A_m(\nu_0)}{M(\nu_0)} + \int_0^1 \left(\frac{c\nu}{2}\right)^2 \frac{A_\ell(\nu)B_m(\nu)}{M(\nu)} e^{-2a/\nu} d\nu + \int_0^1 \left(\frac{c\nu}{2}\right)^2 \frac{A_\ell(\nu)A_m(\nu)}{M(\nu)} d\nu \right\}. \tag{45}$$

$T_{\ell m}$ defines the elements of T square matrix. Since a_ℓ coefficients can't become zero, the determinant of T matrix must be equal to zero. This defines the criticality equation:

$$\det T = 0. \tag{46}$$

4. Numerical results

Equation (46) is the criticality equation. This equation is the function of the secondary neutron number, c, the scattering parameter, t, and τ which corresponds to the thickness of the slab. Equation also contains the discrete and the continuum eigenvalues, ν_0 and ν. The continuum eigenvalue is defined in interval $\nu \in [-1, 1]$. The discrete eigenvalue is found by solving Eq. (21) according to the value of c. Finally, c and t are the independent variables and $\tau = 2a$ is the dependent variable. First calculation is the determination of the discrete eigenvalues. *FindRoot* command in Mathematica 12.2 software [36] is used for this aim. This command uses the *Newton-Raphson method* [37] in the background. The discrete eigenvalues are given in Table 1 for varying c and t. All of values are pure imaginer in Table 1 because of $c > 1$.

The critical thickness value for varying c and varying t parameter is found by numerical solving of Eq. (46) by again *FindRoot* command. Eq. (45 or 46) also includes an integral term, the fourth term RHS, which contains the unknown variable. To calculate this integral term *Gaussian-Quadrature* [37] method was used with 128-point.

The upper value of the power series expansion, G, is selected as 8. The *WorkingPrecision* option was selected as 128. A special code part was also used in this study. This code holds all numbers as rational numbers to avoid the round-off errors.

Tables 2 and 3 represent the critical thickness values. Table 4 represents the interpolated results when $t \rightarrow 0$ limit. Table 4 also includes the literature data for isotropic scattering; thus, we can compare the data calculated here when $t \rightarrow 0$ limit.

The tabulated results are plotted in Figs. 1–5. We can see that the critical thickness values decrease for increasing secondary neutron number as expected. But the increasing rate decreases for the increasing c values.

We can also plot a 3D graphic with Mathematica, Fig. 6. The secondary neutron number and the scattering parameter, which are the independent variables, are the axis of 3D graphic. The last variable, which will correspond to the dependent variable, of 3D graphic will be the critical thickness. Thus, we can see collective behaviour of the criticality problem.

5. Conclusion

In this study the critical thickness values for a slab reactor are investigated for the quadratic AG scattering with H_N method. Table 1 represents the discrete eigenvalues. Tables 2–3 represent the critical thickness values for varying c and t. The critical thickness values in mfp unit, and $\tau = 2a$ According to the results the critical thickness values decrease for increasing c as we expected. For fixed c the critical thickness values increase for increasing t. These results could be seen in Figs. 1–5. It is interesting that the behaviour of the critical thickness is nearly linear up to $c = 1.5$. After $c = 1.5$, the behaviour of the critical thickness becomes parabolic. We can say that the effect of the quadratic scattering appears for big c values. The collective behaviour of the critical thickness for varying c and varying t could be seen in Fig. 6 as an 3D plot.

The AG scattering function includes only t parameter. Therefore, if a researcher wants to study with the quadratic anisotropic scattering in AG scattering just as in this study, then automatically the linear anisotropic scattering will be included the scattering. Therefore, in the tabulated results also include the linear anisotropic contributions.

Besides that, if we think that $t \equiv 3f_1$ in the comparison of the linear anisotropic scattering in the Legendre expansion and the linear AG scattering; then, the next scattering terms in the AG scattering function is the function of $(3f_1)^n$. The quadratic scattering, for instance, can be think $t^2 = (3f_1)^2$. Therefore, we can think that the quadratic AG scattering corresponds to the second order of the linear anisotropic scattering.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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In the first version of the manuscript the scattering parameter was taken in interval $|t| \leq 1$ as in the original paper for AG scattering function. With a short analysis I did thanks to the criticism of the referees, the definition range of the scattering parameter must be $-0.54 \leq t \leq 0.54$. Therefore, it is my debt to thank the referees for contributing to the scientific value of my study. I also want to thank Prof. F. Anlı for helpful discussions.

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