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CERTAIN ASPECTS OF $\mathcal{I}$-LACUNARY ARITHMETIC STATISTICAL CONVERGENCE

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Abstract. In this paper, we firstly presented the definitions of arithmetic $\mathcal{I}$-statistically convergence, $\mathcal{I}$-lacunary arithmetic statistically convergence, strongly $\mathcal{I}$-lacunary arithmetic convergence, $\mathcal{I}$-Cesàro arithmetic summable and strongly $\mathcal{I}$-Cesàro arithmetic summable using weighted density via Orlicz function $\widetilde{\phi}$. Then, we proved some theorems associated with these concepts, and we examined the relationship between them. Finally, we establish some sequential properties of $\mathcal{I}$-lacunary arithmetic statistical continuity.

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1. Introduction

The idea of statistical convergence was first introduced by Fast [9]. Statistical convergence has several applications in different fields of mathematics like number theory, trigonometric series, summability theory, statistics and probability theory, measure theory, optimization, approximation theory and rough set theory. Since then several generalizations and applications of this concept have been investigated by various authors, namely Fridy [10], Gürdal and Huban [15], Nabiev et al. [28], and many others (see [12, 16]). Also, the readers should refer to the monographs [2] and [26] for the background on the sequence spaces and related topics. The idea of arithmetic convergence was introduced by Ruckle [30]. The studies on arithmetic convergence and related results can be found in [20, 21, 22, 23, 34, 35, 36]. Kostyrko et al. [24] extended the notion of statistical convergence to ideal convergence and established some basic theorems. On the other hand, the new form of convergence called $\mathcal{I}$-statistical convergence has been introduced in [5]. Recently lots of interesting developments have occurred in $\mathcal{I}$-statistical convergence and related topics (see [8, 19, 33]).

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In this study, we introduce the concepts of arithmetic $\mathcal{I}$-statistically $\widetilde{\phi}$-convergence of weight g, $\mathcal{I}$-lacunary arithmetic statistically $\widetilde{\phi}$-convergence of weight g, strongly $\mathcal{I}$-lacunary arithmetic $\widetilde{\phi}$-convergence of weight g, $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable of weight g and strongly $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable of weight g and study some relations among these new concepts. We also establish some sequential properties of $\mathcal{I}$-lacunary arithmetic statistical $\widetilde{\phi}$-continuity.

2. Basic Concepts

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left|x_m - x_{\langle m,n\rangle}\right| < \varepsilon$, where the symbol $\langle m, n\rangle$ denotes the greatest common divisor of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC. Firstly, arithmetic convergence was introduced due to Ruckle [30]. Later Yaying and Hazarika was studied this notion in many ways [34, 35, 36].

Statistical convergence depends on the natural density of subsets of the set $\mathbb{N}$ of positive integers. The natural density $\delta(A)$ of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_{t\to\infty} t^{-1} \left|\{m \leq t : m \in A\}\right|$$

where the vertical bars indicate the number of elements in the enclosed set. A sequence (x_m) is said to be statistically convergent to ℓ if for each $\varepsilon > 0$,

$$\lim_{t\to\infty} t^{-1} \left|\{m \leq t : |x_m - \ell| \geq \varepsilon\}\right| = 0.$$

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [11] as follows: A lacunary sequence is an increasing sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Here, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

Definition 2.1. A sequence (x_m) of real numbers is said to be lacunary statistically convergent to ℓ if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1} \left|\{m \in I_r : |x_m - \ell| \geq \varepsilon\}\right| = 0.$$

In [11] the relation between lacunary statistical convergence and statistical convergence was established among other things. And the notion of lacunary convergence has been investigated by many authors [4, 19, 25].

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [29], that is, a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x) \leq M.\widetilde{\phi}(x)$ for every $x \in \mathbb{R}^+$. In [29], Rao and Ren describes some important applications of Orlicz functions in many areas such as economics, stochastic problems etc. The reader can also refer to the paper [6] and recent monograph [3] related with various ways to generalize Orlicz sequence spaces systematically and investigate several structural properties of such spaces. Few examples of Orlicz functions are given below:

Example 2.2. (i) For a fixed $r \in \mathbb{N}$, the function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = |x|^r$ is an Orlicz function.

(ii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

(iv) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^3$ is not an Orlicz function.

Definition 2.3. ([32]) Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be statistically $\widetilde{\phi}$-convergent to L if for each $\varepsilon > 0$,

$$\lim_t \frac{1}{t} \left| \left\{ m \leq t : \widetilde{\phi}\,(x_m - L) \geq \varepsilon \right\} \right| = 0.$$

Recently the notion of statistically $\widetilde{\phi}$-convergent has been investigated by many authors [7, 8, 18, 19].

And the concept of ideal convergence is introduced as a generalization of statistical convergence. First of all this notion was highlighted by Kostyrko et al. [24]. Later on it was studied by Hazarika and Esi and many others [13, 14, 17].

Definition 2.4. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ provided:
(a) $\emptyset \in \mathcal{I}$,
(b) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(c) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.5. A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be an filter of $\mathbb{N}$ provided:
(a) $\emptyset \notin \mathcal{F}$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F$, $A \subset B$ implies $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}\,(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$$

is filter of $\mathbb{N}$. It is called the filter associated with the ideal.

A proper ideal $\mathcal{I}$ is called to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper we consider that $\mathcal{I}$ is a proper admissible ideals of $\mathbb{N}$.

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a sequence of points of elements of $\mathbb{R}$. We say that the sequence x is $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A\,(\varepsilon) = \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}$. Take for $\mathcal{I}$ the class $\mathcal{I}_{\text{fin}}$

of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_{\text{fin}}$ is a non-trivial admissible ideal and $\mathcal{I}_{\text{fin}}$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the references in [27].

We also recall that the concept of $\mathcal{I}$-statistically convergent is studied in [5]. A sequence (x_m) is said to be $\mathcal{I}$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ t \in \mathbb{N} : \frac{1}{t} \left| \{ m \leq t : |x_m - L| \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}$-statistical limit of the sequence (x_m) and we write $\mathcal{I}\text{-}st - \lim_{m\to\infty} x_m = L$.

In the recent times in [1], using the natural density of weight g where $g : \mathbb{N} \to [0,\infty)$ is a function with property that $\lim_{n\to\infty} g(n) = \infty$ and $\dfrac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$, the concept of natural density was extended as follows: The upper density of weight g was defined by

$$\overline{\delta}_g(A) = \lim_{n\to\infty} \sup \frac{A(1,n)}{g(n)}$$

for $A \subset \mathbb{N}$, where $A(1,n)$ denotes the number of elements in $A \cap [1,n]$. The lower density of weight g is defined in a similar manner. Then, the family

$$\mathcal{I}_g = \left\{ A \subset \mathbb{N} : \overline{\delta}_g(A) = 0 \right\}$$

creates an ideal. It was seen in [1] that $\mathbb{N} \in \mathcal{I}_g$ iff $\dfrac{n}{g(n)} \to 0$ as $n \to \infty$. Furthermore, we suppose that $n/g(n) \nrightarrow 0$ as $n \to \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. We denote by G the collection of such weight functions g satisfying the above properties. As a natural consequence we can introduce the following definition.

Definition 2.6. A sequence $x = (x_m)$ of real numbers is said to δ_g-statistically $\widetilde{\phi}$-convergent to L if for any given $\varepsilon > 0$,

$$\overline{\delta}_g\left(\left\{ m \in \mathbb{N} : \widetilde{\phi}(x_m - L) \geq \varepsilon \right\} \right) = 0.$$

Remark 2.1. If we take $\widetilde{\phi}(x) = |x|$, then convergent concepts in above definition coincide with δ_g-statistically convergence in [31].

3. Main results

In this section, we give the definitions of arithmetic $\mathcal{I}$-statistically convergence, $\mathcal{I}$-lacunary arithmetic statistically convergence, strongly $\mathcal{I}$-lacunary arithmetic convergence, $\mathcal{I}$-Cesàro arithmetic summable and strongly $\mathcal{I}$-Cesàro arithmetic summable using weighted density via Orlicz function $\widetilde{\phi}$. Also, we study the relationship between them and obtain some interesting results.

Definition 3.1. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_m)$ is said to be arithmetic statistically $\widetilde{\phi}$-convergent if for $\varepsilon>0$, there is an integer n such that

$$\lim_{t\to\infty}\frac{1}{t}\left|\left\{m\le t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|=0.$$

$ASC\left(\widetilde{\phi}\right)$ is used to denote the set of all arithmetic statistical $\widetilde{\phi}$-convergence sequences. Therefore, for $\varepsilon>0$ and integer n

$$ASC\left(\widetilde{\phi}\right)=\left\{x=(x_m):\lim_{t\to\infty}\frac{1}{t}\left|\left\{m\le t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|=0\right\}.$$

It is written as $ASC\left(\widetilde{\phi}\right)-\lim x_m=x_{\langle m,n\rangle}$.

Definition 3.2. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_m)$ is called to be lacunary arithmetic statistically $\widetilde{\phi}$-convergent if for $\varepsilon>0$ there is an integer n such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|=0.$$

We give

$$ASC_\theta\left(\widetilde{\phi}\right)=\left\{x=(x_m):\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|=0\right\}.$$

It is written as $ASC_\theta\left(\widetilde{\phi}\right)-\lim x_m=x$.

We define the following:

Definition 3.3. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_m)$ is called to be arithmetic $\mathcal{I}$-statistically $\widetilde{\phi}$-convergent of weight g if for $\varepsilon>0$ and $\delta>0$, there is an integer n such that

$$\left\{t\in\mathbb{N}:\frac{1}{g(t)}\left|\left\{m\le t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|\ge\delta\right\}$$

belongs to $\mathcal{I}$.

We use $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g$ to indicate the set of all arithmetic $\mathcal{I}$-statistical $\widetilde{\phi}$-convergent of weight g sequences. Thus, for $\varepsilon>0$, $\delta>0$ and integer n

$$A\mathcal{I}SC\left(\widetilde{\phi}\right)^g=\left\{x=(x_m):\left\{t\in\mathbb{N}:\frac{1}{g(t)}\left|\left\{m\le t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\in\mathcal{I}\right\}.$$

We write $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$ to denote the sequence (x_m) is arithmetic $\mathcal{I}$-statistically $\widetilde{\phi}$-convergent of weight g to $x_{\langle m,n\rangle}$.

Remark 3.1. If we take $\widetilde{\phi}(x) = |x|$, $g(t) = t$ and $\mathcal{I} = \mathcal{I}_\delta = \{A : \delta(A) = 0\}$, then $\mathcal{AISC}\left(\widetilde{\phi}\right)^g$ concepts coincide with arithmetic statistically convergent in [34].

Definition 3.4. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be $\mathcal{I}$-lacunary arithmetic statistically $\widetilde{\phi}$-convergent of weight g if for $\varepsilon > 0$ and $\delta > 0$, there is an integer n such that

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\left\{m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$.

We write

$$\mathcal{AISC}_\theta\left(\widetilde{\phi}\right)^g = \left\{x = (x_m) : \left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\left\{m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right| \right.\right.$$
$$\left.\left. \geq \delta\right\} \in \mathcal{I}\right\}.$$

We will use $\mathcal{AISC}_\theta\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$ to show that the sequence (x_m) is $\mathcal{I}$-lacunary arithmetic statistically $\widetilde{\phi}$-convergent of weight g to $x_{\langle m,n\rangle}$.

Remark 3.2. It should be noted that lacunary statistical $\widetilde{\phi}$-convergence of weight g has not been studied till now. Obviously lacunary statistical $\widetilde{\phi}$-convergence of weight g is a special case of $\mathcal{I}$-lacunary statistical $\widetilde{\phi}$-convergence of weight g when we take $\mathcal{I} = \mathcal{I}_{\text{fin}}$. So, some properties of lacunary statistical $\widetilde{\phi}$-convergence of weight g can be easily obtained from our results with obvious modifications.

Theorem 3.5. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $g_1, g_2 \in G$ be such that there exist $M > 0$ and $j_0 \in \mathbb{N}$ such that $\frac{g_1(x)}{g_2(x)} \leq M$ for all $n \geq j_0$. Then $\mathcal{AISC}\left(\widetilde{\phi}\right)^{g_1} \subset \mathcal{AISC}\left(\widetilde{\phi}\right)^{g_2}$.*

Proof. For any $\varepsilon > 0$,

$$\frac{\left|\left\{m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right|}{g_2(t)} = \frac{g_1(t)}{g_2(t)} . \frac{\left|\left\{m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right|}{g_1(t)}$$
$$\leq M . \frac{\left|\left\{m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right|}{g_1(t)}$$

for $n \geq j_0$. Hence for any $\delta > 0$,

$$\left\{t \in \mathbb{N} : \frac{1}{g_2(t)} \left|\left\{m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subset \left\{t \in \mathbb{N} : \frac{1}{g_1(t)} \left|\left\{m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right| \geq \frac{\delta}{M}\right\} \cup \{1, 2, ..., j_0\}.$$

If $(x_m) \in A\mathcal{I}SC\left(\widetilde{\phi}\right)^{g_1}$ then the set on the right hand side belongs to the ideal $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $A\mathcal{I}SC\left(\widetilde{\phi}\right)^{g_1} \subset A\mathcal{I}SC\left(\widetilde{\phi}\right)^{g_2}$. □

Definition 3.6. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and θ be a lacunary sequence. A sequence $x = (x_m)$ is called to be strongly $\mathcal{I}$-lacunary arithmetic $\widetilde{\phi}$-convergent of weight g if for $\varepsilon > 0$, there is an integer n such that

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{m \in I_r} \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}$$

belongs to $\mathcal{I}$.

We use $AN_\theta[\mathcal{I}]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$ to indicate the sequence (x_m) is strongly $\mathcal{I}$-lacunary arithmetic convergent of weight g to $x_{\langle m,n\rangle}$.

Theorem 3.7. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function, $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I}$ be an admissible ideal.*

(i) If $AN_\theta[\mathcal{I}]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$, then $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$.

(ii) If $x = (x_m) \in \ell_\infty$, we denote the space of all bounded sequences by ℓ_∞ and $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$, then $AN_\theta[\mathcal{I}]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$.

Proof. (i) If $\varepsilon > 0$ and $AN_\theta[\mathcal{I}]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$, we can write

$$\begin{aligned}\sum_{m \in I_r} \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) &\geq \sum_{\substack{m \in I_r \\ \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon}} \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \\ &\geq \varepsilon. \left|\left\{m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right|\end{aligned}$$

and so

$$\frac{1}{\varepsilon g(h_r)} \sum_{m \in I_r} \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \frac{1}{g(h_r)} \left\{m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}.$$

Then, for any $\delta > 0$

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\left\{m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{m \in I_r} \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \geq \varepsilon.\delta\right\} \in \mathcal{I}.\end{aligned}$$

Since $AN_\theta[\mathcal{I}]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$, so it follows that $A\mathcal{I}SC_\theta - \lim x_m = x_{\langle m,n\rangle}$.

(ii) Suppose that $\mathcal{AISC}_\theta\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$ and $x_m \in \ell^\infty$. Then there exists an $M > 0$ such that

$$\widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) \leq M$$

for integer n. Given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{g\left(h_r\right)}\sum_{m\in I_r}\widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right) &= \frac{1}{g\left(h_r\right)}\sum_{\substack{m\in I_r\\ \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\frac{\varepsilon}{2}}}\widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\\ &+ \frac{1}{g\left(h_r\right)}\sum_{\substack{m\in I_r\\ \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)<\frac{\varepsilon}{2}}}\widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\\ &\leq \frac{M}{g\left(h_r\right)}\left|\left\{m\in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{g\left(h_r\right)}\sum_{m\in I_r}\widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\\ &\subseteq\left\{r\in\mathbb{N} : \frac{1}{g\left(h_r\right)}\left|\left\{m\in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\frac{\varepsilon}{2}\right\}\right|\geq\frac{\varepsilon}{2M}\right\}\in\mathcal{I}.\end{aligned}$$

Hence, we get $AN_\theta\left[\mathcal{I}\right]\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$. This proof is completed. □

Theorem 3.8. *Let $\widetilde{\phi} : \mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\theta = \{k_r\}$ be a lacunary sequence. If $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, then*

$$\mathcal{AISC}\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle} \text{ implies } \mathcal{AISC}_\theta\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}.$$

Proof. Suppose that $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, then we can find a $E > 1$ such that for sufficiently large r we have

$$\frac{g\left(h_r\right)}{g\left(k_r\right)}\geq E.$$

Since $\mathcal{AISC}\left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n\rangle}$, then for every $\varepsilon > 0$, sufficiently large r and integer n, we have

$$\begin{aligned}&\frac{1}{g\left(k_r\right)}\left|\left\{m\leq k_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|\\ &\geq\frac{1}{g\left(k_r\right)}\left|\left\{m\in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|\\ &\geq E.\frac{1}{g\left(h_r\right)}\left|\left\{m\in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|.\end{aligned}$$

Then for any $\delta > 0$, we get

$$\left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right| \geq \delta \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{g(k_r)} \left| \left\{ m \leq k_r : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right| \geq E.\delta \right\} \in \mathcal{I}.$$

This shows that $A\mathcal{I}SC_\theta \left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n \rangle}$. □

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$, $\cup \{m : k_{r-1} \leq m \leq k_r,\ r \in C\} \in \mathcal{F}(\mathcal{I})$.

Theorem 3.9. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. If $\theta = \{k_r\}$ is a lacunary sequence with $\sup_r \sum_{i=0}^{r-1} \frac{g(h_{i+1})}{g(k_{r-1})} = E < \infty$, then*

$$A\mathcal{I}SC_\theta \left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n \rangle} \text{ implies } A\mathcal{I}SC \left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n \rangle}.$$

Proof. Assume that $A\mathcal{I}SC_\theta \left(\widetilde{\phi}\right)^g - \lim x_m = x_{\langle m,n \rangle}$, and for $\varepsilon, \delta, \gamma > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right| < \delta \right\}$$

and

$$T = \left\{ t \in \mathbb{N} : \frac{1}{g(t)} \left| \left\{ m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right| < \gamma \right\}.$$

It is obvious from our assumption that $C \in \mathcal{F}(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{g(h_j)} \left| \left\{ m \in I_j : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right| < \delta$$

for all $j \in C$. Let $t \in \mathbb{N}$ be such that $k_{r-1} < t \leq k_r$ for some $r \in C$. Now

$$\frac{1}{g(t)} \left| \left\{ m \leq t : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{g(k_{r-1})} \left| \left\{ m \leq k_r : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$
$$= \frac{1}{g(k_{r-1})} \left| \left\{ m \in I_1 : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$
$$+ \frac{1}{g(k_{r-1})} \left| \left\{ m \in I_2 : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{1}{g(k_{r-1})} \left| \left\{ m \in I_r : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$
$$= \frac{g(k_1)}{g(k_{r-1})} \frac{1}{g(h_1)} \left| \left\{ m \in I_1 : \widetilde{\phi}\left(x_m - x_{\langle m,n \rangle}\right) \geq \varepsilon \right\} \right|$$

$$+\frac{g(k_2-k_1)}{g(k_{r-1})}\frac{1}{g(h_2)}\left|\left\{m\in I_2:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|$$
$$+\ldots+\frac{g(k_r-k_{r-1})}{g(k_{r-1})}\frac{1}{g(h_r)}\left|\left\{m\in I_r:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|$$
$$=\frac{g(k_1)}{g(k_{r-1})}A_1+\frac{g(k_2-k_1)}{g(k_{r-1})}A_2+\ldots+\frac{g(k_r-k_{r-1})}{g(k_{r-1})}A_r$$
$$\leq\left\{\sup_{j\in C}A_j\right\}\sup_r\sum_{i=0}^{r-1}\frac{g(k_{i+1}-k_i)}{g(k_{r-1})}<E\delta.$$

Choosing $\gamma=\frac{\delta}{E}$ and in view of the fact that $\cup\{t:k_{r-1}<t<k_r,\ r\in C\}\subset T$ where $C\in\mathcal{F}(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Now, we examine $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable and strongly $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable using weighted density.

Definition 3.10. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_m)$ is said to be $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable of weight g if for $\varepsilon>0$, there is an integer n such that

$$\left\{t\in\mathbb{N}:\widetilde{\phi}\left(\frac{1}{g(t)}\sum_{m=1}^{t}\left(x_m-x_{\langle m,n\rangle}\right)\right)\geq\varepsilon\right\}$$

belongs to $\mathcal{I}$. In this case, we write $AC_1(\mathcal{I})\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$.

Definition 3.11. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_m)$ is said to be strongly $\mathcal{I}$-Cesàro arithmetic $\widetilde{\phi}$-summable of weight g if for $\varepsilon>0$, there is an integer n such that

$$\left\{t\in\mathbb{N}:\frac{1}{g(t)}\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}$$

belongs to $\mathcal{I}$. In this case, we write $AC_1[\mathcal{I}]\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$.

Theorem 3.12. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. If $AC_1[\mathcal{I}]\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$, then $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$.*

Proof. Let $AC_1[\mathcal{I}]\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$, and $\varepsilon>0$ given. Then

$$\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\sum_{\substack{m=1\\ \widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon}}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)$$
$$\geq\varepsilon.\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon.g\left(t\right)}\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\frac{1}{g\left(t\right)}\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|.$$

Then for any $\delta>0$

$$\left\{t\in\mathbb{N}:\frac{1}{g\left(t\right)}\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{t\in\mathbb{N}:\frac{1}{g\left(t\right)}\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon.\delta\right\}\in\mathcal{I}.$$

This proves the result. □

Theorem 3.13. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $x=(x_m)\in\ell_\infty$. If $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$ then, $AC_1\left[\mathcal{I}\right]\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$.*

Proof. Suppose that $x=(x_m)$ is bounded and $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$. Then there is a B such that $\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\leq B$ for all n. Given $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{g\left(t\right)}\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)&=\frac{1}{g\left(t\right)}\sum_{\substack{m=1\\ \widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon}}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\\&\quad+\frac{1}{g\left(t\right)}\sum_{\substack{m=1\\ \widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)<\varepsilon}}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\\&\leq\frac{1}{g\left(t\right)}.B\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|\\&\quad+\frac{1}{g\left(t\right)}\varepsilon\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)<\varepsilon\right\}\right|\\&\leq\frac{B}{g\left(t\right)}\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|+\varepsilon.\end{aligned}$$

Then for any $\delta>0$,

$$\left\{t\in\mathbb{N}:\frac{1}{g\left(t\right)}\sum_{m=1}^{t}\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\delta\right\}$$
$$\subseteq\left\{t\in\mathbb{N}:\frac{1}{g\left(t\right)}\left|\left\{m\leq t:\widetilde{\phi}\left(x_m-x_{\langle m,n\rangle}\right)\geq\varepsilon\right\}\right|\geq\frac{\delta}{B}\right\}\in\mathcal{I}.$$

Therefore $AC_1\left[\mathcal{I}\right]\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$. □

Now, we shall examine $\mathcal{I}$-lacunary arithmetic statistical $\widetilde{\phi}$-continuity and obtain some interesting results.

Definition 3.14. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A function f defined on a subset K of $\mathbb{R}$ is said to be $\mathcal{I}$-lacunary arithmetic statistical $\widetilde{\phi}$-continuous of weight g if it preserves $\mathcal{I}$-lacunary arithmetic statistically $\widetilde{\phi}$-convergent of weight g, i.e. if $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g-\lim x_m=x_{\langle m,n\rangle}$ implies $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g-\lim f\left(x_m\right)=f\left(x_{\langle m,n\rangle}\right).$

We shall write $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous function to denote $\mathcal{I}$-lacunary arithmetic statistical $\widetilde{\phi}$-continuous function of weight g.

It is easy to see that the sum and the difference of two $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous functions is $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous. Also the composition of two $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous function is again $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous. In the classical case, it is known that the uniform limit of sequentially continuous function is sequentially continuous, now we see that the uniform limit of $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous functions is also $A\mathcal{I}SC\left(\widetilde{\phi}\right)^g$continuous.

Theorem 3.15. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function, $\left(f_m\right)_{m\in\mathbb{N}}$ be a sequence of $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous functions defined on a subset K of $\mathbb{R}$ and $\left(f_m\right)$ be uniformly ideally $\widetilde{\phi}$-convergent of weight g to a function f. Then f is $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous.*

Proof. Let $\varepsilon>0$ and $\left(x_m\right)$ be any $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ convergent sequence on a bounded subset K of $\mathbb{R}$. By the uniform $\widetilde{\phi}$-convergence of f_m, there exists $N\in\mathbb{N}$ such that $\widetilde{\phi}\left(f_m\left(x\right)-f\left(x\right)\right)<\frac{\varepsilon}{3}$ for all $m\geq N$ and for all $x\in K$.

Since f_k is $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous on K, we have for an integer n

$$\left\{r\in\mathbb{N}:\frac{1}{g\left(h_r\right)}\left|\left\{m\in I_r:\widetilde{\phi}\left(f_k\left(x_m\right)-f_k\left(x_{\langle m,n\rangle}\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

On the other hand, for an integer n we have

$$\begin{aligned}&\left|\left\{m\in I_r:\widetilde{\phi}\left(f\left(x_m\right)-f\left(x_{\langle m,n\rangle}\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|\\&\leq\left|\left\{m\in I_r:\widetilde{\phi}\left(f_k\left(x_{\langle m,n\rangle}\right)-f\left(x_{\langle m,n\rangle}\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|\\&+\left|\left\{m\in I_r:\widetilde{\phi}\left(f_k\left(x_{\langle m,n\rangle}\right)-f_k\left(x_m\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|\\&+\left|\left\{m\in I_r:\widetilde{\phi}\left(f_k\left(x_m\right)-f\left(x_m\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|.\end{aligned}$$

Then, for any $\delta>0$, we get

$$\left\{r\in\mathbb{N}:\frac{1}{g\left(h_r\right)}\left|\left\{m\in I_r:\widetilde{\phi}\left(f\left(x_m\right)-f\left(x_{\langle m,n\rangle}\right)\right)\geq\frac{\varepsilon}{3}\right\}\right|\geq\delta\right\}$$

$$\subset \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \widetilde{\phi}\left(f_k\left(x_{\langle m,n\rangle}\right) - f\left(x_{\langle m,n\rangle}\right)\right) \geq \frac{\varepsilon}{3} \right\} \right| \geq \delta \right\}$$
$$\cup \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \widetilde{\phi}\left(f_k\left(x_{\langle m,n\rangle}\right) - f_k\left(x_m\right)\right) \geq \frac{\varepsilon}{3} \right\} \right| \geq \delta \right\}$$
$$\cup \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \widetilde{\phi}\left(f_k\left(x_m\right) - f\left(x_m\right)\right) \geq \frac{\varepsilon}{3} \right\} \right| \geq \delta \right\}.$$

Thus, the set on the right hand side belongs to $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that f is $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous. □

As an immediate consequence of Theorem 3.15, we have the following result.

Corollary 3.16. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. The set of all $A\mathcal{I}SC_\theta\left(\widetilde{\phi}\right)^g$ continuous functions defined on a compact subset K of $\mathbb{R}$ is a closed subset of all continuous function on K.*

4. Conclusion

This study makes three contributions to the field of summability theory: (i) a kind of lacunary statistical and ideal arithmetic convergence for sequences using weighted density via Orlicz function; (ii) the strongly ideal lacunary arithmetic convergence for sequence via Orlicz function; and (iii) the concept of $\mathcal{I}$-Cesàro arithmetic summable and strongly $\mathcal{I}$-Cesàro arithmetic summable for Orlicz function. The conclusions of this study are more general and a natural extension of the conventional arithmetic convergence of sequences.

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