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THE RIGIDITY OF RECTANGULAR FRAMEWORKS AND THE LAPLACIAN MATRICES†

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Abstract. In general, the rigidity problem of braced rectangular frameworks is determined by the connectivity of the bipartite graph induced by given rectangular framework. In this paper, we study how to solve the rigidity problem of the braced rectangular framework using the Laplacian matrix of the matrix induced by a braced rectangular framework.

AMS Mathematics Subject Classification : 65F30, 65F99.
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1. Introduction

Many high-rise buildings are supported by steel frameworks consisting of rectangular arrangements of girder beams and welded or riveted joints. However, the structure is treated as a flat (planar rather than space) structure with pin-joints rather than rigid welds when joining beams together for various reasons([7], p. 59). Figure 1 shows that the simplest structure, consisting of four beams and four pin-joints. This structure is unstable because it can be easily deformed under sufficiently high loads. In order for the structure to be stable, it must be braced by extra beams.

In the case of a larger structure containing many rectangular cells, the determination of the rigidity of the structure is not easy. Nevertheless, it is possible to ensure the rigidity by attaching support rods (extra beams) to all the rectangular cells, but it is too costly. So we have natural mathematical problems; the rigidity problem and the optimization problem.

It is well known that the rigidity problem of braced rectangular frameworks can be solved with the connectivity of the bipartite graph (see, [1], [3], [4], [5], [6], [7]). In fact, Figure 2 shows a rigid braced rectangular framework and its

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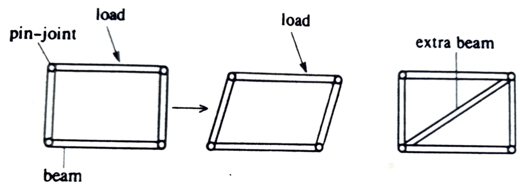


FIGURE 1. Simplest type of rectangular framework([7], p. 60)

connected bipartite graph, and Figure 3 shows a non-rigid braced rectangular framework and its disconnected bipartite graph.

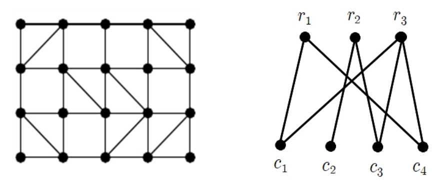


FIGURE 2. Rigid rectangular framework and connected bipartite graph

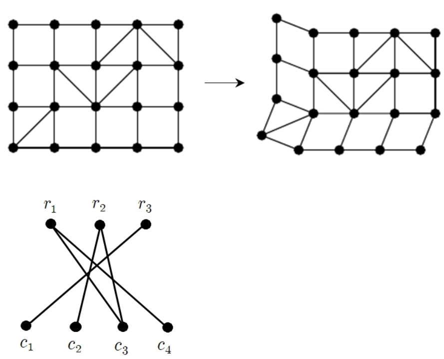


FIGURE 3. Non-rigid rectangular framework and disconnected bipartite graph

In this paper, we study how to solve the stability problem using the Laplacian matrix of the matrix induced by the braced rectangular framework.

2. Laplacian matrix of a matrix

In this section, for convenience, the (i, j)-component of a given matrix M is expressed as $(M)_{ij}$. Generally, the Laplacian matrix L is defined from a graph

$G = (V, E)$. In fact, $L = D - A$, where A is the weighted adjacency matrix of G and D is the degree matrix of G, and is the diagonal matrix such that

$$(D)_{ii} = \sum_{j}^{|V|} (A)_{ij}$$

Now, similar to the definition of the Laplacian matrix of a graph, we can define the Laplacian matrix of a given real symmetric matrix with non-nagative entries.

Definition 2.1. Let S be a $n \times n$ real symmetric matrix with non-nagative entries and let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where

$$d_i = \sum_{j=1}^{n} (S)_{ij}$$

for $i = 1, 2, \ldots, n$. Let's call the matrix $L = D - S$ the *Laplacian of the given matrix S*.

Lemma 2.2. *Let S be a $n \times n$ real symmetric matrix with non-nagative entries. Then the Laplacian matrix $L = D - S$ is a positive semi-definite.*

Proof. For any $\mathbf{x} = [\, x_1 \, x_2 \, \cdots x_n \,]^T \in \mathbb{R}^n$, since the matrix S is symmetric and its entries are all non-negative, we have

$$\begin{aligned}
\mathbf{x}^T L \mathbf{x} = \mathbf{x}^T D \mathbf{x} - \mathbf{x}^T S \mathbf{x} &= \sum_{i=1}^{n} d_i \, x_i^2 - \sum_{i,j=1}^{n} (S)_{ij} \, x_i \, x_j \\
&= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} (S)_{ij} \right\} x_i^2 - \sum_{i=1}^{n} (S)_{ii} x_i^2 - \sum_{i \neq j}^{n} (S)_{ij} x_i x_j \\
&= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} (S)_{ij} - (S)_{ii} \right\} x_i^2 - \sum_{i \neq j}^{n} (S)_{ij} \, x_i \, x_j \\
&= \sum_{i \neq j}^{n} (S)_{ij} \, x_i^2 - \sum_{i \neq j}^{n} (S)_{ij} \, x_i \, x_j \\
&= \frac{1}{2} \left(\sum_{i \neq j}^{n} (S)_{ij} \, x_i^2 - 2 \sum_{i \neq j}^{n} (S)_{ij} \, x_i \, x_j + \sum_{i \neq j}^{n} (S)_{ij} \, x_j^2 \right) \\
&= \frac{1}{2} \sum_{i \neq j}^{n} (S)_{ij} (x_i - x_j)^2 \geq 0 \qquad (1)
\end{aligned}$$

This completes the proof. $\square$

Definition 2.3. (cf. [4]) Let $M = [\mathbf{c}_1 \; \mathbf{c}_2 \cdots \mathbf{c}_n]$ be a $m \times n$ real matrix, where $\mathbf{c}_j$ is a j-th column vector of M for $j = 1, 2, \ldots, n$. And let $C = \{\mathbf{c}_1, \mathbf{c}_2, \cdots \mathbf{c}_n\}$.

Then two vectors $\mathbf{c}_k$ and $\mathbf{c}_l$ of C are called *c–connectable* if there is a sequence $\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \cdots, \mathbf{c}_{j_w}$ in C such that

$$\mathbf{c}_k \cdot \mathbf{c}_{j_1} \neq 0,\ \mathbf{c}_{j_1} \cdot \mathbf{c}_{j_2} \neq 0, \cdots, \mathbf{c}_{j_w} \cdot \mathbf{c}_l \neq 0$$

Moreover, a subset S of C is said to be *c–connected* if every pair of S is c–connectable.

Let M be a $m \times n$ real matrix with non-nagative entries, let $W = M^T M$, and let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where $d_i = \sum_{j=1}^{n} (W)_{ij}$ for $i = 1, 2, \ldots, n$.

Theorem 2.4. *We have the followings*

(1) $(W)_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j$ *for* $1 \leq i, j \leq n$, *where* $\mathbf{c}_i$ *and* $\mathbf{c}_j$ *are* i*-th and* j*-th column vectors of* M, *respectively.*
(2) *The Laplacian matrix* $L = D - W$ *of* W *is a positive semi-definite.*
(3) $\lambda = 0$ *is eigenvalue of* L.

Proof. (1) Obvious.
(2) By Lemma 2.2.
(3) For convenience, let $w_{ij} = (W)_{ij}$. Then

$$L = \begin{bmatrix} \left(\sum_{j=1}^{n} w_{1j} - w_{11}\right) & -w_{12} & \cdots & -w_{1n} \\ -w_{21} & \left(\sum_{j=1}^{n} w_{2j} - w_{22}\right) & \cdots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \cdots & \left(\sum_{j=1}^{n} w_{nj} - w_{nn}\right) \end{bmatrix}$$

Thus $L\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [1\,1\,\cdots\,1]^T$. Hence 0 is the eigenvalue of L corresponding to the eigenvector $\mathbf{x}$. □

Lemma 2.5. *Let* $M = [\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n]$ *be a* $m \times n$ *real matrix, where* $\mathbf{c}_j$ *is a column vector of* M *for* $j = 1, 2, \ldots, n$. *Then we have the followings;*

(1) *The Laplacian matrix* L *of* $W = M^T M$ *is decomposed of the form*

$$L = \frac{1}{2} \sum_{i,j} (W)_{ij} L_{ij} = \sum_{i<j} (W)_{ij} L_{ij},$$

where

$$L_{ij} = L_{ji} = \begin{array}{cc} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & -1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{c} \\ i^{th} \\ \\ j^{th} \\ \\ \end{array} \\ \begin{array}{ccccc} \quad & i^{th} & \quad & j^{th} & \quad \end{array} & \end{array}$$

Here, all dotted entries of L_{ij} are zero and L_{ii} is the $n \times n$ zero matrix.

(2) *If F is a c–connected component in the set of all column vectors of M and I is the index set of the elements in F, then $L\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [x_1\, x_2\, \cdots\, x_n]^T$ and*

$$x_w = \begin{cases} 1, & w \in I \\ 0, & \textit{otherwise} \end{cases}$$

Proof. (1) Notice that W is a symmetric matrix and that $L_{ij} = L_{ji}$ for $i, j(1 \leq i, j \leq n)$. For convenience, we let $(W)_{ij} = w_{ij}$ for $i, j(1 \leq i, j \leq n)$.

$$\begin{aligned}
\sum_{i,j} (W)_{ij} L_{ij} = & \left[\begin{array}{cc|c} w_{12} & -w_{12} & \mathbf{0} \\ -w_{21} & w_{21} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{ccc|c} w_{13} & 0 & -w_{13} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \\ -w_{31} & 0 & w_{31} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
& + \cdots + \left[\begin{array}{ccc} w_{1n} & \mathbf{0} & -w_{1n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -w_{n1} & \mathbf{0} & w_{n1} \end{array}\right] \\
& + \left[\begin{array}{cc|c} w_{12} & -w_{12} & \mathbf{0} \\ -w_{21} & w_{21} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{c|cc|c} 0 & 0 & 0 & \mathbf{0} \\ \hline 0 & w_{23} & -w_{23} & \mathbf{0} \\ 0 & -w_{32} & w_{32} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
& + \cdots + \left[\begin{array}{c|ccc} 0 & 0 & \mathbf{0} & 0 \\ \hline 0 & w_{2n} & \mathbf{0} & -w_{2n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & -w_{n2} & \mathbf{0} & w_{n2} \end{array}\right] \\
& + \left[\begin{array}{ccc} w_{1n} & \mathbf{0} & -w_{1n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -w_{n1} & \mathbf{0} & w_{n1} \end{array}\right] + \left[\begin{array}{c|ccc} 0 & 0 & \mathbf{0} & 0 \\ \hline 0 & w_{2n} & \mathbf{0} & -w_{2n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & -w_{n2} & \mathbf{0} & w_{n2} \end{array}\right] \\
& + \cdots + \left[\begin{array}{c|cc} \mathbf{0} & 0 & 0 \\ \hline \mathbf{0} & w_{n-1\,n} & -w_{n-1\,n} \\ \mathbf{0} & -w_{n\,n-1} & w_{n\,n-1} \end{array}\right] \\
= 2L &
\end{aligned}$$

(2) For $i, j \in I$, if $\mathbf{c}_i \cdot \mathbf{c}_j = 0$, then $(W)_{ij} = 0$. If $\mathbf{c}_i \cdot \mathbf{c}_j \neq 0$, then $L_{ij}\mathbf{x} = 0$. Thus $(W)_{ij} L_{ij}\mathbf{x} = 0$. Notice that if $i, j \notin I$, $L_{ij}\mathbf{x} = 0$. Threrefore by (1), we have $L\mathbf{x} = 0$. $\square$

Theorem 2.6. *Let M be a $m \times n$ real matrix with non-nagative entries, and let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the Laplacian matrix L of M. Then $\lambda_2 > 0$ if and olny if the set of all column vectors of M is c–connected.*

Proof. Let $C = \{\mathbf{c}_1, \mathbf{c}_2, \cdots \mathbf{c}_n\}$ be the set of all column vectors of M. Suppose that the C is not c–connected. We construct the column vector $\mathbf{x} =$

$[x_1\, x_2\, \cdots x_n]^T \in \mathbb{R}^n$ as follows: for fixed c–connected component F,

$$x_i = \begin{cases} 1, & \mathbf{c}_i \in F \\ 0, & \text{otherwise} \end{cases}$$

Then $L\mathbf{x} = \mathbf{0}$ by (1) of Lemma 2.5. We note that the two eigenvectors $\mathbf{x}_1 = [1\,1\, \cdots\, 1]^T$ and $\mathbf{x}$ corresponding to the eigenvalue $\lambda = 0$ are linearly independent. Thus the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 0$ is greater than equal to 2. So, we have $\lambda_2 = 0$.

Conversely, suppose that C is c–connected. Assume $L\mathbf{x} = \mathbf{0}$. That is to say, $\mathbf{x} = [x_1\, x_2\, \cdots x_n]^T \in \mathbb{R}^n$ is a eigenvector corresponding to the eigenvalue $\lambda = 0$. Then by the equation (1) in Lemma 2.2,

$$\mathbf{x}^T L\mathbf{x} = \sum_{i \neq j} (W)_{ij}(x_i - x_j)^2 = 0.$$

In the case of $(W)_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j \neq 0$, we have $x_i = x_j$. If $(W)_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j = 0$, then since for any two column vectors of C is c–connected, there exists a sequence $\mathbf{c}_{c_1}, \mathbf{c}_{c_2}, \cdots, \mathbf{c}_{c_k}$ such that

$$\mathbf{c}_i \cdot \mathbf{c}_{c_1} \neq 0,\ \mathbf{c}_{c_1} \cdot \mathbf{c}_{c_2} \neq 0, \cdots,\ \mathbf{c}_{c_k} \cdot \mathbf{c}_j \neq 0$$

Thus we have $x_i = x_{c_1} = \cdots = x_{c_k} = x_j$, and hence $x_i = x_j$. If $x_i = 0$ for some i $(1 \leq i \leq n)$, then $x_i = 0$ for all i $(1 \leq i \leq n)$, i.e., $\mathbf{x}$ is a zero vector that is contradicts to $\mathbf{x} \neq \mathbf{0}$. Thus we have $\mathbf{x} = k\,[1\,1\, \cdots\, 1]^T$ $(k \neq 0)$. It follows that the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 0$ is 1, and hence $\lambda_2 > 0$. □

3. Rigidity of rectangular frameworks

Definition 3.1. Let C be a finite set, and F a non-empty family of subsets of C, that is $F \subseteq 2^C$. The ordered pair (C, F) is said to be a *matroid* if

- (M1) If $B \in F$ and $A \subseteq B$, then $A \in F$.
- (M2) If $A, B \in F$ and $|A| < |B|$, then there exists c in $B \setminus A$ such that $A \cup \{c\}$ in F.

Theorem 3.2. *Let M be a $m \times n$ real matrix, let C be a set of column vectors of M, and let and*

$$F = \{S : S \subseteq C,\ S \text{ is } c\text{–connected}\}$$

Then (C, F) is a matroid if and only if C is c–connected or all one-point subsets of C are c–connected components.

Proof. Suppose that (C, F) is a matroid and assume that C is not c–connected and also there is c–connected component that is not one-point subset of C, say, A. For convenience, let $\{a\}$ be a proper subset of A. Let B be a c–connected component in C differ from A of C. Then $A, B \in F$ and $A \cap B = \emptyset$.

Case 1: $|A| < |B|$, for any $b \in B$, $A \cup \{b\}$ is not c–connected, and hence $A \cup \{b\} \notin F$. This is a contradiction to the fact (C, F) is a matroid.

Case 2: $|A| > |B|$, we can easily deduce a contradiction using the similar method as in the case 1.

Case 3: $|A| = |B|$, by (M1), $\{a\} \in F$, $|\{a\}| < |A| = |B|$, and for any $b \in B$, $\{a\} \cup \{b\} \notin F$. This is a contradiction to the fact (C, F) is a matroid.

Conversely, if C is a union of trivial c–connected components, then

$$F = \{\{c\} : c \in C\} \cup \emptyset.$$

Clearly, F satisfies (M1) and (M2), so that (C, F) is a matroid. If C is c–connected, then $F = 2^C$, and hence F satisfies (M1) and (M2), so that (C, F) is a matroid.

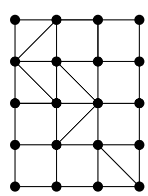
□

Remark 3.1. Let M be a $m \times n$ real matrix, let C_0 be the set of all column vectors of M, and let $F = \{S : S \subseteq C_0, S \text{ is } c\text{-connected}\}$. Then if C_0 is a c–connected component in C, then (C_0, F) is a matroid.

Let R be a $m \times n$ rectangular framework. Define the matrix $M_R = [a_{ij}]_{m\times n}$ of R by

$$a_{ij} = \begin{cases} 1, & (i,j)\text{–entry has extra-beam in } R \\ 0, & \text{otherwise.} \end{cases}$$

For example,

$R =$  $\Longleftrightarrow M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

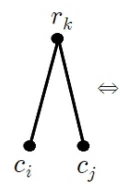
Theorem 3.3. *Let R be a $m \times n$ rectangular framework and let L_R be the Laplacian Matrix of the matrix M_R. Then R is a rigid rectangular framework if and only if*

(1) *The eigenvalue λ_2 of the matrix L_R is positive and*
(2) *There are no zero row vectors in M_R.*

Proof. Suppose that R is a rigid rectangular framework. Then the bipartite graph corresponding to R is connected. Clearly, (2) is satisfied and the column space of M_R is c–connected. By Theorem 2.6, (1) is satisfied.

Conversely, if (1) are (2) are satisfied, then we can easily show that the bipartite graph corresponding to R is connected by Theorem 2.6 and (2). Thus R is a rigid rectangular framework. □

Remark 3.2. Let R be a braced rectangular framework. Figure 4 says that the fundamental relationship of bipartite graph, matrix, and the Laplacian matrix of R.



$$\Leftrightarrow \quad \overset{(M_R)_{ki} = 1 = (M_R)_{kj}}{\begin{array}{c c} & \\ k^{th} & \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ & \begin{array}{ccccc} & i^{th} & & j^{th} & \end{array} \end{array}} \quad \Leftrightarrow \quad \overset{(L_R)_{ij}}{\begin{array}{c c} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & -1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{c} \\ i^{th} \\ \\ j^{th} \\ \\ \end{array} \\ \begin{array}{ccccc} & i^{th} & & j^{th} & \end{array} & \end{array}}$$

FIGURE 4. Bipartite graph, matrix, and the Laplacian matrix

Remark 3.3. For an example of python coding using the Laplacian matrix of the matrix induced by given rectangular framework, which can determine the stability of rectangular framework, see Appendix.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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Appendix: python code

```
import numpy as np
from sympy import Matrix
from sympy import Symbol

def Rect_Fra(*args):
    M = []
    for i in args:
        if sum(i) != 1 or len(i) == 1:
            M.append(i)
    if len(M) == 0:
        print('The rectangular framework is NOT rigid')
    else:
        Mt = list(map(list,zip(*M)))
        W = np.dot(Mt, M)

        if np.trace(W) < len(M) + len(Mt) -1:
            print('The rectangular framework is NOT rigid')

        else:
            S = []
            for i in W:
                S.append(sum(i))
            D = np.diag(S)
            L = D - W

            x = Symbol('lambda')
            if Matrix(L).charpoly() % x**2 == 0:
                print('The rectangular framework is NOT rigid')
            else:
                if np.trace(W) == len(M) + len(Mt) -1:
                    print('The rectangular framework is rigid with
                    minimum bracings')
                else:
                    print('The rectangular framework is rigid')
```