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SPIN HALF-ADDER IN $\mathcal{B}_3$

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ABSTRACT. This study is about spin half add operations in $\mathcal{B}_2$ and $\mathcal{B}_3$. The burden of technological structures has increased due to the increase in the use of today's technological applications or the processes in the digital systems used. This has increased the importance of fast transactions and storage areas. For this, less transactions, more gain and storage space are foreseen. We have handle tit (triple digit) system instead of bit (binary digit). 729 is reached in 3^6 in $\mathcal{B}_3$ while 256 is reached with 2^8 in B_2. The volume and number of transactions are shortened in $\mathcal{B}_3$. The limited storage space at the maximum level is storaged. The logic connectors and the complement of an element in $\mathcal{B}_2$ and the course of the connectors and the complements of the elements in B_3 are examined. "Carry" calculations in calculating addition and "borrow" in calculating difference are given in $\mathcal{B}_3$. The logic structure $\mathcal{B}_2$ is seen to embedded in the logic structure $\mathcal{B}_3$. This situation enriches the logic structure. Some theorems and lemmas and properties in logic structure $\mathcal{B}_2$ are extended to logic structure $\mathcal{B}_3$.

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1. Introduction

In this part, the theoretical part of the structure is given. Nabiyev showed such that transactions are narrowed and on the other hand, more numbers are produced in the ternary system in [12]. Experiences in producing computers using triple digital logic values "Setun" and "Setun 70" at Moscow State University confirmed this logic in [13].

Definition 1.1. (See [1] - [3]) A group is a set G equipped with a binary operation $* : G \times G \to G$ that associates an element $a * b \in G$ to every pair of elements $a, b \in G$, and having the following properties:$*$ is associative, has an identity element $e \in G$, and every element in G is invertible. The elements

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$x \in G$ satisfying the following equation are called the k^{th} potent elements of the set G.

$$\underbrace{x * ... * x}_{k-\text{ times}} = x. \tag{1}$$

If $k = 3$, then $x \in G$ is called threempotent element of the set G.

The set of all k^{th} potent elements is denoted by $k^{(G,*)}$ which is,

$$k^{(G,*)} = \{x \in G \mid \underbrace{x * ... * x}_{k-\text{ times}} = x, k \in \mathbb{Z}^+ \setminus \{1\}\} = \mathcal{P}(G).$$

It is known that $\mid k^G \mid = \mid \mathcal{P}(G) \mid = k^{|G|} = k^m$.

Theorem 1.2. *For any $k \in \mathbb{Z}^+$ the following statements hold.*

(i) $(2k+1)^{(\mathbb{R},.)} = 3^{(\mathbb{R},.)}$.

(ii) $(2k)^{(\mathbb{R},.)} \subset 3^{(\mathbb{R},.)}$.

Proof. A.

(i) For any $x \in (2k+1)^{(\mathbb{R},.)}$

$$x^{2k+1} = \underbrace{x^{2k}}_{x^2} x = x^2 x = x^3 = x \Rightarrow x \in 3^{(\mathbb{R},.)}$$

$$(2k+1)^{(\mathbb{R},.)} \subseteq 3^{(\mathbb{R},.)}$$

(ii) For any $x \in 3^{(\mathbb{R},.)}$

$$x^3 = \underbrace{x^2}_{x^{2k}} x = x^{2k} x = x^{2k+1} = x \Rightarrow x \in 2k+1)^{(\mathbb{R},.)}$$

$$3^{(\mathbb{R},.)} \subseteq (2k+1)^{(\mathbb{R},.)}$$

by A.(i), A.(ii).

$$(2k+1)^{(\mathbb{R},.)} = 3^{(\mathbb{R},.)}.$$

(iii) Let us give the proof with the induction method. The statement $k = 1$ says that

$$(2)^{(\mathbb{R},.)} \subset 3^{(\mathbb{R},.)}$$

which is true. Fix $k - 1 \geq 1$, and suppose that it holds for $k - 1$, that is,

$$(2(k-1))^{(\mathbb{R},.)} \subset 3^{(\mathbb{R},.)}$$

It remains to show for k, that is,

$$x^{2k} = \underbrace{x^{2(k-1)}}_{x^3} x^2$$

$$x^{2(k-1)} = x^3 = x$$

by Theorem 1.2.(1). Then,

$$x^{2k} = \underbrace{x^{2(k-1)}}_{\text{x}} x^2 = x^3 = x \Rightarrow x \in 3^{(\mathbb{R},.)}$$

So,

$$(2k)^{(\mathbb{R},.)} \subset 3^{(\mathbb{R},.)}.$$

□

Real numbers are sufficient for most of equations. The solutions of equations $x^2 = x$ and $x^3 = x$ are real numbers. But the roots (or zeros) of the following equations are the same as in complex numbers:

$$z^2 = z \text{ and } z^3 = z, \text{ where } z \in \mathbb{C}.$$

Thus, the sets of idempotent elements for any $z \in \mathbb{C}$ are:

$$2^{(\mathbb{C},+)} = \{z \in \mathbb{C} \mid z + z = z\} = \{0\}.$$

$$2^{(\mathbb{C},.)} = \{z \in \mathbb{C} \mid z^2 = z\}.$$

$$3^{(\mathbb{C},.)} = \{z \in \mathbb{C} \mid z^3 = z\}.$$

If $x = 1$ and $x = 0$, for any complex number $z = x + iy$, where $x, y \in \mathbb{R}$, then,

$$2^{(\mathbb{C},.)} = \{0, 1\} = 2^{(\mathbb{R},.)}.$$

And

$$3^{(\mathbb{C},.)} = \{-1, 0, 1\} = 3^{(\mathbb{R},.)}.$$

2. Main results

This chapter is about the logical relationship and purposes of threempotent elements. The aim is to determine the relationship between the logic created by the idempotent elements used and the logical formation of the newly obtained structure. The basic structure in $\mathcal{B}_3$ is based on the triple number system, which is balanced and compact. Suppose that the base value of a number system is 3, then the digits are -1, 0 and +1 in Ternary Number System(TNS). The set of these elements is equal to the set of treempotent elements.

The operations to be performed are on real numbers.

$$x \in 3^{(\mathbb{R},.)}, \forall x \in B = \{x \in \mathbb{R} | x^k = x, k \geq 2\}. \tag{2}$$

Definition 2.1. ([4]). A Boolean algebra $\mathcal{B}$ is a system $\mathcal{B} = (\mathcal{B}, \wedge, \vee, ', 0, 1\ )$ such that $\wedge$ and $\vee$ are binary operations on B, $'$ is a unary operation on $\mathcal{B}$, $0, 1 \in \mathcal{B}$, and the following conditions hold for all $x, y, z \in \mathcal{B}$:

(i) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;
(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$;
(iii) $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$;
(iv) $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$ and $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$;
(v) $x \wedge x' = 0$ and $x \vee x' = 1$.

Proposition 2.2. *([5, 6]). Let $x, y \in \mathcal{B}$. The followings are equivalent.*

(i) $x \leq y$.
(ii) $x \vee y = y$.
(iii) $x \wedge y = x$.

(iv) $x' = 1 - x$.

An ordered structure $(\mathcal{B}, \vee, \wedge, \leq)$ is a Boolean lattice if and only if an ordered structure satisfies (i), (ii) and (iii) of Proposition 2.2.

In this section, the basic concept of logic is discussed by considering the idempotent elements which are the basic building blocks of the known lattice theory. If $\mathcal{B}$ is an arbitrary ring then its set of *central idempotents*, which is the set

$$Cen(\mathcal{B}) = \{e \in \mathcal{B} | e^2 = e, xe = ex = x \text{ } for \text{ } all \text{ } x \in \mathcal{B}\}$$

becomes a Boolean algebra when its operations are defined by

$$x \vee y := x + y - xy = maks\{x, y\}, x \wedge y := xy = min\{x, y\}.$$

Let $\mathcal{B}$ be an arbitrary ring. It is clear that $Cen(\mathcal{B}) \subseteq k^{\mathcal{B}}$.
The current lattice is:

$$\mathcal{B}_2 = (\mathcal{B},', \wedge, \vee, 0, 1\ ) \equiv (\mathcal{B},', \wedge, \vee, 2^{\mathcal{B}}\ ).$$

If the equation $x^k = x$ is taken into account instead of the equation $x^2 = x$, then

$$\mathcal{B}_k \equiv (\mathcal{B},', \wedge, \vee, k^{\mathcal{B}}\ ), where\ k \geq 2, k \in Z^+.$$

In $\mathcal{B}_2$

$$x \in \mathcal{B} \Rightarrow x^2 = x \Rightarrow 2^{\mathcal{B}} = \{0, 1\}.$$

If the number of elements of the set $\mathcal{B}$ is m, then total number of elements in power set is 2^m in $\mathcal{B}_2$. This number is 3^m in $\mathcal{B}_3$.

Example 2.3. Let $x \in \mathbb{R}$ and $B = \{x\}$. The power set of $\mathcal{B}$ in $\mathcal{B}_2$ is $\mathbb{P}(\mathcal{B}) = \{\emptyset, \{x\}\}$. The power set of $\mathcal{B}$ in $\mathcal{B}_3$ is $\mathbb{P}(\mathcal{B}) = \{\emptyset, \{x\}, \{-x\}\}$.

Proposition 2.4. *Let $\mathcal{B}$ be a finite set in $\mathcal{B}_3$ with $|\mathcal{B}| = m$. Then*

$$|\mathbb{P}(\mathcal{B})| = 3^m.$$

Proof. Let $|\mathcal{B}| = m$ be the finite set in $\mathcal{B}_3$. Then,

$$\sum_{i=0}^{m} \binom{m}{i} \sum_{k=0}^{i} \binom{i}{k} = \sum_{i=0}^{m} \binom{m}{i} 1^{m-i} 2^i = (1+2)^m = 3^m.$$

□

Let T be true, F be false, and L lie be in a logical expression.

TABLE 1. The logic values in $\mathcal{B}_2$ [7].

x	x'	$x \wedge x'$	$x \vee x'$
0	1	0	1
1	0	0	1
F	T	F	T
T	F	F	T

In $\mathcal{B}_3$

$$x \in B \Rightarrow x^3 = x \Rightarrow 3^B = \{-1, 0, 1\}.$$

If $x = 0 \Rightarrow x' = 1 \vee x' = -1$.

$$x \vee x' = \begin{cases} 0, if & x' = 1 \\ 1, if & x' = -1 \end{cases}$$

$$x \wedge x' = \begin{cases} 0, if & x' = 1 \\ -1, if & x' = -1. \end{cases}$$

$$\mathcal{B}_3 = (\mathcal{B},', \wedge, \vee, -1, 0, 1\ ) \equiv (\mathcal{B},', \wedge, \vee, 3^{\mathcal{B}}\ ).$$

$$\mathcal{B}_2 \subseteq \mathcal{B}_3.$$

TABLE 2. The logic values in $\mathcal{B}_3$.

x	x'_1	x'_2	$x \vee x'_1$	$x \wedge x'_1$	$x \vee x'_2$	$x \wedge x'_2$
0	-1	1	0	-1	1	0
1	0	-1	1	0	1	-1
-1	0	1	0	-1	1	-1
F	L	T	F	L	T	F
T	F	L	T	F	T	L
L	F	T	F	L	T	L

Proposition 2.5. *Let $\mathcal{B}_3$ be a Boolean Lattice. Then the followings are equivalent for any $\mathcal{B}$.*

(i) $x' = x^2 - 1$.

(ii) *One the x' of x is in $\mathcal{B}_2$.*

(iii) $x' = 1$,$x' = -1$.

Proof. Let $\mathcal{B}_3$ be a Boolean Lattice and $x \in \mathcal{B}$. Then

$$x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0$$

If (i) holds, then

$$x' = x^2 - 1 \Rightarrow (x + 1)(x - 1) = 0 \Rightarrow x' = 1 \Rightarrow (ii).$$

If (ii) and (i) hold, then

$$x' = -1, x' = -1 \Rightarrow (iii).$$

If (iii) holds, then

$$x' = (x - 1), x' = (x + 1) \Rightarrow x' = x^2 - 1. \Rightarrow (i).$$

□

Lemma 2.6. *Let $\mathcal{B}_k$ be a Boolean ring. If $k \in \mathbb{Z}^+$ is odd, then*

$$\mathcal{B}_{2k} \subseteq \mathcal{B}_3.$$

Proof. The proof is done by the induction method. For $k = 1$, $\mathcal{B}_2 \subseteq \mathcal{B}_3$. Let us assume $\mathcal{B}_{2k-2} \subseteq \mathcal{B}_3$ is true for $k - 1$. We have to prove that $\mathcal{B}_{2k} \subseteq \mathcal{B}_3$.

$$x \in \mathcal{B}_{2k-2} \Rightarrow x^{2k-2} = x.$$
$$x^k = (x^{2k-2})(x^{2-k}) = x(x^2)(x^k)^{-1} = x^3(x^{-1}) = x^2 = x.$$
$$\Rightarrow x \in \mathcal{B}_3 \Rightarrow \mathcal{B}_{2k} \subseteq \mathcal{B}_3.$$

□

Lemma 2.7. *Let $\mathcal{B}_3$ be a Boolean ring. Then*

(i) $x \vee x' = 1, x \vee x' = 0.$
(ii) $x \wedge x' = -1, x \wedge x' = 0.$

Proof. Let us assume it is $x = 0$, without loss of the generality. Then, Proof of (i):

$$x = 0 \Rightarrow x \vee x' = 1, x \vee x' = 0.$$

And Proof of (ii):

$$x = 0 \Rightarrow x \wedge x' = -1, x \wedge x' = 0.$$

□

Let us explain that it is necessary to use $\mathcal{B}_3$ in practice. $-1 \equiv 2(mod 3)$ in $\mathcal{B}_3$. Although $-1 \equiv 1(mod 2)$ in $\mathcal{B}_2$, Likewise, $1 \neq 2$. $\mathcal{B}_3$ necessitates from this situation.

3. Spin half-adder in $\mathcal{B}_3$

In this section, the operation values and logic values in $\mathcal{B}_2$ and $\mathcal{B}_3$ are given. Among these, some procedural studies are discussed. Different approaches of known or applied logical structures are included here. The in-depth path is followed on its applications in computer science. The emphasis is on the acquisition of the new contribution to logical operations in $\mathcal{B}_3$. $2^{(\mathbb{C},.)}$, $2^{(\mathbb{R},.)}$ and $3^{(\mathbb{R},.)}$, $3^{(\mathbb{C},.)}$ are equal to the same set in complex numbers $(\mathbb{C}, .)$ and real numbers $\mathbb{R}$. $k^{(\mathbb{C},.)}$ are different sets of complex numbers for $k \geqslant 4$.

The Boolean expression of the Half adder, where s total, c Carry, and x and y are the input values, is given as follows:

$$s = x \bigoplus y, c = x.y$$

in [8].

Some rules of trinary addition are given below.

(i) $0 = 1 - 10$, in $\mathcal{B}_3$.
(ii) $2 = 1 - 1$, in $\mathcal{B}_3$
(iii) $1 = 1 - 11$, in $\mathcal{B}_3$.
(iv) $0 + 0 = 0$, (result$= 0$, carry $= 0$)

(v) $0 + 1 = 1,$ (result= 1, carry = 0)
(vi) $1 + 1 = -1,$ (result= -1, carry = 0)
(vii) $1 + 1 + 1 = 10,$ (result= 0, carry = 1)
(viii) $1 + 1 + 1 + 1 = 11,$ (result= 1, carry = 1)
(ix) $1 + (-1) = 11,$ (result= 0, carry = 1)
(x) $(-1) + 1 = 11,$ (result= 0, carry = 1)
(xi) $(-1) + (-1) = 11,$ (result= 0, carry = 1)
(xii) $(-1) + (-1) + (-1) = -10,$ (result= 0, carry = -1)

Table 3. The Sum, Carry, Difference and Borrow logic values in $\mathcal{B}_2$ [9, 10].

Input		Half-Adder		Half-Subtracter		
x	y	$\wedge, .$ Carry(c)	$\vee, \bigoplus$ Sum(s)	(-) Difference)	Borrow(x-y)	Borrow(y-x)
0	0	0	0	0	0	0
0	1	0	1	1	1	0
1	0	0	1	1	0	1
1	1	1	0	0	0	0

Table 4. The Sum, Carry, Difference and Borrow logic values in $\mathcal{B}_3$ [9, 10].

Input		Output				
		Half-Adder		Half-Subtracter		
x	y	$\wedge, .$ Carry(c)	$\vee, \bigoplus$ Sum(s)	(-) Difference)	Borrow(x-y)	Borrow(y-x)
0	0	0	0	0	0	0
0	1	0	1	-1	1	0
0	-1	0	-1	1	1	1
1	0	0	1	1	0	1
1	1	1	-1	0	0	0
1	-1	-1	0	-1	1	1
-1	0	0	-1	-1	1	0
-1	1	-1	0	1	1	1
-1	-1	0	0	0	0	0

Table 5. The operations multiplication and addition in $\mathcal{B}_3$.

.	-1	0	1	$\bigoplus$	-1	0	1
-1	0	0	-1	-1	0	-1	0
0	0	0	0	0	-1	0	1
1	-1	0	1	1	0	1	-1

The symmetrical structure is formed as the result of the processes.

Table 6. The values Sum, Carry and Difference in $\mathcal{B}_2$ and $\mathcal{B}_3$ [9], [10], [11].

Input		$\mathcal{B}_2$			$\mathcal{B}_3$		
x	y	. Carry(c)	$\bigoplus$ Sum(s)	(-) Difference (x-y)	. Carry(c)	$\bigoplus$ Sum(s)	(-) Difference (x-y)
0	0	0	0	0	0	0	0
0	1	0	1	-1	0	1	-1
0	2	0	2	-2	0	2	-2
0	3	0	3	-3	0	3	-3
1	0	0	1	1	0	1	1
1	1	1	2	0	1	2	0
1	2	2	3	-1	2	3	-1
1	3	3	4	-2	3	4	-2
2	0	0	2	2	0	2	2
2	1	2	3	1	2	3	1
2	2	4	4	0	4	4	0
2	3	6	5	-1	6	5	-1
3	0	0	3	3	0	3	3
3	1	3	4	2	3	4	2
3	2	6	5	1	6	5	1
3	3	9	6	0	9	6	0

The same results are obtained for difference values $\mathcal{B}_3$ and $\mathcal{B}_2$.

The following two lemmas without proof are given.

Lemma 3.1. *Suppose x, y are any two elements in $\mathcal{B}_3$. Then the following statements hold.*

(i) *If $x = y = -1$ and $x = y = 0$ then $x.y \in \mathcal{B}_2$.*
(ii) *If at least one of the products $x.y$ is 0, then $x.y = 0$.*
(iii) *If $1.y = y$ and $x.1 = x$, where $x, y \in \mathcal{B}_3$.*
(iv) *If $x = -1$, $y = 1$, then $x.y = -1$, $y.x = -1$.*

Corollary 3.2. *Let $(\mathcal{B}_3, .)$. 1 is the identity element according to the multiplication operation in $\mathcal{B}_3$.*

Lemma 3.3. *Suppose x, y are any two elements in $\mathcal{B}_3$. Then the following statements hold.*

(i) *If $x = -1$, $y = -1$ and $x = 0$, $y = 0$ then $x \bigoplus y \in \mathcal{B}_2$.*
(ii) *If $x = 0$ then $x \bigoplus y = y$, where $x, y \in \mathcal{B}_3$.*
(iii) *If $0 \bigoplus x = x$ and $y \bigoplus 0 = y$, where $x, y \in \mathcal{B}_3$.*
(iv) *If $x = -1$, $y = 1$, then $x \bigoplus y = 0$, $y \bigoplus x = 0$.*

Corollary 3.4. *Let $(\mathcal{B}_3, \bigoplus)$.*

(i) *0 is the identity according to the addition operation in $\mathcal{B}_3$.*
(ii) *0 is the absorbing element according to the multiplication operation.*

Lemma 3.5. *The following assertions hold.*

(i) *If $x = -1$ then $x^{-1} \in \{-1, 1\}$ in $(\mathcal{B}_3, \bigoplus)$.*

(ii) *If* $x = 1$ *then* $x^{-1} \in \{-1, 1\}$ *in* $(\mathcal{B}_3, \bigoplus)$.

Proof. The proofs are clear by Tablo 5. □

Theorem 3.6. *Let* $(\mathcal{B}_3, \bigoplus, .)$ *be a binary two operation. Then the following assertions are true.*

(i) *For any* $x, y \in \mathcal{B}_3, x \bigoplus y \in \mathcal{B}_3, (closure)$.
(ii) *For any* $x, y \in \mathcal{B}_3, x.y \in \mathcal{B}_3$*,(closure).*

4. Conclusion

In existing applications, the three-values logic in B are operated. This strengthens the common point of mechanical logic and digital logic. The B_3 structure greatly contributes to the new mechanical structure of present day computers.

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