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THE STUDY ON GENERALIZED (p, q)-POLY-GENOCCHI POLYNOMIALS WITH VARIABLE $a^{\dagger}$

H.Y. LEE

ABSTRACT. In this paper, the generalized (p, q)-poly-Genocchi polynomials with variable a is defined by generalizing it more, and various properties of this polynomial are introduced. To do this, we define a generating function and use the definition to introduce some interesting properties as follows: basic properties, relation between Stirling numbers of the second kind and generalized (p, q)-poly-Genocchi polynomials with variable a and symmetric properties.

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1. Introduction

Recently, among the various fields of mathematics, one of the researchers' interests is the applications of following polynomials and numbers : [Bernoulli, Euler, Genocchi, Tangent, poly-Bernoulli, poly-Euler, poly-Genocchi, poly-Tangent and so on] numbers and polynomials. Many mathematicians have studied Genocchi numbers and polynomials and focus on expansion and generalization of theirs with generating function. Specially, it is being studied about poly-Genocchi numbers and polynomials concerned with polylogarithm function(cf. [1-11]).

The main symbols used in this paper are as follows. $\mathbb{N}$: the set of natural numbers, $\mathbb{Z}_+$: the set of nonnegative integers, $\mathbb{Z}$: the set of integers and $\mathbb{C}$: the set of complex numbers, respectively.

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As already well known, the classical Genocchi polynomials $G_n(x)$ are given by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad \text{(cf. [1- 3], [6- 10])}. \tag{1.1}$$

When $x = 0, G_n = G_n(0)$ are called the Genocchi numbers.

Definition 1.1. For $a \in \mathbb{C} \setminus \{0\}$, we define a generalized Genocchi polynomials $G_n(x; a)$ with variable a by the following generating function

$$\frac{2t}{e^{at}+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x; a)\frac{t^n}{n!}, \quad |t| < \frac{\pi}{|a|}. \tag{1.2}$$

When $a = 1$, it is equal to the classical Genocchi polynomials.

The q-polylogarithm function $Li_{k,q}(t)$ is defined as follows:

$$Li_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k} \quad (k \in \mathbb{Z}) \text{ (cf.[4,5,8])}, \tag{1.3}$$

where $[n]_q = \frac{1-q^n}{1-q}$ and are called q-numbers.

In this paper, we define extended the q-polylogarithm function $Li_{k,p,q}$ as belows:

$$Li_{k,p,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_{p,q}^k} \quad (k \in \mathbb{Z}). \tag{1.4}$$

For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the q-poly-Bernoulli polynomials $B_{n,q}^{(k)}(x)$ with variable a are defined by means of the following generating function

$$\frac{Li_{k,q}(1-e^{-t})}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!}.$$

For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the q-poly-Genocchi polynomials $G_n^{(k)}(x)$ with variable a are defined by means of the following generating function.

$$\frac{2Li_{k,q}(1-e^{-t})}{e^{at}+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{1.4}$$

When $k = 1$, $Li_{1,q}(x) = -log(1-x)$ and $Li_{1,q}(1-e^{-t}) = t$. Using the result of polylogarithm function, we deduce that the poly-Genocchi polynomials is the Genocchi polynomials when $k = 1$ and $a = 1$.

The numbers $S_2(n, m)$ are the classical Stirling numbers of the second kind defined by the following relations:

$$x^n = \sum_{m=0}^{n} S_2(n, m)(x)_m,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is falling factorial.

Generally, the Stirling numbers of the second kind is defined as:

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ (cf. [4, 6, 7, 9, 11])}. \tag{1.5}$$

In this paper, we introduce a generalized poly-Genocchi polynomials and numbers with variable a. The properties of the Genocchi polynomials with parameters were studied in [5, 8]. We define a generalized poly-Genocchi polynomials with variable a and give various and interesting relations between the generalized poly-Genocchi polynomials and the classical Genocchi polynomials. We also investigate several identities that are connected with the Stirling numbers of the second kind. Symmetry is a very important problem in mathematics. In this paper, we find symmetric properties using special functions and power sum polynomials.

2. Generalized q-poly-Genocchi polynomials with variable a

In this section, we introduce a generalized (p, q)-poly-Genocchi polynomials $G^{(k)}_{n,p,q}(x; a)$ and numbers $G^{(k)}_{n,p,q}(a)$ with variable a by the generating functions. We provide various identities for the polynomials $G^{(k)}_{n,p,q}(x; a)$ and find relations associated with classical Genocchi polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the generalized (p, q)-poly-Genocchi polynomials $G^{(k)}_{n,p,q}(x; a)$ with variable a are defined by means of the following generating function

$$\frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} = \sum_{n=0}^{\infty} G^{(k)}_{n,q}(x; a)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $G^{(k)}_{n,p,q}(a) = G^{(k)}_{n,p,q}(0; a)$ are called the generalized (p, q)-poly-Genocchi numbers with variable a. When the condition allow $a = 1$ and $p = 1$, it is trivial that the generalized (p, q)-poly-Genocchi polynomials is reduced to q-poly-Genocchi polynomials.

From (2.1), we have a relation between the generalized q-poly-Genocchi numbers and polynomials.

Theorem 2.1. *Let n, m be nonnegative integers and $k \in \mathbb{Z}$. We have*

$$G^{(k)}_{n,p,q}(mx; a) = \sum_{l=0}^{n} \binom{n}{l} (m-1)^{n-l} G^{(k)}_{l,p,q}(x; a)x^{n-l}.$$

proof.

For $n, m \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(mx;a)\frac{t^n}{n!} &= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} \times e^{(m-1)xt}\\ &= \sum_{n=0}^{\infty} G_{n,p,q}^{k}(x:a)\frac{t^n}{n!} \times \sum_{n=1}^{\infty}(m-1)^n x^n \frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(m-1)^{n-l}G_{l,p,q}^{(k)}(x;a)x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain theorem 2.1.

Theorem 2.2. *Let n, m be a nonnegative integers and $k \in \mathbb{Z}$. We have*

$$G_{n,p,q}^{(k)}(x;a) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i E_{n-i}(x;a)\sum_{s=0}^{\infty}\frac{1}{[s+1]_{p,q}^{k}}\sum_{l=0}^{s+1}\binom{s+1}{l}(-1)^l l^i.$$

proof.

For $n, m \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a)\frac{(mt)^n}{n!} = \frac{2Li_{k,p,q}(1-e^{-mt})}{e^{amt}+1}e^{xmt}\\ &= \frac{2}{e^{amt}+1}e^{xmt} \times Li_{k,p,q}(1-e^{-mt})\\ &= \sum_{n=0}^{\infty} E_n(x;a)\frac{(mt)^n}{n!} \times \sum_{s=0}^{\infty}\frac{1}{[s+1]_{p,q}^{k}}\sum_{l=0}^{s+1}\binom{s+1}{l}(-1)^l\sum_{n=0}^{\infty}(-ml)^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}E_{n-i}(x;a)m^{n-i}\sum_{s=0}^{\infty}\frac{1}{[s+1]_{p,q}^{k}}\sum_{l=0}^{s+1}\binom{s+1}{l}(-1)^{l+i}m^i l^i\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient on both sides, we get:

$$G_{n,p,q}^{k}(x;a)m^n = \sum_{i=0}^{n}\binom{n}{i}E_{n-i}(x;a)m^{n-i}\sum_{s=0}^{\infty}\frac{1}{[s+1]_{p,q}^{k}}\sum_{l=0}^{s+1}\binom{s+1}{l}(-1)^{l+i}m^i l^i.$$

Therefore, eliminating m on both sides gives the Theorem 2.2.

Now let's look at the change according to the sign.

Using the Definition 2.1, we get some interesting properties from Theorem 2.3 to Theorem 2.4 for the change the sign.

Theorem 2.3. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned}G_{n,p,q}^{(k)}(x;-a) &= \sum_{l=0}^{n}\binom{n}{l}G_{l,p,q}^{(k)}(x;a)a^{n-l}\\ &= G_{n,p,q}^{(k)}(x+a;a).\end{aligned}$$

Corollary 2.3.1. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned} G_{n,p,q}^{(k)}(x;a) &= \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}(x;-a)(-a)^{n-l} \\ &= G_{n,p,q}^{(k)}(x-a;-a). \end{aligned}$$

Theorem 2.4. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned} G_{n,p,q}^{(k)}(-x;-a) &= \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}(a;a)(-x)^{n-l} \\ &= G_{n,p,q}^{(k)}(a-x;a). \end{aligned}$$

If x is replaced $x+y$ in (2.1), we get the next addition theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned} G_{n,p,q}^{(k)}(x+y;a) &= \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}(x;a)y^{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}(y;a)x^{n-l}. \end{aligned}$$

proof.
Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we get

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x+y;a)\frac{t^n}{n!} &= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{(x+y)t} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{l,p,q}^{(k)}(x;a)y^{n-l}\right)\frac{t^n}{n!}. \end{aligned}$$

Thus, we get the explicit result.

Theorem 2.6. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p,q}^{(k)}(x;a+b) = \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}((a+b)x;a+b)(1-a-b)^{n-l}x^{n-l}.$$

Also, if $a+b=\alpha$, we get

$$G_{n,p,q}^{(k)}(x;\alpha) = \sum_{l=0}^{n} \binom{n}{l} G_{l,p,q}^{(k)}(\alpha x;\alpha)(1-\alpha)^{n-l}x^{n-l}.$$

proof.

Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a+b)\frac{t^n}{n!}\\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{(a+b)t}+1}e^{xt}\\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{(a+b)t}+1}e^{(a+b)xt} \times \frac{e^{xt}}{e^{(a+b)xt}}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{l,p,q}^{(k)}((a+b)x;a+b)(1-a-b)^{n-l}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we get the explicit result.

Theorem 2.7. *For $n \geq 1$ and $k \in \mathbb{Z}$, we derive*

$$G_{n,p,q}^{(k)}(x+1;a) - G_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{n-1}\binom{n}{l}G_{l,p,q}^{(k)}(x;a).$$

proof.
Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x+1;a)\frac{t^n}{n!} - \sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}\\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} \times (e^t-1)\\
&= \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\binom{n}{l}G_{l,p,q}^{(k)}(x;a)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result.

Theorem 2.8. *For $n \geq 1$ and $k \in \mathbb{Z}$, we derive*

$$\begin{aligned}
G_{n,p,q}^{(k)}(x+a;a) + G_{n,p,q}^{(k)}(x;a) &= 2\sum_{i=0}^{\infty}\frac{1}{[i]_{p,q}^{k}}\sum_{l=0}^{i}\binom{i}{l}(-1)^l(x-l)^n\\
&= G_{n,p,q}^{(k)}(x;-a) + G_{n,p,q}^{(k)}(x;a).
\end{aligned}$$

proof.

Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x+a;a)\frac{t^n}{n!} + \sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} \\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} \times (e^{at}+1) \\
&= \sum_{i=1}^{\infty} \frac{1}{[i]_{p,q}^k} \sum_{l=0}^{i} \binom{i}{l}(-1)^l e^{(x-l)t} \\
&= \sum_{n=0}^{\infty} \left(2\sum_{i=0}^{\infty} \frac{1}{[i]_{p,q}^k} \sum_{l=0}^{i} \binom{i}{l}(-1)^l (x-l)^n \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result.

By using the binomials series and the definition of q-polylogarithm function, we derive the result as below.

Theorem 2.9. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p,q}^{(k)}(x;a) = 2\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{m+1} \frac{(-1)^{l-m+s}}{[m+1]_{p,q}^k}\binom{m+1}{s}(x-s+al-am)^n.$$

proof.
Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} \\
&= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} \\
&= 2Li_{k,q}(1-e^{-t})\sum_{m=0}^{\infty}(-1)^m e^{mat}e^{xt} \\
&= 2\left(\sum_{m=0}^{\infty}(-1)^m e^{(ma+x)t}\right)\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_{p,q}^k}\right) \\
&= 2\sum_{l=0}^{\infty}\sum_{m=0}^{l}(-1)^{l-m}e^{(al-am+x)t}\frac{(1-e^{-t})^{m+1}}{[m+1]_{p,q}^k} \\
&= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{l-m+r}(x-r+al-am)^n}{[m+1]_{p,q}^k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Similarly, we find next result that is related with the generalized Genocchi polynomials with variable a.

Theorem 2.10. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have*

$$G_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{\infty} \sum_{r=0}^{l+1} \binom{l+1}{r} \frac{(-1)^r}{[l+1]_{p,q}^k} E_{n,a}(x-r).$$

proof.
For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} &= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt} \\
&= \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{[l]_{p,q}^k} \times \frac{2e^{xt}}{e^{at}+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{r=0}^{l} \binom{l}{r}(-1)^r \times \sum_{n=0}^{\infty} E_{n,a}(x-r)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{r=0}^{l+1} \binom{l+1}{r} \frac{(-1)^r}{[l+1]_{p,q}^k} E_{n,a}(x-r) \right) \frac{t^n}{n!}.
\end{aligned}$$

3. Relation between Stirling numbers of the second kind and generalized q-poly-Genocchi polynomials with variable a

In this section, by using the generationg function of the Stirling numbers of the second kind, we obtain some interesting relations that is associated with the generalized q-poly-Genocchi polynomials with variable a. Recall that the Stirling numbers of the second kind are given by

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}.$$

By the definitions of the q-polylogarithm function $Li_{k,q}(x)$ and the Stirling numbers of the second kind, we get the following result.

$$Li_{k,q}(1-e^{-t}) = \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(-1)^{l+n}}{[l]_q^k} l! S_2(n,l)\frac{t^n}{n!}. \tag{3.1}$$

From the Equation (3.1), we have the next theorem which is connected with the Stirling numbers.

Theorem 3.1. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$Li_{k,p,q}(1-e^{-t}) = \sum_{n=1}^{\infty} \sum_{l=0}^{n} \frac{(-1)^{n+l}}{[l]_{p,q}^k} l! S_2(n,l)\frac{t^n}{n!}.$$

proof.

Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned} Li_{k,p,q}(1-e^{-t}) &= \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{[n]_{p,q}^k} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_{p,q}^k} n! \sum_{l=n}^{\infty} S_2(l,n) \frac{(-t)^l}{l!} \\ &= \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} \frac{(-1)^{n+l}}{[n]_{p,q}^k} n! S_2(l,n) \frac{t^l}{l1} \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(-1)^{n+1}}{[l]_{p,q}^k} l! S_2(n,l) \frac{t^n}{n!}. \end{aligned}$$

Theorem 3.2. *For $n \in (\mathbb{Z}_+)$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p,q}^{(k)}(x;a) = \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} \binom{n}{m} l! E_m(x;a) \frac{(-1)^{l+n-m}}{[l]_{p,q}^k} S_2(n-m,l).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a) \frac{t^n}{n!} &= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1} e^{xt} \\ &= \frac{2}{e^{at}+1} e^{xt} \times Li_{k,p,q}(1-e^{-t}) \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} \binom{n}{m} l! E_m(x;a) \frac{(-1)^{l+n-m}}{[l]_{p,q}^k} S_2(n-m,l) \frac{t^n}{n!}. \end{aligned}$$

□

Theorem 3.3. *For $n \in (\mathbb{Z}_+)$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p,q}^{(k)}(x;a) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} l! G_{n-m}(s;a) \frac{(-1)^{l+n+m}}{[l]_{p,q}^k} \frac{S_2(m+1,l)}{m+1}.$$

Proof. Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,p,q}^{(k)}(x;a) \frac{t^n}{n!} &= \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1} e^{xt} \\ &= \frac{2t}{e^{at}+1} e^{xt} \times \frac{Li_{k,p,q}(1-e^{-t})}{t} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} l! G_{n-m}(x;a) \frac{(-1)^{1+n+m}}{[l]_{p,q}^k} \frac{S_2(m+1,l)}{m+1} \frac{t^n}{n!}. \end{aligned}$$

□

By Theorem 2.12 and Theorem 2.13, we get the following corollary

Corollary 3.3.1. *For $n \in (\mathbb{Z}_+)$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned}&\sum_{m=0}^{n-1}\sum_{l=1}^{n-m}\binom{n}{m}l!E_m(x;a)\frac{(-1)^{l+n-m}}{[l]_{p,q}^k}S_2(n-m,l)\\&=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}l!G_{n-m}(s;a)\frac{(-1)^{l+n+m}}{[l]_{p,q}^k}\frac{S_2(m+1,l)}{m+1}.\end{aligned}$$

Theorem 3.4. *For $n \in (\mathbb{Z}_+)$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p,q}^{(k)}(x+2a;a)-G_{n,p,q}^{(k)}(x:a)=2\sum_{m=1}^{\infty}\frac{1}{[m]_{p,q}^k}\sum_{l=0}^{m}\binom{m}{l}(-1)^l((x+a-l)^n-(x-l)^n).$$

proof. Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(x+2a;a)-G_{n,p,q}^{(k)}(x:a)\frac{t^n}{n!}\\&=\frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}+1}e^{xt}(e^{2at}-1)\\&=2Li_{k,p,q}(1-e^{-t})e^{xt}(e^{at}-1)\\&=\sum_{n=0}^{\infty}2\sum_{m=1}^{\infty}\frac{1}{[m]_{p,q}^k}\sum_{l=0}^{m}\binom{m}{l}(-1)^l((x+a-l)^n-(x-l)^n)\frac{t^n}{n!}.\end{aligned}$$

4. Symmetric properties of the generalized (p,q)-Genocchi polynomials involving special functions

In this section, we consider several special functions and investigate some symmetric properties of the generalized (p,q)-Genocchi polynomials with variable a.

Theorem 4.1. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, $m_1, m_2 > 0$ with $m_1 \neq m_2$. Then we obtain*

Type 1:

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}m_1^l m_2^{n-l}G_{l,p,q}^{(k)}(m_2x;a)G_{n-l,p,q}^{(k)}(m_1x;a)\\&=\sum_{l=0}^{n}\binom{n}{l}m_1^{n-l}m_2^{l}G_{n-l,p,q}^{(k)}(m_2x;a)G_{l,p,q}^{(k)}(m_1x;a)\end{aligned}$$

and

Type 2:

$$\sum_{l=0}^{n}\binom{n}{l}m_1^l m_2^{n-l}G_{l,p,q}^{(k)}(m_2x;\frac{m_2}{m_1}a)G_{n-l,p,q}^{(k)}(m_1x;\frac{m_1}{m_2}a)$$
$$=\sum_{l=0}^{n}\binom{n}{l}m_1^{n-l}m_2^{l}G_{n-l,p,q}^{(k)}(m_2x;\frac{m_2}{m_1}a)G_{l,p,q}^{(k)}(m_1x;\frac{m_1}{m_2}a).$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $m_1, m_2 > 0 (m_1 \neq m_2)$, we consider a special function as follows

$$F(t)=\frac{4Li_{k,p,q}(1-e^{-m_1t})Li_{k,p,q}(1-e^{-m_2t})}{(e^{am_1t}+1)(e^{am_2t}+1)}e^{2m_1m_2xt}. \tag{4.1}$$

The Equation (4.1) is appeared by

$$\begin{aligned}F(t)&=\frac{2Li_{k,p,q}(1-e^{-m_1t})}{(e^{am_1t}+1)}e^{m_1m_2xt}\frac{2Li_{k,p,q}(1-e^{-m_2t})}{(e^{am_2t}+1)}e^{m_1m_2xt}\\&=\sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(m_2x;a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(m_1x;a)\frac{(m_2t)^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}m_1^{n-l}m_2^lG_{l,p,q}^{(k)}(m_1x;a)G_{n-l,p,q}^{(k)}(m_2x;a)\frac{t^n}{n!}.\end{aligned} \tag{4.2}$$

Similarly, we can see that

$$F(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}m_1^{l}m_2^{n-l}G_{n-l,p,q}^{(k)}(m_1x;a)G_{l,p,q}^{(k)}(m_2x;a)\frac{t^n}{n!}. \tag{4.3}$$

Comparing the coefficient of Equation (4.2) and (4.3), it is clear to get Theorem 4.1. □

Type 2 can be obtained in a similar way if we think of it as follows:

$$\begin{aligned}&\frac{2Li_{k,p,q}(1-e^{-m_1t})}{e^{am_1t}+1}e^{m_1m_2xt}\\&=\frac{2Li_{k,p,q}(1-e^{-m_1t})}{e^{\frac{m_2}{m_1}a\times m_1t}+1}e^{m_2x\times m_1t}.\end{aligned}$$

Let m be an odd number. Then we have

$$\frac{e^{mt}+1}{e^t+1}=\sum_{m=0}^{\infty}\mathbf{A}_k(m)\frac{t^k}{k!} \tag{4.4}$$

where $\mathbf{A}_k(m)=\sum_{i=1}^{m-1}(-1)^i i^k$ is called the alternating power sum polynomials(cf, [4, 6, 11]).

Using Equation (4.4), we have the symmetric identity of the generalized q-poly-Genocchi polynomials.

Theorem 4.2. *Let m_1 and m_2 be odd numbers. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned}
&\sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{l=0}^{s}\binom{n}{r}\binom{r}{s}\binom{s}{l}a^{n-s}m_1^{l+r-s+1}m_2^{n+s-r-l} \\
&\qquad \times G_{l,p,q}^{(k)}(a)G_{s-l,p,q}^{(k)}(a)\mathbf{A}_{r-s}(m_2)G_{n-r}(m_1x) \\
&= \sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{l=0}^{s}\binom{n}{r}\binom{r}{s}\binom{s}{l}a^{n-s}m_1^{l+n-r}m_2^{r-l+1} \\
&\qquad \times G_{l,p,q}^{(k)}(a)G_{s-l,p,q}^{(k)}(a)\mathbf{A}_{r-s}(m_1)G_{n-r-l}(m_2x).
\end{aligned}$$

proof. Let $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$ and $m_1, m_2 > 0$ with$m_1 \neq m_2$. Then we consider the generating function as follows:

$$F(t) = \frac{8Li_{k,p,q}(1-e^{-m_1t})Li_{k,p,q}(1-e^{-m_2t})(e^{am_1m_2t}+1)(e^{am_1m_2xt})t}{(e^{am_1t}+1)^2(e^{am_2t}+1)^2}.$$

From the generating function $F(t)$ and Equation (4.4), we get

$$\begin{aligned}
F(t) &= \frac{2Li_{k,p,q}(1-e^{-m_1t})2Li_{k,p,q}(1-e^{-m_2t})(e^{am_1m_2t}+1)2t(e^{am_1m_2xt})}{(e^{am_1t}+1)^2(e^{am_2t}+1)^2} \\
&= \sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(a)\frac{(m_2t)^n}{n!} \\
&\times \sum_{n=0}^{\infty}\mathbf{A}_n(m_2)\frac{(am_1t)^n}{n!}\sum_{n=0}^{\infty}G_n(m_1x)\frac{(am_2t)^n}{n!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{l=0}^{s}\binom{n}{r}\binom{r}{s}\binom{s}{l}a^{n-s-1}m_1^{l+r-s}m_2^{n+s-r-l-1} \\
&\qquad \times G_{l,p,q}^{(k)}(a)G_{s-l,p,q}^{(k)}(a)\mathbf{A}_{r-s}(m_2)G_{n-r}(m_1x)\frac{t^n}{n!}.
\end{aligned}$$

In similar method, $F(t)$ is expressed by

$$\begin{aligned}
F(t) &= \frac{2Li_{k,p,q}(1-e^{-m_1t})2Li_{k,p,q}(1-e^{-m_2t})(e^{am_1m_2t}+1)2t(e^{am_1m_2xt})}{(e^{am_1t}+1)^2(e^{am_2t}+1)^2} \\
&= \sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}G_{n,p,q}^{(k)}(a)\frac{(m_2t)^n}{n!} \\
&\times \sum_{r=0}^{\infty}\mathbf{A}_n(m_1)\frac{(am_2t)^n}{n!}\sum_{n=0}^{\infty}G_n(m_2x)\frac{(am_1t)^n}{n!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{r}\sum_{l=0}^{s}\binom{n}{r}\binom{r}{s}\binom{s}{l}a^{n-s-1}m_1^{l+n-r-1}m_2^{r-l} \\
&\qquad \times G_{l,p,q}^{(k)}(a)G_{s-l,p,q}^{(k)}(a)\mathbf{A}_{r-s}(m_1)G_{n-r}(m_2x)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we get the symmetric identity Theorem 4.2.

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Hui Young Lee received Ph.D. from Hannam University. He is currently a professor at Hannam University since 2015. His research interests are various polynomials, numbers and symmetry properties.

Department of Mathematics, Hannam University, 70 Hannamro, Daedeok-Gu Daejeon 34430, Korea.

e-mail: `hylee@hnu.kr`