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IRRESOLUTE TOPOLOGICAL RING WITH INHERENT PROPERTIES

SHALLU SHARMA*, TSERING LANDOL AND SAHIL BILLAWRIA

Abstract. We studied new notions of analogues of topological rings. Salih [10] acquaints us with the notion of irresolute topological ring in 2018. In this paper, we further studied the space closely and characterized indispensable properties of the space. We prove that every open subset of an irresolute topological ring is irresolute topological ring. We also obtained the equivalent condition of neighborhood of an element in an irresolute topological ring. It is proved that ring homeomorphism of an irresolute topological ring is irresolute if it is irresolute at identity element e in the irresolute topological ring $\mathcal{R}$.

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1. Introduction

The theory of Topological ring is one of the most recent branch of analysis. Topological ring is a ring endowed with a topology on which the algebraic operation $(a, b) \to a-b$, $(a, b) \to a.b$ are continuous. The foundation of the Topological ring was laid by Van Dantzig [2] in his thesis in 1931. Its study was initiated by Kothe, Lorch, Kalpinsky [4, 5, 6] and many more. In the last few decades, the concept of Topological ring found its own way of development in different direction. Recently, M. Ram et.al. developed the concept of α-irresolute topological ring [11] and semi-topological ring [12]. The reader can find in details about the topological ring and its extension in [4, 5, 6].

Definition 1.1. [13] A pair $(\mathcal{R}, \tau)$, where $\mathcal{R}$ is a ring and τ is a topology on $\mathcal{R}$, is called topological ring if the following conditions are satisfied:
(1) the mapping $\mathcal{R} \times \mathcal{R} \to \mathcal{R}, (x, y) \to x + y$ is continuous;

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(2) the mapping $\mathcal{R} \to \mathcal{R}, x \to -x$, is continuous;
(3) the mapping $\mathcal{R} \times \mathcal{R} \to \mathcal{R}, (x, y) \to xy$ is continuous.
The axioms of topological ring can be written in terms of neighborhoods of elements as follows:
(1) For any two elements $x, y \in \mathcal{R}$ and each neighborhood U_{x+y} of x+y, there exist neighborhoods U_x, U_y of x and y respectively in such that $U_x + U_y \subseteq U_{x+y}$;
(2) For each element $x \in \mathcal{R}$ and each neighborhood U_{-x} of $-x$, there exist neighborhood V_x of x such that $-V_x \subseteq U_{-x}$;
(3) For any two elements $x, y \in \mathcal{R}$ and each neighborhood U_{xy} of xy, there exist neighborhoods U_x, U_y of x and y respectively in such that $U_x U_y \subseteq U_{xy}$.

Our aim here is to complement the sources and study a new class of topological ring called irresolute-topological ring [10] which was introduced by M. Salih in 2018.

Definition 1.2. [10] Let $(\mathcal{R}, \tau)$ be a ring endowed with a topology τ such that the following conditions are satisfied:
(i) For each $a, b \in \mathcal{R}$ and each semi-open set C in $\mathcal{R}$ containing $a + b$, there exist semi-open sets A and B in $\mathcal{R}$ containing a and b respectively, such that $A + B \subseteq C$;
(ii) For each $a \in \mathcal{R}$ and semi-open set B in $\mathcal{R}$ containing $-a$, there exist semi-open set A in $\mathcal{R}$ containing a such that $-A \subseteq B$; and
(iii) For each $a, b \in \mathcal{R}$ and each semi-open set C in $\mathcal{R}$ containing ab, there exist semi-open sets A and B in $\mathcal{R}$ containing a and b respectively, such that $AB \subseteq C$.
Then the pair $(\mathcal{R}, \tau)$ is called irresolute topological ring.

Note that $(\mathcal{R}, \tau)$ is said to be irresolute topological ring with unity if $(\mathcal{R}, \tau)$ is irresolute topological ring and $\mathcal{R}$ is ring with unity. We represents $\mathcal{R}^*$ as set of all invertible elements in $\mathcal{R}$.

2. Preliminaries

Definition 2.1. A subset P of a topological space X is called semi-open [8] if $P \subseteq \overline{P^\circ}$, where "$\overline{P}$" and "$P^\circ$" denotes closure and interior of set P respectively.

The complement of a semi-open set is called semi-closed set [8]. Throughout the paper, we denote the collection of semi-open sets of $\mathcal{R}$ by $\mathcal{SO}(\mathcal{R})$.
The intersection of all semi-closed sets in X containing a subset P of X is called semi-closure[8] of P and is denoted by $\mathcal{C}l_s(P)$. A subset P of X is semi-closed if and only if $P = \mathcal{C}l_s(P)$. A point $a \in \mathcal{C}l_s(P) \Leftrightarrow$ there exist $Q \in \mathcal{SO}(\mathcal{R})$ containing a such that $Q \cap P \setminus \{a\} \neq \emptyset$. A point $a \in X$ is termed as semi-interior point[8] of a subset P of X if there exist semi-open set Q in X containing a such that $Q \subseteq P$. The union of all semi-open set in X that are contained in $P \subseteq X$ is called semi-interior [8]. We denote set of all semi-interior points of P by $\mathcal{I}nt_s(P)$. $\mathcal{N}_a(R)$ denote the semi-open neighborhood of a in $\mathcal{R}$.

3. Characterizations

Theorem 3.1. *Let O be a semi-open set in an irresolute topological ring $\mathcal{R}$. Then*
(i) $-O \in \mathcal{SO}(\mathcal{R})$.
(ii) $c + O \in \mathcal{SO}(\mathcal{R})$, for each $c \in \mathcal{R}$.

Proof. (i) Let a be an element of $-O$. Then by hypothesis, there exist semi-open neighborhood M of a such that $-M \subseteq O$. This gives $a \in M \subseteq -O$, which implies a is semi-interior point of $-O$, i.e. $a \in \mathcal{I}nt_s(-O)$. Hence, $-O = \mathcal{I}nt_s(-O)$. Therefore, $-O \in \mathcal{SO}(\mathcal{R})$.

(ii) Consider an element $a \in c+O$. Then by definition of irresolute topological ring $\mathcal{R}$, there exist semi-open neighborhoods M and N of $-c$ and a respectively such that $M + N \subseteq O$. In particular, $-c + N \subseteq O$. This means, $a \in N \subseteq c + O$, which implies a is semi-interior point of $c + O$, i.e. $a \in \mathcal{I}nt_s(c + O)$. Hence, $c + O = \mathcal{I}nt_s(c + O)$. Therefore, $c + O \in \mathcal{SO}(\mathcal{R})$. □

Corollary 3.2. *Let O be any semi-open set in irresolute topological ring $\mathcal{R}$. Then $M + O \in \mathcal{SO}(\mathcal{R})$, for any $M \subseteq \mathcal{R}$.*

Corollary 3.3. *Let O be any semi-open set in irresolute topological ring $\mathcal{R}$. Then*
(i) $-O \subseteq \overline{(-O)^\circ}$.
(ii) $c + O \subseteq \overline{(c + O)^\circ}$, for each $c \in \mathcal{R}$.

Corollary 3.4. *For any semi-closed set C in irresolute topological ring $\mathcal{R}$. Then*
(i) $-C \subseteq (\overline{-C})^\circ$
(ii) $a + C \subseteq (\overline{a + C})^\circ$, for each $a \in \mathcal{R}$.

Theorem 3.5. *Let $(\mathcal{R}, \tau)$ be irresolute topological ring. For any subset O of $(\mathcal{R}, \tau)$, we have*
(i) $\mathcal{C}l_s(-O) = -\mathcal{C}l_s(O)$, and
(ii) $\mathcal{C}l_s(c + O) = c + \mathcal{C}l_s(O)$, for each $c \in \mathcal{R}$.

Proof. (i) Consider $b \in \mathcal{C}l_s(-O)$. Let $a = -b$ and M be semi-open neighborhood of a in $\mathcal{R}$, i.e. $M \in \mathcal{N}_a(\mathcal{R})$. Then, there exist $N \in N_{-a}(\mathcal{R})$ such that $-N \subseteq M$. By assumption, $N \cap (-O) \neq \emptyset$, which implies there is some $x \in (-O) \cap N$. That is, $-x \in O \cap M \neq \emptyset$. Hence, $a \in \mathcal{C}l_s(O)$ implies that $-b \in \mathcal{C}l_s(O)$. Thus, $b \in -\mathcal{C}l_s(O)$.
Conversely, suppose $a \in -\mathcal{C}l_s(O)$. Then $a = -b$, for some $b \in \mathcal{C}l_s(O)$. Let M be semi-open neighborhood of a in $\mathcal{R}$. Then by hypothesis, there exist $N \in \mathcal{N}_b(\mathcal{R})$ such that $-N \subseteq M$. Now, $b \in \mathcal{C}l_s(O)$ implies $O \cap N \neq \emptyset$. Hence, there is some $x \in O \cap N$ which gives $-x \in (-O) \cap M$. It implies that $a \in \mathcal{C}l_s(-O)$, which completes the proof.

(ii) Consider an element $b \in \mathcal{C}l_s(c + O)$. Let $a = -c + b$ and P be any semi-open neighborhood of a in $\mathcal{R}$. Then, by definition of irresolute topological ring

$\mathcal{R}$, there exist semi-open neighborhood M and N such that $-c \in M$, $b \in N$ and $M + N \subseteq P$. Also, $b \in \mathcal{C}l_s(c+O)$ implies $(c+O) \cap N \neq \emptyset$. Hence, there exist some $x \in (c+O) \cap N$ which gives $-c+x \in O \cap (M+N) \subseteq O \cap P$, which implies $O \cap P \neq \emptyset$ and consequently, $a \in \mathcal{C}l_s(O)$, *i.e*, $-c+b \in \mathcal{C}l_s(O)$. It implies that $b \in c + \mathcal{C}l_s(O)$. Hence $\mathcal{C}l_s(c+O) \subseteq c + \mathcal{C}l_s(O)$.
For the converse part, let $a \in c + \mathcal{C}l_s(O)$. Then $a = c + b$, where $b \in \mathcal{C}l_s(O)$. Let P be any semi-open neighborhood of a in $\mathcal{R}$, then by hypothesis, we have $M, N \in \mathcal{SO}(\mathcal{R})$ such that $c \in M, b \in N$ and $M + N \subseteq P$. Since $b \in \mathcal{C}l_s(O)$ implies $O \cap N \neq \emptyset$. Consequently, $(c+O) \cap P \neq \emptyset$, i.e. $a \in \mathcal{C}l_s(c+O)$. Hence the proof. □

Theorem 3.6. *Let $(\mathcal{R}, \tau)$ be irresolute topological ring with unity. If $O \in \mathcal{SO}(\mathcal{R})$, then $rO, Or, rOr \in \mathcal{SO}(\mathcal{R})$, for all $r \in \mathcal{R}^*$, where $\mathcal{R}^*$ denotes the irresolute topological ring with unity.*

Proof. Firstly, we will show that $rO \in \mathcal{SO}(\mathcal{R})$. Choose an element $a \in rO$. We need to show that $a \in \mathcal{I}nt_s(rO)$. Since $a \in rO$ and $r \in \mathcal{R}^*$ then there exist $r^{-1} \in \mathcal{R}$ such that $r^{-1}a \in O$. By definition of irresolute topological ring $\mathcal{R}$, there exists semi-open neighborhood M of r^{-1} and N of a such that $MN \subseteq O$. This means that $r^{-1}N \subseteq O$ which gives $N \subseteq rO$. Thus, there exists semi-open neighborhood N of a such that $a \in N \subseteq rO$, which implies that a is a semi-interior point of rO. That is, $a \in \mathcal{I}nt_s(rO)$. Hence, $rO \in \mathcal{SO}(\mathcal{R})$.

Similarly, we can also show that $Or, rOr \in \mathcal{SO}(\mathcal{R})$.

□

Theorem 3.7. *Let $(\mathcal{R}, \tau)$ be irresolute topological ring with unity. Then, for each $r \in \mathcal{R}^*$ and $O \in \mathcal{SO}(\mathcal{R})$, the following holds*
(a) $\mathcal{C}l_s(rO) = r\mathcal{C}l_s(O)$.
(b) $\mathcal{C}l_s(Or) = \mathcal{C}l_s(O)r$.
(c) $\mathcal{C}l_s(rOr) = r\mathcal{C}l_s(O)r$.

Proof. (a) Let $a \in \mathcal{C}l_s(rO)$ and consider $b = r^{-1}a$. Let C be a semi-open set in $\mathcal{R}$ containing b. Then, we obtain semi-open sets A and B containing r^{-1} and a respectively such that $AB \subseteq C$. Since $a \in \mathcal{C}l_s(rO)$, it implies $(rO) \cap B \neq \emptyset$. Let $x \in rO \cap B$ and $\mathcal{R}$ is an irresolute topological ring with unity so there exists $r^{-1} \in \mathcal{R}$ such that $r^{-1}r = e = rr^{-1}$. Hence, $r^{-1}x \in O \cap AB \subseteq O \cap C$. It implies that $O \cap C \neq \emptyset$. Thus, $b \in \mathcal{C}l_s(O)$. That is, $r^{-1}a \in \mathcal{C}l_s(O)$ implies that $a \in r\mathcal{C}l_s(O)$, which completes the direct part.
For the converse part, let $b \in r\mathcal{C}l_s(O)$, where $r \in \mathcal{R}^*$. Then $b = ra$, where $a \in \mathcal{C}l_s(O)$. Let C be any semi-open neighborhood of b in $\mathcal{R}$. Then there exist semi-open neighborhoods A and B of r and a respectively such that $AB \subseteq C$. Also, $a \in \mathcal{C}l_s(O)$ implies $O \cap B \neq \emptyset$. So there exists some $x \in O \cap B$ which gives $rx \in (rO) \cap (AB) \subseteq (rO) \cap C$. That is, $rO \cap C \neq \emptyset$ which gives $b \in \mathcal{C}l_s(rO)$. Thus $\mathcal{C}l_s(rO) = r\mathcal{C}l_s(O)$ for each $r \in \mathcal{R}^*$.
(b) and (c) are trivial, hence omitted. □

Theorem 3.8. *Let $(\mathcal{R}, \tau)$ be an irresolute topological ring, then for any subset O of $\mathcal{R}$, the following assertions hold*
(a) $\mathcal{I}nt_s(-O) = -\mathcal{I}nt_s(O)$, *for each* $r \in \mathcal{R}^*$.
(b) $\mathcal{I}nt_s(a+O) = a + \mathcal{I}nt_s(O)$, *for each* $r \in \mathcal{R}^*$.

Proof. (a) Firstly, we shall show that $\mathcal{I}nt_s(-O) \subseteq -\mathcal{I}nt_s(O)$. For, let $b \in \mathcal{I}nt_s(-O)$, then we can write $b = -a$, for some $a \in O$. By hypothesis, there exists semi-open neighborhood A of a such that $-A \subseteq \mathcal{I}nt_s(-O)$ which implies that $A \subseteq O$ i.e, $a \in A \subseteq O$, which means a is semi-interior point of O. Hence $-b \in \mathcal{I}nt_s(O)$. We get $b \in -\mathcal{I}nt_s(O)$. Thus, $\mathcal{I}nt_s(-O) \subseteq -\mathcal{I}nt_s(O)$.
Next, we shall show that $-\mathcal{I}nt_s(O) \subseteq \mathcal{I}nt_s(-O)$. Consider an element $b \in -\mathcal{I}nt_s(O)$, then we can write b as $b = -a$, for some $a \in \mathcal{I}nt_s(O) \subseteq O$. Thus, there exists semi-open neighborhood A of a in $\mathcal{R}$ such that $-A \subseteq -O$. Since $A \in \mathcal{SO}(\mathcal{R})$ then by Theorem 3.1, we have $-A \in \mathcal{SO}(\mathcal{R})$. Therefore, $b = -a \in -A \subseteq -O$, which implies that b is a semi-interior point of $-O$. Thus $b \in \mathcal{I}nt_s(-O)$, which completes the reverse part. Hence $\mathcal{I}nt_s(-O) = -\mathcal{I}nt_s(O)$.

(b) Let $c \in \mathcal{I}nt_s(a+O)$. Then $c = a+b$, for some $b \in O$. Then, by hypothesis, we get semi-open neighborhoods A and B containing a and b respectively, such that $A+B \subseteq \mathcal{I}nt_s(a+O)$. In particular, $c = a+b \in \mathcal{I}nt_s(a+O) \subseteq a+O$. This gives $\mathcal{I}nt_s(a+O) \subseteq a + \mathcal{I}nt_s(O)$, which completes the direct part.
Conversely, let $b \in a + \mathcal{I}nt_s(O)$. So, we have $-a+b \in \mathcal{I}nt_s(O)$. Since $(\mathcal{R}, \tau)$ is an irresolute topological ring, there exist semi-open neighborhoods A of $-a$ and B containing b such that $A+B \subseteq \mathcal{I}nt_s(O) \subseteq O$. It implies that $-a+B \subseteq O$ which also imply that $B \subseteq a+O$. Thus, b is a semi-interior point of $a+O$. Therefore, $a + \mathcal{I}nt_s(O) \subseteq \mathcal{I}nt_s(a+O)$, which completes the proof. □

Theorem 3.9. *Let O be a subset of an irresolute topological ring $\mathcal{R}$ with unity. Then, the following holds:*
(a) $\mathcal{I}nt_s(rO) = r\mathcal{I}nt_s(O)$ *for each* $r \in \mathcal{R}^*$.
(b) $\mathcal{I}nt_s(O.r) = \mathcal{I}nt_s(O)r$ *for each* $r \in \mathcal{R}^*$.
(c) $\mathcal{I}nt_s(rOr) = r\mathcal{I}nt_s(O)r$ *for each* $r \in \mathcal{R}^*$

Proof. (a) Consider an element $b \in \mathcal{I}nt_s(rO)$. Then, $b = ra$, for some $a \in O$. By definition of an irresolute topological ring $\mathcal{R}$, we obtain semi-open neighborhood A of r and B of a such that $AB \subseteq rO$. Particularly, $rB \subseteq rO$. Also, B is semi-open set, which implies rB is semi-open by Theorem 3.6 and hence $b \in rB \subseteq \mathcal{I}nt_s(O)$. Conversely, suppose $b \in r\mathcal{I}nt_s(O)$. Since $\mathcal{R}$ is ring with unity, $\exists r \in \mathcal{R}^*$ such that $r^{-1}b \in \mathcal{I}nt_s(O)$. By hypothesis, there exist semi-open neighborhood A and B of r^{-1} and b respectively such that $AB \subseteq \mathcal{I}nt_s(O) \Rightarrow r^{-1}B \subseteq \mathcal{I}nt_s(O) \subseteq O \Rightarrow B \subseteq rO$, i.e. $b \in B \subseteq rO$, which implies b is semi-interior point of rO. Hence, $b \in \mathcal{I}nt_s(rO)$. Hence the proof. □

Definition 3.10. [9] Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a single valued function between topological spaces $\mathcal{X}$ and $\mathcal{Y}$. Then $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is termed as irresolute if for each semi-open set V in $\mathcal{Y}$, $f^{-1}(V)$ is semi-open in $\mathcal{X}$.

Definition 3.11. [1] Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a single valued function between topological spaces. Then $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is termed as pre-semi-open if for every semi-open set A of $\mathcal{X}$, $f(A)$ is semi-open in $\mathcal{Y}$.

Definition 3.12. [1] A mapping f from a topological space to itself is called semi-homeomorphism, if it is bijective, irresolute and pre-semi open.

Definition 3.13. [3] A space $\mathcal{X}$ is called semi-compact if every cover of $\mathcal{X}$ by semi open sets has a finite subcover.

Lemma 3.14. *Let $(\mathcal{R}, \tau)$ be an irresolute topological ring with unity. Then the mapping*
(a) $\phi_a : \mathcal{R} \to \mathcal{R}$ defined by $\phi_a(x) = a + x$
(b) $\varphi : \mathcal{R} \to \mathcal{R}$ defined by $\varphi(a) = -a$
(c) $h : \mathcal{R} \to \mathcal{R}$ defined by $h(a) = r.a \ \forall \ a \in R$ and $r \in \mathcal{R}^*$ are irresolute.

Proof. The proof of part (a) and (b) follows directly from Theorem 3.1 and part(c) follows from the Theorem 3.6. □

4. Main Results

Theorem 4.1. *Let $(\mathcal{R}, \tau)$ be an irresolute topological ring with unity. Then the mapping ϕ_a, φ and h defined in Lemma 3.14 is irresolute homeomorphism onto itself.*

Proof. Clearly, ϕ_a is irresolute and bijective by Lemma 3.14. We need to show that ϕ_a is pre-semi open. For, let U be any semi-open set in $\mathcal{R}$, then $\phi_a(U) = a+U$ is semi-open by Theorem 3.1, therefore for any semi-open set U in $\mathcal{R}$, $\phi_a(U)$ is semi-open. Next, we shall show that $\varphi(a) = -a$ is semi-homoemorphism. In reference to Lemma 3.14, we only need to show that φ is pre-semi open. Let U be any semi-open set in $\mathcal{R}$. Then $\varphi(U) = -U$ and $-U$ is semi-open for any semi-open set U in irresolute topological ring $\mathcal{R}$. Therefore, φ is also semi-hoemeomorphism. Similarly, using Lemma 3.14 and Theorem 3.6, h is also semi-homeomorphism. □

Theorem 4.2. *Let $\mathcal{R}$ be an irresolute topological ring with unitary element, $r \in \mathcal{R}$ be an invertible element and $x \in \mathcal{R}$. Then the following statements are equivalent*
(a) U is semi-open neighborhood of element x in $\mathcal{R}$.
(b) $U.r$ is semi-open neighborhood of element $x.r$ in $\mathcal{R}$.
(c) $r.U$ is semi-open neighborhood of element $r.x$ in $\mathcal{R}$.

Proof. (a)$\Rightarrow$ (b)
Consider the function $\psi_r : \mathcal{R} \to \mathcal{R}$ defined by $\psi_r(x) = x.r$.
Clearly ψ_r is bijection and irresolute. Let U be semi-open subset of $\mathcal{R}$. Let

$b' \in \psi_r(U)$, then $b' = \psi_r(b) = b.r$, for some $b \in U$ i.e. $b' = b.r$ or $b = b'.r^{-1} \in U$. By definition of irresolute topological ring $\mathcal{R}$, there exist semi-open set V containing b' such that $V.r^{-1} \subseteq U \Rightarrow V \subseteq U.r = \psi_r(U)$. But $U.r$ is semi-open which implies $\psi_r(U)$ is semi-open. Hence, $\psi_r(U)$ is semi-open, for every semi-open set U. Therefore, ψ_r is semi-homeomorphism. Since U is semi-open neighborhood of x in $\mathcal{R}$ and $x.r = \psi_r(x)$ and $U.r = \psi_r(U)$ is semi-open neighborhood of $x.r$ in $\mathcal{R}$. Hence, (a) $\Rightarrow$ (b).
(b)$\Rightarrow$ (c)
Let $U.r$ is semi-open neighborhood of element $x.r$ in $\mathcal{R}$.
To prove: $r.U$ is semi-open neighborhood of element $r.x$ in $\mathcal{R}$. Consider the function $\psi'_r : \mathcal{R} \to \mathcal{R}$ defined by $\psi'_r(x) = r.x$. ψ'_r is semi-homeomorphism in the similar way as above.
Define $\theta_r : \mathcal{R} \to \mathcal{R}$ by $\theta_r(x) = rzr^{-1}$, for every $z \in \mathcal{R}$, which is composition of semi-homeomorphic mapping ψ_r and ψ'_r. And hence θ_r is a semi-homeomorphism. Since $\theta_r(x.r) = r(xr)r^{-1} = rx$ and $\theta_r(Ur) = rU$.
Hence, $r.U$ is semi-open neighborhood of the element $r.x$ in $\mathcal{R}$.
(c)$\Rightarrow$ (a)
Let $r.U$ is a semi-open neighborhood of the element $r.x$ in $\mathcal{R}$.
To prove: U is a semi-open neighborhood of the element x in $\mathcal{R}$.
Since $\psi'_{r^{-1}}(r.x) = r^{-1}rx = x$ and $\psi'_{r^{-1}}(r.U) = U$.
It follows that U is a semi-open neighborhood of the element x in $\mathcal{R}$. □

Lemma 4.3. [7] *Let $(\mathcal{X}, \tau)$ be a topological space and $A \subseteq \mathcal{X}$. If A is open and U is semi open, then $A \cap U \in \mathcal{SO}(\mathcal{X})$.*

Theorem 4.4. *Every open subset of an irresolute topological ring $\mathcal{R}$ is also an irresolute topological ring.*

Proof. (1) Let $x, y \in H$ and let $W \subseteq H$ be semi-open nbd of x-y.
Since H is open and W is semi-open in $\mathcal{R}$ then $W \cap H = W$ is semi-open in $\mathcal{R}$ containing $x - y$. By definition of irresolute topological ring, there exist semi-open neighborhoods A and B of $\mathcal{R}$ containing x and $-y$ respectively such that $A - B \subseteq W$.
Set $U = A \cap H$ and $V = B \cap H$ be semi-open subsets of H containing x and y and satisfies $U - V = (A \cap H) - (B \cap H) = (A - B) \cap H \subseteq W \cap H = W$.
(2) Let $x, y \in H$ and let $W \subseteq H$ be semi-open neighborhood of xy. As H is open and W is semi-open in $\mathcal{R}$ and $W \cap H = W$ is semi-open in $\mathcal{R}$. By definition of irresolute topological ring $\mathcal{R}$, there exist semi-open neighborhoods A and B of $\mathcal{R}$ containing x and y such that $A.B \subseteq W$. Set $U = A \cap H, V = B \cap H$ be semi-open in H containing x and y respectively (Lemma 4.3) and satisfies $U.V = (A \cap H).(B \cap H) = A.B \cap H \subseteq W \cap H \subseteq W$. □

Theorem 4.5. *Let $\mathcal{R}$ and $\mathcal{S}$ be two irresolute topological rings and let $\rho : \mathcal{R} \to \mathcal{S}$ be a ring homomorphism. If ρ is irresolute at identity, then ρ is irresolute on $\mathcal{R}$.*

Proof. Let $a \in \mathcal{R}$ and C be any semi-open neighborhood of $b = \rho(a)$ in $\mathcal{S}$. Since $\phi_b : \mathcal{S} \to \mathcal{S}$ is irresolute from Lemma 3.14, $\exists$ a semi-open neighborhood B of 0 such that $\phi_b(B) = B + b \subset C$. Also, ρ is irresolute at $0 \in \mathcal{R}$, there exists semi-open neighborhood A of 0 in $\mathcal{R}$ such that $\rho(A) \subseteq B$. By Lemma 3.14, $\phi_a : \mathcal{R} \to \mathcal{R}$ is irresolute, so $A + a$ is semi-open neighborhood of a. Hence, $\rho(A + a) = \rho(A) + \rho(a) = \rho(A) + b \subseteq B + b \subseteq C$.
This proves that ρ is irresolute at arbitrary element $a \in \mathcal{R}$, and hence on $\mathcal{R}$. □

Theorem 4.6. *Let $(\mathcal{R}, \tau)$ be an irresolute topological ring. Let A and B be subsets of R such that A is semi-compact and B is semi-closed satisfying $A \cap B = \phi$. Then there exists a set $S \in \mathcal{SO}(\mathcal{R})$ with the property $(A + S) \cap (B + S) = \phi$.*

Proof. Consider $\mathcal{S}_o$ be the class of all semi-open sets containing "0". Let $x \in A$. This gives $x \notin B$. That is, $x \in B^c$. Now $0 + 0 = 0 \in B^c - \{x\}$. By Theorem 3.1, $B^c - \{x\}$ is semi-open. Thus $\exists\ S_1,\ S_2 \in \mathcal{S}_o$ such that $S_1 + S_2 \subseteq B^c - \{x\}$.
Let $S_x = S_1 \cap S_2 \cap (-S_1) \cap (-S_2)$. Then clearly S_x is symmetric and $S_x \in \mathcal{S}_o$ such that

$$x + S_x + S_x \subseteq B^c$$
$$\text{i.e, } x + S_x + S_x \cap B = \phi$$
$$\text{Or } x + S_x \cap B - S_x = \phi$$
$$\text{Or } x + S_x \cap B + S_x = \phi. \text{ ——(1)}$$

We can get a family of semi-open sets $\pi = \{x + S_x : S_x \in \mathcal{S}_o,\ x \in A\}$. Since A is semi-compact, we have finite set, say, $D \subseteq A$ such that $A \subseteq \cup\{x \in S_x : S_x \in \mathcal{S}_o,\ x \in D\}$.
Consider the set $S = \cap\{S_x : x \in D\}$. Thus S is semi-open in $\mathcal{R}$.
From (1), $(A + S) \cap (B + S) = \phi$. □

5. Conclusion

In this paper, we continued the research on irresolute topological ring introduced by Salih [10]. Section 1 and 2 familiarize the basics and origin of irresolute topological ring along with the important definition. In section 3, significant characterizations of irresolute topological ring are obtained. Section 4 includes the main results of the paper. We obtained the equivalent condition of neighborhood of an element in an irresolute topological ring and many more.

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Shallu Sharma received Ph.D. from University of Jammu. She is currently assistant professor at University of Jammu since 2008. Her research interest include functional analysis, operator theory and topological vector space.

Department of Mathematics, University of Jammu, JK-180006, India.
e-mail: shallujamwal09@gmail.com

Tsering Landol received M.Sc. from University of Jammu. She is currently pursuing a Ph.D. from University of Jammu since 2018. Her research interests are functional analysis and topology with algebra.

Department of Mathematics, University of Jammu, JK-180006, India.
e-mail: tseringlandol09@gmail.com

Sahil Billawria received M.Sc. from University of Jammu. He is currently pursuing Ph.D. from University of Jammu since 2018. His research interest are functional analysis and topological vector spaces.

Department of Mathematics, University of Jammu, JK-180006, India.
e-mail: sahilbillawria2@gmail.com