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SOME IDENTITIES OF (p, q)-POLY-COSINE TANGENT AND (p, q)-POLY-SINE TANGENT POLYNOMIALS†

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Abstract. In this paper we give some prperties of the (p, q)-poly-cosine tangent polynomials and (p, q)-poly-sine tangent polynomials.

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1. Introduction

Mathematicians have been working on Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials (see [1, 2, 3, 4, 5, 6, 10, 11, 12, 14]). It is well known that the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x = 0, B_n = B_n(0)$ are called the Bernoulli numbers. The tangent polynomials are given by the generating function to be

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0, T_n = T_n(0)$ are called the tangent numbers (see [5, 6, 7]).

The Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order r are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi). \tag{3}$$

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The Frobenius–Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{4}$$

The values at $x=0$ are called Frobenius-Euler numbers of order r; when $r=1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers. The cosine-tangent polynomials $T_n^{(C)}(x,y)$ and sine-tangent polynomials $T_n^{(S)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\cos yt, \tag{5}$$

and

$$\sum_{n=0}^{\infty} T_n^{(k,S)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\sin yt, \tag{6}$$

respectively.

For any integer k and $0 \le q < p \le 1$, let $Li_{k,q}(t)$ be the power series given by

$$Li_{k,p,q}(t) = \sum_{m=1}^{\infty} \frac{t^m}{[m]_{p,q}^k}, \tag{7}$$

where $[n]_{p,q} = \dfrac{p^n - q^n}{p-q}$ is the (p,q)-integer.

Note that if $p=1$, then $\lim_{q\to 1}[x]_{p,q} = x$ and $\lim_{q\to 1} Li_{k,p,q}(t) = Li_k(t)$, where $Li_k(t)$ is the kth polylogarithm function. In this paper, we introduce some special polynomials which are related to tangent polynomials. In addition, we give some identities for these polynomials. Finally, we investigate the distribution of zeros of these polynomials.

2. (p,q)-poly-cosine tangent and (p,q)-poly-sine tangent polynomials

In this section, we define the (p,q)-poly-cosine tangent and (p,q)-poly-sine tangen polynomials. In [9, 11, 12], we introduced poly-tangent numbers and polynomials, poly-cosine tangent and poly-sine tangent polynomials, and q-poly-tangent numbers and polynomials. After that we investigated some their properties. We also obtained some relationships both between these polynomials and tangent polynomials and between these polynomials and cauchy numbers. For any integer k, the modified poly-tagent polynomials $T_n^{(k)}(z)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty} T_n^{(k)}(z)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{zt}. \tag{8}$$

The numbers $T_n^{(k)}(0) := T_n^{(k)}$ are called the poly-tagent numbers.

For $0 < q < p \leq 1$ and any integer k, the (p,q)-poly-Bernoulli polynomials $B_{n,p,q}^{(k)}(x)$, the (p,q)-poly-Euler polynomials $E_{n,p,q}^{(k)}(x)$, and the (p,q)-poly-tangent polynomials $T_{n,p,q}^{(k)}(x)$ are defined by means of the following generating functions:

$$\begin{aligned} \frac{\mathrm{Li}_{k,p,q}(1-e^{-t})}{1-e^{-t}}e^{xt} &= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x)\frac{t^n}{n!}, \\ \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{e^{t}+1}e^{xt} &= \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x)\frac{t^n}{n!}, \\ \frac{2\mathrm{Li}_{k,p,q}(1-e^{-t})}{e^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty} T_{n,p,q}^{(k)}(x)\frac{t^n}{n!}. \end{aligned} \tag{9}$$

Now, we define modified (p,q)-poly-tangent numbers and polynomials.

Definition 2.1. For any integer k and $0 < q < p \leq 1$, the modified (p,q)-poly-tagent polynomials $T_{n,p,q}^{(k)}(x)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k)}(z)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{zt}. \tag{10}$$

The numbers $\mathcal{T}_{n,p,q}^{(k)}(0) := \mathcal{T}_{n,p,q}^{(k)}$ are called the modified (p,q)-poly-tagent numbers. If $p = 1$, then

$$\lim_{q\to 1} \mathcal{T}_{n,p,q}^{(k)}(z) = T_n^{(k)}(z), \lim_{q\to 1} \mathcal{T}_{n,p,q}^{(k)} = T_n^{(k)}.$$

Theorem 2.2. *For $n > 0$, we have*

$$T_{n-1,p,q}^{(k)}(z) = \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}\left(T_l(z) - T_l(z-1)\right)B_{n-l,p,q}^{(k)}.$$

Proof. By (2), (9), and (10) and by using Cauchy product, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,p,q}^{(k)}(z)\frac{t^{n+1}}{n!} &= \left(\frac{\mathrm{Li}_{k,p,q}(1-e^{-t})}{1-e^{-t}}\right)\frac{2(1-e^{-t})}{e^{2t}+1}e^{zt} \\ &= \left(\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(T_n(z) - T_n(z-1)\right)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\left(T_l(z) - T_l(z-1)\right)B_{n-l,p,q}^{(k)}\right)\frac{t^n}{n!}. \end{aligned} \tag{11}$$

By comparing the coefficients on both sides of (11), we have the theorem related the (p,q)-poly-tangent polynomial, (p,q)-poly-Bernoulli polynomials, and tangent polynomials. □

Now, we consider the (p,q)-poly-tagent polynomials that are given by the generating function to be

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k)}(x+iy)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{(x+iy)t}. \tag{12}$$

On the other hand, we note that

$$e^{(x+iy)t} = e^{xt}e^{iyt} = e^{xt}(\cos yt + i\sin yt). \tag{13}$$

From (12) and (13), we obtain

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k)}(x+iy)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(\cos yt + i\sin yt), \tag{14}$$

and

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k)}(x-iy)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(\cos yt - i\sin yt). \tag{15}$$

Hence, by (14) and (15), we obtain

$$\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt = \sum_{n=0}^{\infty}\left(\frac{\mathcal{T}_{n,p,q}^{(k)}(x+iy)+\mathcal{T}_{n,p,q}^{(k)}(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{16}$$

and

$$\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\sin yt = \sum_{n=0}^{\infty}\left(\frac{\mathcal{T}_{n,p,q}^{(k)}(x+iy)-\mathcal{T}_{n,p,q}^{(k)}(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{17}$$

It follows that we define the following (p,q)-poly-cosine tangent and (p,q)-poly-sine-tangent polynomials.

Definition 2.3. The (p,q)-poly-cosine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,C)}(x,y)$ and (p,q)-poly-sine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,S)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k,C)}(x,y)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt, \tag{18}$$

and

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}^{(k,S)}(x,y)\frac{t^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\sin yt, \tag{19}$$

respectively.

Note that $\mathcal{T}_{n,p,q}^{(k,C)}(x,0) = \mathcal{T}_{n,p,q}^{(k)}(x), \mathcal{T}_{n,p,q}^{(k,S)}(x,0) = 0, (n \geq 0)$.

By (16)-(19), we have

$$\mathcal{T}_{n,p,q}^{(k,C)}(x,y) = \frac{\mathcal{T}_{n,p,q}^{(k)}(x+iy)+\mathcal{T}_{n,p,q}^{(k)}(x-iy)}{2},$$

$$\mathcal{T}_{n,p,q}^{(k,S)}(x,y) = \frac{\mathcal{T}_{n,p,q}^{(k)}(x+iy)-\mathcal{T}_{n,p,q}^{(k)}(x-iy)}{2i}.$$

Clearly, we obtain the following explicit representations of $\mathcal{T}_{n,p,q}^{(k)}(x+iy)$

$$\mathcal{T}_{n,p,q}^{(k)}(x+iy)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k)}(x+iy)^{n-l},$$

$$\mathcal{T}_{n,p,q}^{(k)}(x+iy)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k)}(x)i^{n-l}y^{n-l}.$$

Let

$$e^{xt}\cos yt=\sum_{l=0}^{\infty}C_l(x,y)\frac{t^l}{l!},\qquad e^{xt}\sin yt=\sum_{l=0}^{\infty}S_l(x,y)\frac{t^l}{l!}. \tag{20}$$

Then, by Taylor expansions of $e^{xt}\cos yt$ and $e^{xt}\sin yt$, we get

$$e^{xt}\cos yt=\sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m}\right)\frac{t^l}{l!} \tag{21}$$

and

$$e^{xt}\sin yt=\sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}\right)\frac{t^l}{l!}, \tag{22}$$

where [ ] denotes taking the integer part (see [4]). By (20), (21) and (22), we get

$$C_l(x,y)=\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m},$$

and

$$S_l(x,y)=\sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}, (l\geq 0).$$

Now, we observe that

$$\begin{aligned}\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt&=\left(\sum_{l=0}^{\infty}\mathcal{T}_{l,p,q}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_m(x,y)\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k)}C_{n-l}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain the following theorem

Theorem 2.4. *For $n\geq 0$, we have*

$$\mathcal{T}_{n,p,q}^{(k,C)}(x,y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k)}C_{n-l}(x,y)$$

and

$$\mathcal{T}_{n,p,q}^{(k,S)}(x,y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k)}S_{n-l}(x,y).$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [12])

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k \text{ and } (x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{23}$$

respectively. Here, $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}. \tag{24}$$

Let

$$\begin{aligned} Li_{k,p,q}(1-e^{-t})e^{xt}\cos yt &= \sum_{l=0}^{\infty} C_{l,p,q}^{(k)}(x,y)\frac{t^l}{l!}, \\ Li_{k,p,q}(1-e^{-t})e^{xt}\sin yt &= \sum_{l=0}^{\infty} S_{l,p,q}^{(k)}(x,y)\frac{t^l}{l!}. \end{aligned} \tag{25}$$

Then, by (21), we get

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} &= \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_{p,q}^k} e^{xt}\cos yt, \\ &= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i e^{(x-i)t}\cos(yt) \\ &= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \sum_{n=0}^{\infty} C_n(x-i,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i C_n(x-i,y) \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

By (25) and (26), we get

$$\begin{aligned} C_{n,p,q}^{(k)}(x,y) &= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i C_n(x-i,y) \\ &= \sum_{l=0}^{n} \sum_{i=0}^{l} \sum_{m=0}^{[\frac{n-l}{2}]} \binom{n}{l} \binom{n-l}{2m} \frac{(-1)^{l+i+n} i! S_2(l,i)}{[i]_{p,q}^k} x^{n-l-2m} y^{2m}, \end{aligned}$$

and

$$\begin{aligned}S^{(k)}_{n,p,q}(x,y) &= \sum_{l=0}^{\infty}\frac{1}{[l+1]^k_{p,q}}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i S_n(x-i,y)\\ &= \sum_{l=0}^{n}\sum_{i=0}^{l}\sum_{m=0}^{[\frac{n-l-1}{2}]}\binom{n}{l}\binom{n-l}{2m+1}\frac{(-1)^{l+i+n}i!S_2(l,i)}{[i]^k_{p,q}}x^{n-l-2m-1}y^{2m+1}.\end{aligned}$$

A few of them are

$$\begin{aligned}&C^{(k)}_{0,q}(x,y)=1,\quad C^{(k)}_{1,q}(x,y)=1,\\ &C^{(k)}_{2,q}(x,y)=-1+\frac{2}{[2]^k_{p,q}}+2x,\\ &C^{(k)}_{3,q}(x,y)=1-\frac{6}{[2]^k_{p,q}}+\frac{6}{[3]^k_{p,q}}-3x+\frac{6}{[2]^k_{p,q}}x+3x^2-3y^2,\end{aligned}$$

and

$$\begin{aligned}&S^{(k)}_{0,q}(x,y)=0,\quad S^{(k)}_{1,q}(x,y)=0,\\ &S^{(k)}_{2,q}(x,y)=2y,\\ &S^{(k)}_{3,q}(x,y)=-3y+\frac{6}{[2]^k_{p,q}}y+6xy.\\ &S^{(k)}_{4,q}(x,y)=4y-\frac{24}{[2]^k_{p,q}}y+\frac{24}{[3]^k_{p,q}}y-12xy+\frac{24}{[2]^k_{p,q}}xy+12x^2y-4y^3.\end{aligned}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}^{(k,C)}_{n,q}(x,y)\frac{t^{n+1}}{n!} &= Li_{k,p,q}(1-e^{-t})e^{xt}\cos yt\frac{2}{e^{2t}+1}\\ &= \left(\sum_{n=0}^{\infty}C^{(k)}_{n,p,q}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}C^{(k)}_{l,p,q}(x,y)T_{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain the following theorem

Theorem 2.5. *For $n>0$, we have*

$$n\mathcal{T}^{(k,C)}_{n-1,p,q}(x,y)=\sum_{l=0}^{n}\binom{n}{l}C^{(k)}_{l,p,q}(x,y)T_{n-l}$$

and

$$n\mathcal{T}^{(k,S)}_{n-1,p,q}(x,y)=\sum_{l=0}^{n}\binom{n}{l}S^{(k)}_{l,p,q}(x,y)T_{n-l}.$$

From (18), we have

$$\begin{aligned}&2Li_{k,p,q}(1-e^{-t})e^{xt}\cos yt\\&=\left(\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(x,y)\frac{t^{n+1}}{n!}\right)\left(e^{2t}+1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}l\binom{n}{l}\mathcal{T}_{l-1,p,q}^{(k,C)}(x,y)2^{n-l}+n\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned}\tag{27}$$

By (18) and (27), we get

$$C_{n,p,q}^{(k)}(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}l\binom{n}{l}\mathcal{T}_{l-1,p,q}^{(k,C)}(x,y)2^{n-l}+n\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)\right).$$

Therefore, we obtain the following theorem

Theorem 2.6. *For $n>0$, we have*

$$C_{n,p,q}^{(k)}(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}l\binom{n}{l}\mathcal{T}_{l-1,p,q}^{(k,C)}(x,y)2^{n-l}+n\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)\right),$$

and

$$S_{n,p,q}^{(k)}(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}l\binom{n}{l}\mathcal{T}_{l-1,p,q}^{(k,S)}(x,y)2^{n-l}+n\mathcal{T}_{n-1,p,q}^{(k,S)}(x,y)\right).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(x+2,y)\frac{t^n}{n!}&=\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{(x+2)t}\cos yt\\&=\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(e^{2t}-1+1)\cos yt\\&=\frac{2}{t}Li_{k,p,q}(1-e^{-t})e^{xt}\cos yt-\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\end{aligned}$$

Hence we have

$$\sum_{n=0}^{\infty}\left(\mathcal{T}_{n,p,q}^{(k,C)}(x+2,y)+\mathcal{T}_{n,p,q}^{(k,C)}(x,y)\right)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\left(2C_{n,p,q}^{(k)}(x,y)\right)\frac{t^n}{n!}.$$

By comparing the coefficients on the both sides, we get

$$\mathcal{T}_{n-1,p,q}^{(k,C)}(x+2,y)+\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)=\frac{2}{n}C_{n,p,q}^{(k)}(x,y),\ (n\geq 1).$$

Therefore, we obtain the following theorem:

Theorem 2.7. *For $n\geq 1$, we have*

$$\mathcal{T}_{n-1,p,q}^{(k,C)}(x+2,y)+\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)=\frac{2}{n}C_{n,p,q}^{(k)}(x,y),$$

and

$$\mathcal{T}_{n-1,p,q}^{(k,S)}(x+2,y)+\mathcal{T}_{n-1,p,q}^{(k,S)}(x,y)=\frac{2}{n}S_{n,p,q}^{(k)}(x,y).$$

By (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(x+r,y)\frac{t^n}{n!}&=\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\right)e^{rt}\\&=\left(\sum_{l=0}^{\infty}\mathcal{T}_{l,p,q}^{(k,C)}(x,y)\frac{t^l}{l!}\right)\left(\sum_{l=0}^{\infty}r^l\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k,C)}(x,y)r^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on the both sides, we obtain the following theorem:

Theorem 2.8. *For $n\geq 0, r\in\mathbb{N}$, we have*

$$\mathcal{T}_{n,p,q}^{(k,C)}(x+r,y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k,C)}(x,y)r^{n-l},$$

and

$$\mathcal{T}_{n,p,q}^{(k,S)}(x+r,y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,p,q}^{(k,S)}(x,y)r^{n-l}.$$

By (18), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}\mathcal{T}_{n,p,q}^{(k,C)}(x,y)\frac{t^n}{n!}&=\frac{\partial}{\partial x}\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\right)\\&=\frac{2Li_{k,p,q}(1-e^{-t})}{e^{2t}+1}e^{xt}\cos yt\\&=\sum_{n=1}^{\infty}\left(n\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned}\tag{28}$$

Comparing the coefficients on the both sides of (28), we have

$$\frac{\partial}{\partial x}T_{n,q}^{(k,C)}(x,y)=nT_{n-1,q}^{(k,C)}(x,y).$$

Similarly, for $n\geq 1$, we have

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{T}_{n,p,q}^{(k,S)}(x,y)&=n\mathcal{T}_{n-1,p,q}^{(k,S)}(x,y),\\\frac{\partial}{\partial y}\mathcal{T}_{n,p,q}^{(k,C)}(x,y)&=-n\mathcal{T}_{n-1,p,q}^{(k,S)}(x,y),\\\frac{\partial}{\partial y}\mathcal{T}_{n,p,q}^{(k,S)}(x,y)&=n\mathcal{T}_{n-1,p,q}^{(k,C)}(x,y).\end{aligned}$$

By (18), (24) and by using Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)(1-(1-e^{-t}))^{-x}\cos yt \\ &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)\cos yt\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-e^{-t})^l \\ &= \sum_{l=0}^{\infty}<x>_l\ \frac{(e^t-1)^l}{l!}\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)e^{-lt}\cos yt \\ &= \sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{n,p,q}^{(k,C)}(-l,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)\mathcal{T}_{n-i,p,q}^{(k,C)}(-l,y)<x>_l\right)\frac{t^n}{n!},\end{aligned} \tag{29}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (29), we have the following theorem:

Theorem 2.9. *For $n>0$, we have*

$$\mathcal{T}_{n,p,q}^{(k,C)}(x,y)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)\mathcal{T}_{n-i,p,q}^{(k,C)}(-l,y)<x>_l,$$
$$T_{n,q}^{(k,S)}(x,y)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)\mathcal{T}_{n-i,p,q}^{(k,S)}(-l,y)<x>_l\ .$$

Now, we define the new type polynomials that are given by the generating functions to be

$$\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\cos yt=\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(y)\frac{t^n}{n!}, \tag{30}$$

and

$$\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\sin yt=\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,S)}(y)\frac{t^n}{n!}, \tag{31}$$

respectively.

Note that $\mathcal{T}_{n,p,q}^{(k,C)}(0,y)=\mathcal{T}_{n,p,q}^{(k,C)}(y),\mathcal{T}_{n,p,q}^{(k,S)}(0,y)=\mathcal{T}_{n,p,q}^{(k,S)}(y),\mathcal{T}_{n,p,q}^{(k,C)}(0)=\mathcal{T}_{n,p,q}^{(k)}$, $\mathcal{T}_{n,p,q}^{(k,S)}(0)=0$. The new type polynomials can be determined explicitly. A few of them are

$$\mathcal{T}^{(k,C)}_{0,p,q}(y)=1,\quad \mathcal{T}^{(k,C)}_{1,p,q}(y)=-\frac{3}{2}+\frac{1}{[2]^k_{p,q}},$$
$$\mathcal{T}^{(k,C)}_{2,p,q}(y)=\frac{4}{3}-\frac{4}{[2]^k_{p,q}}+\frac{2}{[3]^k_{p,q}}-y^2,$$
$$\mathcal{T}^{(k,C)}_{3,p,q}(y)=\frac{3}{4}+\frac{19}{2[2]^k_{p,q}}-\frac{15}{[3]^k_{p,q}}+\frac{6}{[4]^k_{p,q}}+\frac{9y^2}{2}-\frac{3}{[2]^k_{p,q}}y^2,$$

and

$$\mathcal{T}^{(k,S)}_{0,p,q}(y)=0,\quad \mathcal{T}^{(k,S)}_{1,p,q}(y)=y,$$
$$\mathcal{T}^{(k,S)}_{2,p,q}(y)=-3y+\frac{2}{[2]^k_{p,q}}y,$$
$$\mathcal{T}^{(k,S)}_{3,p,q}(y)=4y-\frac{12}{[2]^k_{p,q}}y+\frac{6}{[3]^k_{p,q}}y-y^3.$$

From (20), we derive the following equations:

$$\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\cos yt=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m\mathcal{T}^{(k)}_{k-2m,p,q}y^{2m}\right)\frac{t^k}{k!},\tag{32}$$

and

$$\begin{aligned}&\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\sin yt\\&=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m\mathcal{T}^{(k)}_{k-2m-1,p,q}y^{2m+1}\right)\frac{t^k}{k!}.\end{aligned}\tag{33}$$

By (30), (31), (32), (33), we get

$$\mathcal{T}^{(C)}_{n,p,q}(y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m y^{2m}\mathcal{T}^{(k)}_{n-2m,p,q},$$

and

$$\mathcal{T}^{(S)}_{n,p,q}(y)=\sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m y^{2m+1}\mathcal{T}^{(k)}_{n-2m-1,p,q}.$$

From (18), (19), (30), and (31), we derive the following theorem:

Theorem 2.10. *For $n\geq 0$, we have*

$$\mathcal{T}^{(k,C)}_{n,p,q}(x,y)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\mathcal{T}^{(k,C)}_{l,p,q}(y),$$

and

$$\mathcal{T}^{(k,S)}_{n,p,q}(x,y)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\mathcal{T}^{(k,S)}_{l,p,q}(y).$$

By (18), (30), and by using Cauchy product, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)((e^t-1)+1)^x \cos yt \\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \cos yt \sum_{l=0}^{\infty}\binom{x}{l}(e^t-1)^l \\
&= \sum_{l=0}^{\infty}(x)_l \frac{(e^t-1)^l}{l!}\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\cos yt\right) \\
&= \sum_{l=0}^{\infty}(x)_l \sum_{n=0}^{\infty} S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)\mathcal{T}_{n-i,p,q}^{(k,C)}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{34}
$$

By comparing the coefficients on both sides of (34), we have the following theorem:

Theorem 2.11. *For $n \geq 0$, we have*

$$
\begin{aligned}
T_{n,q}^{(k,C)}(x,y) &= \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l) T_{n-i,q}^{(k,C)}(y), \\
T_{n,q}^{(k,S)}(x,y) &= \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l) T_{n-i,q}^{(k,S)}(y).
\end{aligned}
$$

By (3), (24) and by using Cauchy product, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(x,y)\frac{t^n}{n!} \\
&= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)e^{xt}\cos(yt) \\
&= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(y)\frac{t^n}{n!} \\
&= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}^{(k,C)}(y)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x)\mathcal{T}_{n-l-i,p,q}^{(k,C)}(y)\right)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.12. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\mathcal{T}^{(k,C)}_{n,p,q}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} \mathcal{T}^{(k,C)}_{n-l-i,p,q}(y)\mathbf{B}^{(r)}_i(x),$$

$$\mathcal{T}^{(k,S)}_{n,p,q}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} \mathcal{T}^{(k,S)}_{n-l-i,p,q}(y)\mathbf{B}^{(r)}_i(x).$$

By (4), (18), (24) and by using the Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{T}^{(k,C)}_{n,p,q}(x,y)\frac{t^n}{n!} = \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right) e^{xt}\cos(yt) \\
&= \frac{(e^t-u)^r}{(1-u)^r}\left(\frac{1-u}{e^t-u}\right)^r e^{xt}\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)\cos yt \\
&= \sum_{n=0}^{\infty}\mathbf{H}^{(r)}_n(u,x)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i}e^{it}(-u)^{r-i}\frac{1}{(1-u)^r}\left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)}\right)\cos yt \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}^{(r)}_n(u,x)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}^{(k,C)}_{n,p,q}(i,y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}^{(r)}_l(u,x)\mathcal{T}^{(k,C)}_{n-l,p,q}(i,y)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.13. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\mathcal{T}^{(k,C)}_{n,p,q}(x,y) = \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathcal{T}^{(k,C)}_{n-l,p,q}(i,y)\mathbf{H}^{(r)}_l(u,x),$$

$$\mathcal{T}^{(k,S)}_{n,p,q}(x,y) = \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathcal{T}^{(k,S)}_{n-l,p,q}(i,y)\mathbf{H}^{(r)}_l(u,x).$$

By Theorem 2.11, Theorem 2.12, and Theorem 2.13 we have the following corollary.

Corollary 2.14. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)\mathcal{T}^{(k,C)}_{n-i,p,q}(y) \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}^{(r)}_l(u,x)\mathcal{T}^{(k,C)}_{n-l,p,q}(i,y) \\
&= \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r)\binom{n-l}{i}\mathbf{B}^{(r)}_i(x)\mathcal{T}^{(k,C)}_{n-l-i,p,q}(y).
\end{aligned}$$

3. Zeros of the (p,q)-poly-cosine tangent and (p,q)-poly-sine polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the (p,q)-poly-cosine tangent polynomials $\mathcal{T}^{(k,C)}_{n,p,q}(x,y)$ and (p,q)-poly-sine tangent polynomials $\mathcal{T}^{(k,S)}_{n,p,q}(x,y)$. The (p,q)-poly-cosine tangent polynomials $\mathcal{T}^{(k,C)}_{n,p,q}(x,y)$ and (p,q)-poly-sine tangent polynomials $\mathcal{T}^{(k,S)}_{n,p,q}(x,y)$ can be determined explicitly. A few of them are

$$\begin{aligned}
\mathcal{T}^{(k,C)}_{0,p,q}(x,y) &= 1,\\
\mathcal{T}^{(k,C)}_{1,p,q}(x,y) &= -\frac{3}{2}+\frac{1}{[2]^k_{p,q}}+x,\\
\mathcal{T}^{(k,C)}_{2,p,q}(x,y) &= \frac{4}{3}-\frac{4}{[2]^k_{q,q}}+\frac{2}{[3]^k_{p,q}}-3x+\frac{2}{[2]^k_{p,q}}x+x^2-y^2\\
\mathcal{T}^{(k,C)}_{3,p,q}(x,y) &= \frac{3}{4}+\frac{19}{2[2]^k_{p,q}}-\frac{15}{3[2]^k_{p,q}}+\frac{6}{[4]^k_{p,q}}+4x-\frac{12}{[2]^k_{p,q}}x\\
&\quad+\frac{6}{[3]^k_{p,q}}x-\frac{9x^2}{2}+\frac{3}{[2]^k_{p,q}}x^2+x^3+\frac{9y^2}{2}-\frac{3}{[2]^k_{p,q}}y^2-3xy^2,\\
\mathcal{T}^{(k,C)}_{4,p,q}(x,y) &= -\frac{14}{5}-\frac{12}{[2]^k_{p,q}}+\frac{66}{[3]^k_{p,q}}-\frac{72}{[4]^k_{p,q}}+\frac{24}{[5]^k_{p,q}}+3x+\frac{38}{[2]^k_{p,q}}x\\
&\quad-\frac{60}{[3]^k_{p,q}}x+\frac{24}{[4]^k_{p,q}}x+8x^2-\frac{24}{[2]^k_{p,q}}x^2+\frac{12}{[3]^k_{p,q}}x^2-6x^3+\frac{4}{[2]^k_{p,q}}x^3\\
&\quad+x^4-8y^2+\frac{24}{[2]^k_{p,q}}y^2-\frac{12}{[3]^k_{p,q}}y^2+18xy^2-\frac{12}{[2]^k_{p,q}}xy^2-6x^2y^2+y^4.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}^{(k,S)}_{0,p,q}(x,y) &= 0,\\
\mathcal{T}^{(k,S)}_{1,p,q}(x,y) &= y,\\
\mathcal{T}^{(k,S)}_{2,p,q}(x,y) &= -3y+\frac{1}{[2]^k_{p,q}}y+2xy\\
\mathcal{T}^{(k,S)}_{3,p,q}(x,y) &= 4y-\frac{12}{[2]^k_{p,q}}y+\frac{6}{[3]^k_{p,q}}y-9xy+\frac{6}{[2]^k_{p,q}}xy+3x^2y-y^3,\\
\mathcal{T}^{(k,S)}_{4,p,q}(x,y) &= 3y+\frac{38}{[2]^k_{p,q}}y-\frac{60}{[3]^k_{p,q}}y+\frac{24}{[4]^k_{p,q}}y+16xy-\frac{48}{[2]^k_{p,q}}xy+\frac{24}{[3]^k_{p,q}}xy\\
&\quad-18x^2y+\frac{12}{[2]^k_{p,q}}x^2y+4x^3y+6y^3-\frac{4}{[2]^k_{p,q}}y^3-4xy^3,
\end{aligned}$$

We investigate the beautiful zeros of the (p,q)-poly-cosine tangent polynomials $\mathcal{T}^{(k,C)}_{n,p,q}(x,y)$ by using a computer. We plot the zeros of the (p,q)-poly-cosine tangent polynomials $\mathcal{T}^{(k,C)}_{n,p,q}(x,y)$ for $n=50, k=2, x=2$ (Figure 1). In Figure

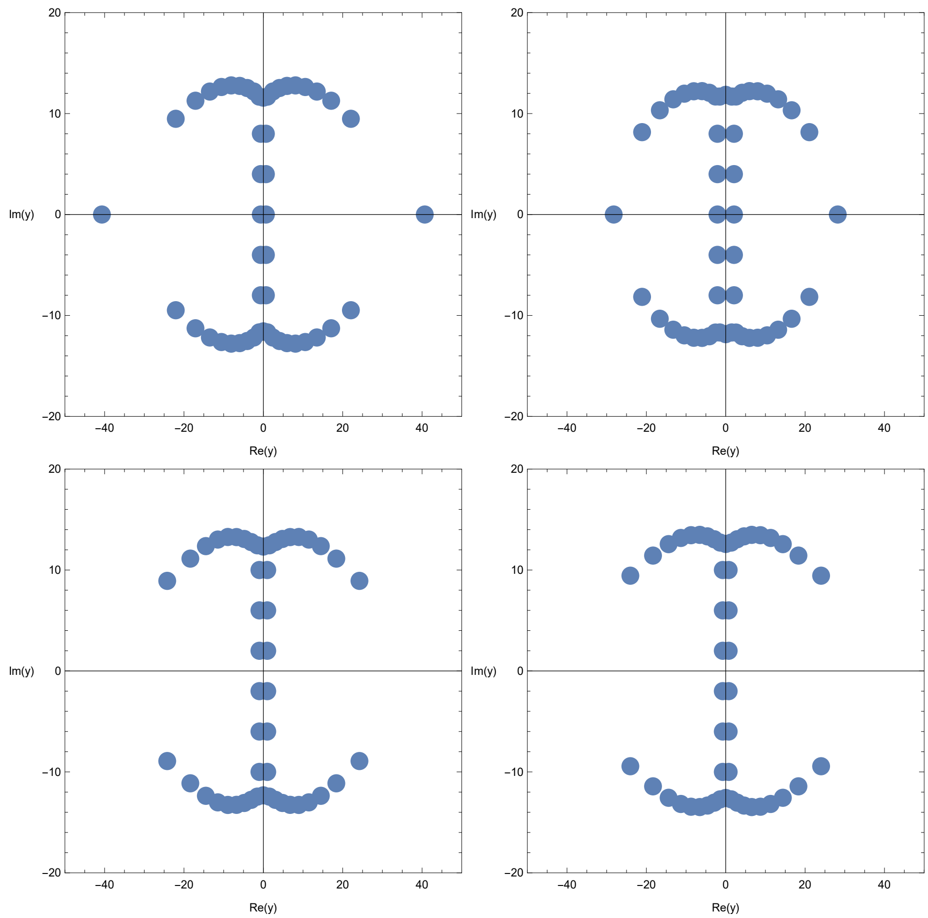

FIGURE 1. Zeros of $\mathcal{T}^{(k,C)}_{n,p,q}(x,y)$

1(top-left), we choose $q=\frac{1}{10}$. In Figure 1(top-right), we choose $q=\frac{3}{10}$. In Figure 1(bottom-left), we choose $q=\frac{7}{10}$. In Figure 1(bottom-right), we choose $q=\frac{9}{10}$.

Stacks of zeros of $\mathcal{T}_{n,p,q}^{(k,C)}(x,y)$ for $1 \leq n \leq 50, k = 2, x = 2$ from a 3-D structure are presented(Figure 2). In Figure 2(top-left), we choose $q = \frac{1}{10}$. In

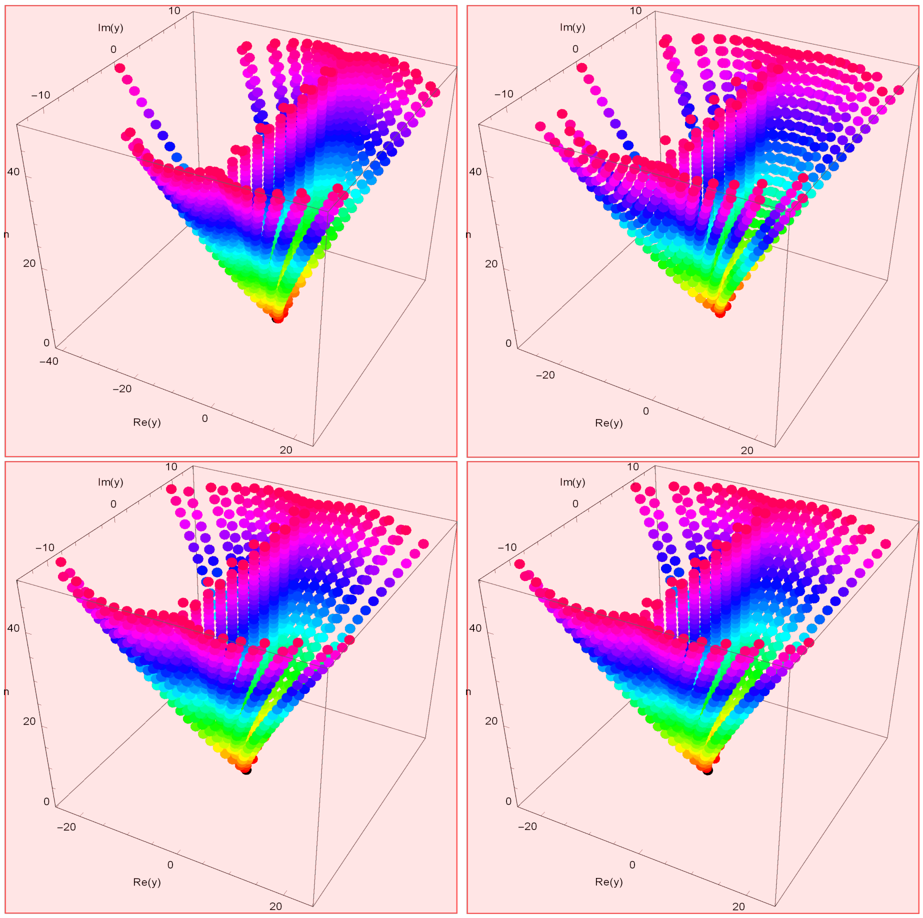

FIGURE 2. Stacks of zeros of $\mathcal{T}_{n,p,q}^{(k,C)}(x,y)$ for $1 \leq n \leq 50$

Figure 2(top-right), we choose $q = \frac{3}{10}$. In Figure 2(bottom-left), we choose $q = \frac{7}{10}$. In Figure 2(bottom-right), we choose $q = \frac{9}{10}$.

The plot of real zeros of $\mathcal{T}_{n,p,q}^{(k,C)}(x, y), k = 2, y = 2$ for $1 \leq n \leq 50$ structure are presented(Figure 3). In Figure 3(top-left), we choose $q = \frac{1}{10}$. In Figure

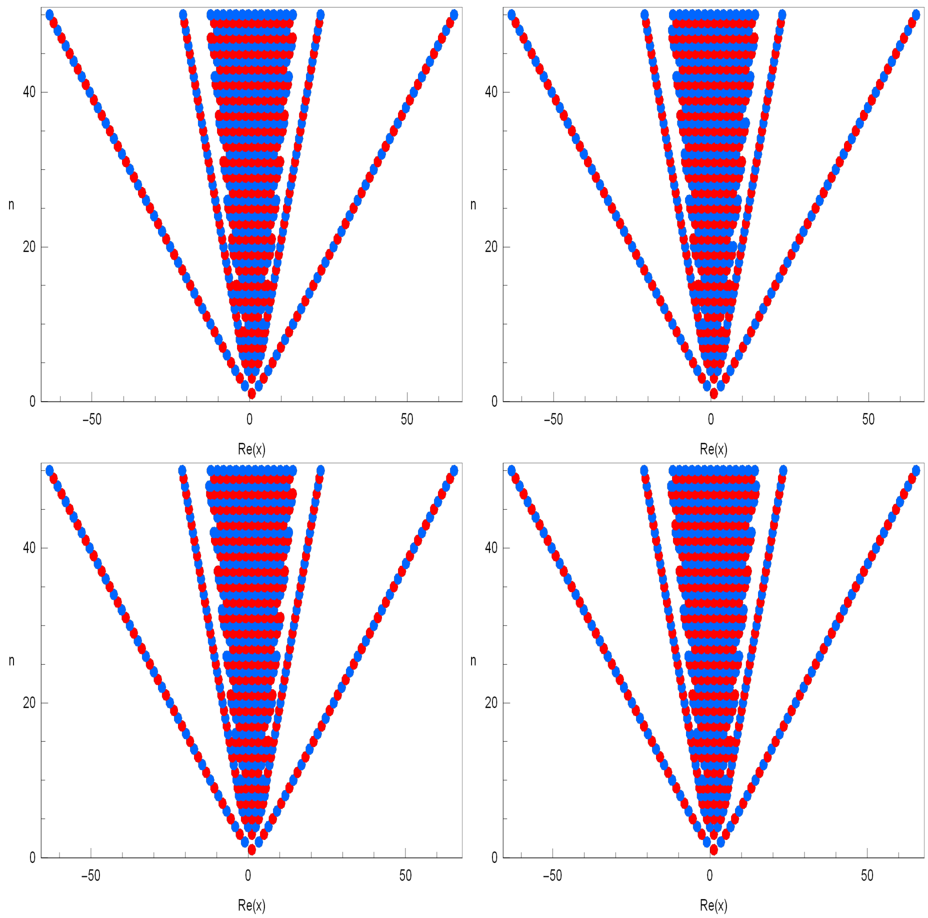

FIGURE 3. Stacks of zeros of $\mathcal{T}_{n,p,q}^{(k,C)}(x, y)$ for $1 \leq n \leq 50$

3(top-right), we choose $q = \frac{3}{10}$. In Figure 3(bottom-left), we choose $q = \frac{7}{10}$. In Figure 3(bottom-right), we choose $q = \frac{9}{10}$.

We investigate the beautiful zeros of the (p,q)-poly-sine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,S)}(x,y)$ by using a computer. We plot the zeros of the (p,q)-poly-sine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,S)}(x,y)$ for $n=50$ (Figure 4). In Figure 4(top-left), we choose

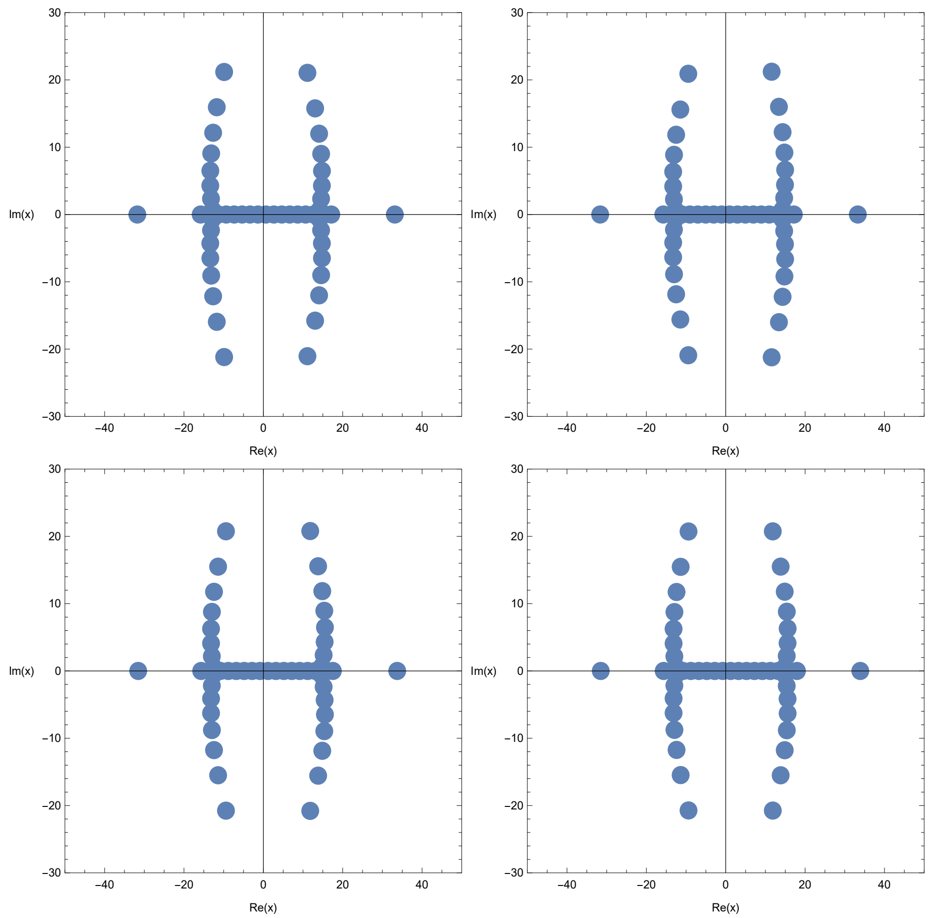

FIGURE 4. Zeros of $\mathcal{T}_{n,p,q}^{(k,S)}(x,y)$

$q=\frac{1}{10}$. In Figure 4(top-right), we choose $q=\frac{3}{10}$. In Figure 4(bottom-left), we choose $q=\frac{7}{10}$. In Figure 4(bottom-right), we choose $q=\frac{9}{10}$.

Stacks of zeros of $\mathcal{T}_{n,p,q}^{(S)}(x, y)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 5). In Figure 5(top-left), we choose $k = 2, y = 2$ and $q = \frac{1}{10}$. In

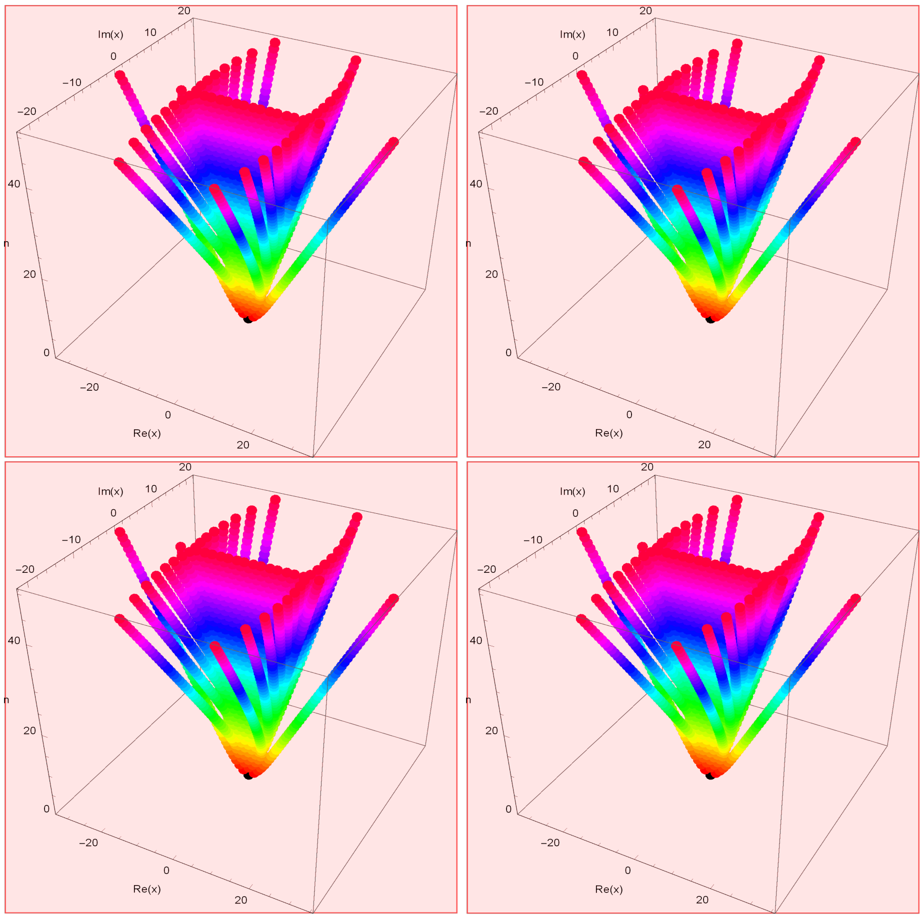

FIGURE 5. Stacks of zeros of $\mathcal{T}_{n,p,q}^{(k,S)}(x, y)$ for $1 \leq n \leq 50$

Figure 5(top-right), we choose $k = 2, y = 2$ and $x = \frac{3}{10}$. In Figure 5(bottom-left), we choose $k = 2, y = 2$ and $x = \frac{7}{10}$. In Figure 5(bottom-right), we choose $k = 2, y = 2$ and $\frac{9}{10}$.

The plot of real zeros of $\mathcal{T}^{(S)}_{n,p,q}(x,y)$ for $1 \leq n \leq 40$ structure are presented(Figure 6). In Figure 6(top-left), we choose $k = 2, y = 2$ and $q = \frac{1}{10}$.

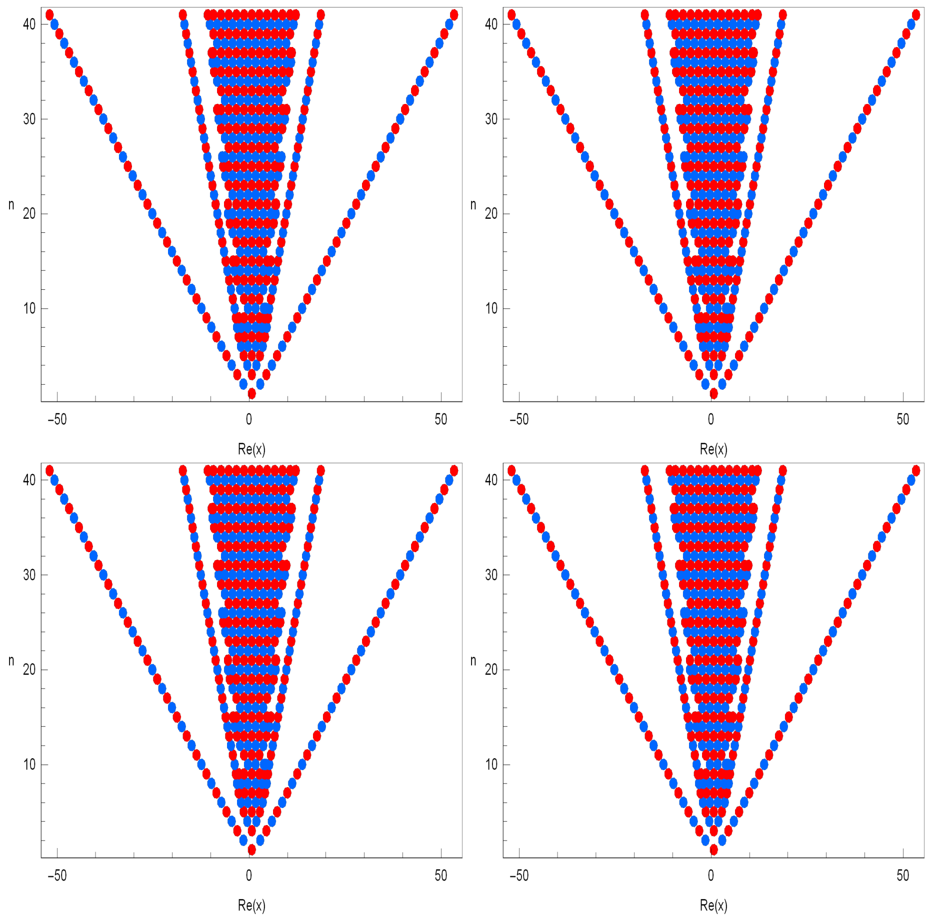

FIGURE 6. Stacks of zeros of $\mathcal{T}^{(k,S)}_{n,p,q}(x,y)$ for $1 \leq n \leq 40$

In Figure 6(top-right), we choose $k = 2, y = 2$ and $q = \frac{3}{10}$. In Figure 6(bottom-left), we choose $k = 2, y = 2$ and $q = \frac{7}{10}$. In Figure 6(bottom-right), we choose $k = 2, y = 2$ and $q = \frac{9}{10}$.

Next, we calculated an approximate solution satisfying (p, q)-poly-sine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,S)}(x, y) = 0$ for $y \in \mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $\mathcal{T}_{n,p,q}^{(2,S)}(4, y) = 0, q = \frac{1}{10}$

degree n	y
1	0
2	0
3	−5.5486, 0, 5.5486
4	−2.9407, 0, 2.9407
5	−9.9476, −1.9154, 0, 1.9154, 9.9476
6	−5.1857, −1.3911, 0, 1.3911, 5.1857
7	−14.250, −3.3160, −1.2003, 0, 1.2003, 3.3160, 14.250

We also calculated an approximate solution satisfying (p, q)-poly-cosine tangent polynomials $\mathcal{T}_{n,p,q}^{(k,C)}(x, y) = 0$ for $x \in \mathbb{R}$.

Table 2. Approximate solutions of $\mathcal{T}_{n,p,q}^{(2,C)}(x, 4) = 0, q = \frac{1}{10}$

degree n	y
1	0.67355
2	−3.4256, 4.7727
3	−6.4261, 0.67298, 7.7737
4	−9.1928, −1.1874, 2.5333, 10.541
5	−11.868, −2.5983, 0.67061, 3.9466, 13.217
6	−14.499, −3.7995, −0.65949, 1.9997, 5.1513, 15.849
7	−17.105, −4.8799, −1.7461, 0.66689, 3.0889, 6.2356, 18.455

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

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