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ON PAIR MEAN CORDIAL GRAPHS

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Abstract. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p - 1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling behavior of few graphs including the closed helm graph, web graph, jewel graph, sunflower graph, flower graph, tadpole graph, dumbbell graph, umbrella graph, butterfly graph, jelly fish, triangular book graph, quadrilateral book graph.

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1. Introduction

In this paper, a simple, finite and undirected graph is a pair $G = (V, E)$, where V and E respectively are the set of all vertices and edges. The order and size of G respectively are the number of vertices and edges in G. A graph labeling is the assignment of labels or weights, traditionally represented by integers to vertices and/or edges or both subject to certain conditions. The concept of graph labeling can be applied to used in many applications like communication

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network addressing, data base management, circuit design, astronomy, radar, X-ray crystallography, coding theory, network security, channel assignment process and social networks. For the survey of graph labeling, we refer [4]. Most methods of graph labeling track their origin to one introduced by Rosa in [15]. The concept of cordial labeling was initiated by I. Cahit in [2] and the various kinds of cordial labeling were studied by several authors[1,3,5,7,13,14,17-23]. For the basic terms and definitions related to the graph theory we refer [6]. The notion of mean labeling was introduced in [16] and the idea of pair difference cordial labeling was first discussed in [8]. We introduce the concept of pair mean cordial labeling in [9] and examined the pair mean cordial labeling behavior of several graphs in [10-12]. In this paper, we obtain some new pair mean cordial graphs.

2. Preliminaries

Definition 2.1. A closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.

Definition 2.2. The web graph Wb_n is the graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.

Definition 2.3. The Jewel graph J_n is the graph with vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leq i \leq n\}$. Obviously the Jewel graph J_n has $n + 4$ vertices and $2n + 5$ edges.

Definition 2.4. The flag graph FL_n is obtained by joining one vertex of C_n to an extra vertex called root.

Definition 2.5. The sunflower graph S_n is obtained by taking a wheel with central vertex u and the cycle $C_n : u_1u_2...u_nu_1$ and new vertices $v_1v_2...v_n$ where v_i is joined by vertices $u_i, u_{i+1(modn)}$. Thus the sunflower graph S_n has $2n + 1$ vertices and $4n$ edges.

Definition 2.6. The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.

Definition 2.7. The tadpole graph $T(m, n)$ is the graph obtained by joining a cycle C_m to a path P_n with a bridge.

Definition 2.8. The graph obtained by joining two disjoint cycles $u_1u_2 \ldots u_mu_1$ and $v_1v_2 \ldots v_nv_1$ with an edge u_1v_1 is called dumbbell graph and it is denoted by $Db(m, n)$.

Definition 2.9. A umbrella graph $U(m, n)$ is the graph obtained by joining a path P_n with the cental vertex of a fan F_m.

Definition 2.10. The triangular book graph $B(3, n)$ with n-pages is defined as n copies of cycle C_3 sharing a common edge. The common edge is called the spine or base of the book.

Definition 2.11. The quadrilateral book graph $B(4, n)$ with n-pages is defined as n copies of cycle C_4 sharing a common edge. The common edge is called the spine or base of the book.

Definition 2.12. Jelly fish graphs $J(m, n)$ obtained from a cycle $C_4 : u_1u_2u_3u_4u_1$ by joining u_1 and u_3 with an edge and appending m pendent edges to u_2 and n pendent edges to u_4.

3. Pair Mean Cordial Labeling

Definition 3.1. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph.

Theorem 3.2. *A helm H_n is not a pair mean cordial for all $n \geq 3$.[9]*

Theorem 3.3. *A closed helm graph CH_n is not a pair mean cordial for all $n \geq 3$.*

Proof. Let $V(CH_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(CH_n) = \{uu_i, u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1, v_nv_1\}$. Then the closed helm graph CH_n has $2n + 1$ vertices and $4n$ edges. Suppose the closed helm CH_n is pair mean cordial. Then if the edge uv get the label 1, the only possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $2n - 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n - 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - (2n - 1) = 2n + 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n + 1 - (2n - 1) = 2 > 1$, a contradiction. □

Theorem 3.4. *The web graph Wb_n is pair mean cordial for all $n \geq 3$.*

Proof. Let us define $V(Wb_n) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(Wb_n) = \{u_iv_i, v_iw_i1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, u_nu_1, v_nv_1 : 1 \leq i \leq n-1\}$. Then the web graph Wb_n has $3n$ vertices and $4n$ edges. We have the following two cases:

Case 1: n is odd

First assign the labels $-1, -3, \ldots, -n$ to the vertices $u_1, u_3, \ldots, u_n$ respectively and assign the labels $3, 5, \ldots, n$ to the vertices $u_2, u_4, \ldots, u_{n-1}$ respectively. Then we assign the labels $2, 4, \ldots, n+1$ to the vertices $v_1, v_3, \ldots, v_n$ respectively

and assign the labels $-2, -4, \dots, -n+1$ respectively to the vertices $v_2, v_4, \dots, v_{n-1}$. Next we assign the labels $-n-1, -n-2, \dots, \frac{-3n+1}{2}$ to the vertices $w_1, w_2, \dots, w_{\frac{n-1}{2}}$ respectively and assign the labels $n+2, n+3, \dots, \frac{3n-1}{2}$ respectively to the vertices $w_{\frac{n+1}{2}}, w_{\frac{n+3}{2}}, \dots, w_{n-2}$. Finally assign the labels $1, -n$ to the vertices w_{n-1}, w_n respectively.
Case 2: n is even
Give the labels $-1, -3, \dots, -n+1$ to the vertices $u_1, u_3, \dots, u_{n-1}$ respectively and give the labels $3, 5, \dots, n+1$ to the vertices $u_2, u_4, \dots, u_n$ respectively. Next we give the labels $2, 4, \dots, n$ to the vertices $v_1, v_3, \dots, v_{n-1}$ respectively and give the labels $-2, -4, \dots, -n$ respectively to the vertices $v_2, v_4, \dots, v_n$. Furthermore we give the labels $-n-1, -n-2, \dots, \frac{-3n}{2}$ to the vertices $w_1, w_2, \dots, w_{\frac{n}{2}}$ respectively and give the labels $n+3, n+4, \dots, \frac{3n}{2}$ respectively to the vertices $w_{\frac{n+2}{2}}, w_{\frac{n+4}{2}}, \dots, w_{n-2}$. Finally give the labels $1, n+2$ to the vertices w_{n-1}, w_n respectively. In both cases, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 2n$.

□

Theorem 3.5. *The Jewel graph J_n is not a pair mean cordial for all $n \geq 1$ and except for $n = 3$ and 5.*

Proof. The vertex set and edge set of J_n are defined in Definition 2.3. The proof is divided into two cases:
Case 1: n is odd
There are four subcases arises:
Subcase 1: $n = 1$
Suppose that J_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 2 = 5$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 5 - 2 = 3 > 1$, a contradiction.
Subcase 2: $n = 3$
Let $\lambda(x) = -3$, $\lambda(y) = 1$, $\lambda(u) = -1$,$\lambda(v) = -2$, $\lambda(u_1) = 2$, $\lambda(u_2) = 3$, and $\lambda(u_3) = 3$. Then $\bar{\mathbb{S}}_{\lambda_1} = 5$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 6$.
Subcase 3: $n = 5$
Let $\lambda(x) = 2$, $\lambda(u) = 3$,$\lambda(v) = 4$, $\lambda(y) = -1$, $\lambda(u_1) = -2$, $\lambda(u_2) = -3$, $\lambda(u_3) = -4$, $\lambda(u_4) = 1$ and $\lambda(u_5) = 1$. Then $\bar{\mathbb{S}}_{\lambda_1} = 7$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 8$.
Subcase 4: $n > 5$
Then $n \geq 7$. Suppose that J_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 7. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 7$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 7 = 2n - 2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - 2 - 7 = 2n - 9 \geq 5 > 1$, a contradiction.
Case 2: n is even
There are two subcases arises:
Subcase 1: $n = 2$
Suppose that J_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum

number of edges label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 3 = 6$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 6 - 3 = 3 > 1$, a contradiction.

Subcase 2: $n > 2$

Then $n \geq 4$. Suppose that J_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 5. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 5 = 2n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - 5 \geq 3 > 1$, a contradiction. □

Theorem 3.6. *The flag graph FL_n is pair mean cordial for all $n \geq 4$.*

Proof. Let $V(FL_n) = \{u, u_i : 1 \leq i \leq n\}$ and $E(FL_n) = \{uu_1, u_iu_{i+1} : 1 \leq i \leq n - 1\}$. Then the flag graph FL_n has $n + 1$ vertices and $n + 1$ edges. Thus we have the following four cases:

Case 1: n is odd

There are two subcases arises:

Subcase 1: $n = 3$

Suppose that FL_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3 - 1 = 2 > 1$, a contradiction.

Subcase 2: $n > 3$

Let us define $\lambda(u) = 1$, $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Next we assign the labels $-3, -4, \ldots, \frac{-n-1}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_n$ and assign the labels $4, 5, \ldots, \frac{n+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{n-1}$ respectively.

Case 2: n is even

Let us define $\lambda(u) = 1$. Then we give the labels $1, 2, \ldots, \frac{n}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{n-1}$ and assign the labels $-1, -2, \ldots, \frac{-n}{2}$ to the vertices $u_2, u_4, \ldots, u_n$ respectively.

The following table given that this vertex labeling λ is a pair mean cordial of FL_n for all $n \geq 4$. □

Nature of n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
n is odd	$\frac{n+1}{2}$	$\frac{n+1}{2}$
n is even	$\frac{n}{2}$	$\frac{n+2}{2}$

Table 1

Theorem 3.7. *The sunflower graph S_n ia not a pair mean cordial for all $n \geq 3$.*

Proof. The vertex set and edge set of S_n are defined in Definition 2.1. Suppose that S_n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $2n - 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n - 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - (2n - 1) = 2n + 1$.

Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n+1-(2n-1) = 2 > 1$, a contradiction. □

Theorem 3.8. *The flower graph Fl_n is pair mean cordial for all $n \geq 3$.*

Proof. Define $V(Fl_n) = \{u, u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(Fl_n) = \{uu_i, uv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1, u_nu_1\}$. Then it has $3n+1$ vertices and $5n$ edges. We have the following two cases:

Case 1: n is odd

Let $\lambda(u) = \frac{3n+1}{2}$. Now assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ to the vertices $u_1, u_3, \ldots, u_{n-2}$ respectively and assign the labels $-2, -5, \ldots, \frac{-3n+5}{2}$ respectively to the vertices $u_2, u_4, \ldots, u_{n-1}$. Then we assign the label $\frac{-3n+1}{2}$ to the vertex u_n. Next we assign the labels $2, 5, \ldots, \frac{3n-5}{2}$ to the vertices $v_1, v_3, \ldots, v_{n-2}$ respectively and assign the labels $-3, -6, \ldots, \frac{-3n+3}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{n-1}$. Furthermore assign the label $\frac{-3n-1}{2}$ to the vertex v_n. We assign the labels $-1, -4, \ldots, \frac{-3n+7}{2}$ to the vertices $w_1, w_3, \ldots, w_{n-2}$ respectively and assign the labels $4, 7, \ldots, \frac{3n-1}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_{n-1}$. Finally assign the label 1 to the vertex w_n.

Case 2: n is even

Let $\lambda(u) = \frac{-3n+4}{2}$. Give the labels $3, 6, \ldots, \frac{3n}{2}$ to the vertices $u_1, u_3, \ldots, u_{n-1}$ respectively and also give the labels $-2, -5, \ldots, \frac{-3n+2}{2}$ to the vertices $u_2, u_4, \ldots, u_n$ respectively. We now give the labels $2, 5, \ldots, \frac{3n-2}{2}$ to the vertices $v_1, v_3, \ldots, v_{n-1}$ respectively and give the labels $-3, -6, \ldots, \frac{-3n}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_n$. Furthermore we give the labels $-1, -4, \ldots, \frac{-3n+4}{2}$ to the vertices $w_1, w_3, \ldots, w_{n-1}$ respectively and give the labels $4, 7, \ldots, \frac{3n-4}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_{n-2}$. Finally assign the label 1 to the vertex w_n.

The following table given that this vertex labeling λ is a pair mean cordial of Fl_n for all $n \geq 3$. □

Nature of n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
n is odd	$\frac{5n-1}{2}$	$\frac{5n+1}{2}$
n is even	$\frac{5n}{2}$	$\frac{5n}{2}$

Table 2

Theorem 3.9. *The tadpole graph $T(m, n)$ is pair mean cordial for all $m \geq 3$ and $n \geq 1$ and except for $m = 3$ and $n = 1$.*

Proof. Let $V(T(m, n)) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(T(m, n)) = \{u_iu_{i+1} : 1 \leq i \leq m-1\} \cup \{u_mu_1, u_1v_1\} \cup \{v_jv_{j+1} : 1 \leq j \leq n-1\}$. Obviously the tadpole graph $T(m, n)$ has $m+n$ vertices and $m+n$ edges. We have the following four cases:

Case 1: $m \equiv 0(\text{mod}4)$

There are two subcases arises:

Subcase 1: n is odd

If $n = 1$, then $\lambda(u_1) = 1$ and if $n > 1$, then $\lambda(u_1) = \frac{m+2}{2}$. Next we assign the labels $2, 3, \ldots, \frac{m}{2}$ respectively to the vertices $u_3, u_5, \ldots, u_{m-1}$ and assign the labels $-1, -2, \ldots, \frac{-m}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. Also we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and assign the labels $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Finally assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively.

Subcase 2: n is even

If $m = 4$, then $\lambda(u_1) = -2$ and if $m > 4$, then $\lambda(u_1) = \frac{m+2}{2}$. Now we give the labels $2, -1, 3, -2$ respectively to the vertices u_2, u_3, u_4, u_5. Next we give the labels $-3, -4, \ldots, \frac{-m}{2}$ to the vertices $u_6, u_8, \ldots, u_m$ respectively and give the labels $4, 5, \ldots, \frac{m}{2}$ respectively to the vertices $u_7, u_9, \ldots, u_{m-1}$. Also we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and give the labels $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally give the label 1 to the vertex v_n.

Case 2: $m \equiv 1 (\text{mod } 4)$

There are two subcases arises:

Subcase 1: n is odd

Let $\lambda(u_1) = \frac{-m-1}{2}$. Now we give the labels $2, -1, 3, -2$ respectively to the vertices u_2, u_3, u_4, u_5. Next we give the labels $-3, -4, \ldots, \frac{-m+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{m-1}$ respectively and give the labels $4, 5, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_7, u_9, \ldots, u_m$. Also we give the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and give the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally give the label 1 to the vertex v_n.

Subcase 2: n is even

Let $\lambda(u_1) = \frac{-m-1}{2}$. Next we assign the labels $2, 3, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_3, u_5, \ldots, u_m$ and assign the labels $-1, -2, \ldots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. We assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively.

Case 3: $m \equiv 2 (\text{mod } 4)$

There are two subcases arises:

Subcase 1: n is odd

Let us assign the labels the vertices $u_i, 1 \leq i \leq m$ and $v_j, 1 \leq j \leq n$ as in Subcase 1 of Case 1.

Subcase 2: n is even

Let $\lambda(u_1) = \frac{m+2}{2}$. Then we assign the labels to the vertices $u_i, 1 \leq i \leq m$ and $v_j, 1 \leq j \leq n$ as in Subcase 2 of Case 1.

Case 4: $m \equiv 3 (\text{mod } 4)$

There are two subcases arises:

Subcase 1: n is odd
If $m = 3$ and $n = 1$, Then $T(3,1)$ is not a pair mean cordial. Suppose that $T_{3,1}$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3 - 1 = 2 > 1$, a contradiction.
If $m = 3$ and $n > 1$, Then we assign the labels $3, 2, -1, -2$ to the vertices u_1, u_2, u_3, v_1 respectively. Furthermore we assign the labels $-3, -4, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{n-1}$ and assign the labels $4, 5, \ldots, \frac{m+n}{2}$ to the vertices $v_3, v_5, \ldots, v_{n-2}$ respectively. Finally assign the label 1 to the vertex v_n. If $m > 3$, then assign the labels to the vertices $u_i, 1 \leq i \leq m$ and $v_j, 1 \leq j \leq n$ as in Subcase 1 of Case 2.
Subcase 2: n is even
Let us assign the labels to the vertices $u_i, 1 \leq i \leq m$ and $v_j, 1 \leq j \leq n$ as in Subcase 2 of Case 2.
The table given below establish that this vertex labeling λ is a pair mean cordial of $T(m,n)$ for all $m \geq 3$ and $n \geq 1$ and except for $m = 3$ and $n = 1$.

□

Nature of m and n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m and n are both odd	$\frac{m+n}{2}$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$
m and n are both even	$\frac{m+n}{2}$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$

Table 3

Theorem 3.10. *The Dumbbell graph* $Db(m,n)$ *is pair mean cordial for all* $m, n \geq 3$.

Proof. Let $V(Db(m,n)) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(Db(m,n)) = \{u_i u_{i+1} : 1 \leq i \leq m\} \cup \{v_j v_{j+1} : 1 \leq j \leq n\} \cup \{u_m u_1, u_1 v_1, v_n v_1\}$. Then clearly the Dumbbell graph Dbm, n has $m + n$ vertices and $m + n + 1$ edges. We have the following two cases:
Case 1: m is odd
There are two cases arises:
Subcase 1: n is odd
If $m = 3$, Assign the labels $3, 2, -1, -2$ to the vertices u_1, u_2, u_3, v_1 respectively. Also we assign the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_3, v_5, \ldots, v_{n-2}$ and assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally assign the label 1 to the vertex v_n.
If $m > 3$, Give the labels $\frac{-m-1}{2}, 2-1, 3, -2$ to the vertices u_1, u_2, u_3, u_4, u_5 respectively. Then we give the labels $-3, -4, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_6, u_8, \ldots, u_{m-1}$ and give the labels $4, 5, \ldots, \frac{m+1}{2}$ to the vertices $u_7, u_9, \ldots, u_m$

respectively. Also we give the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and give the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally assign the label 1 to the vertex v_n.

Subcase 2: n is even

Assign the labels to the vertices $u_i, 1 \leq i \leq m$ as in Subcase 1 of Case 1. Now we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally assign the labels $1, 1$ to the vertices v_{n-1}, vn respectively.

Case 2: m is even

There are two cases arises:

Subcase 1: n is odd

If $m = 4$, Assign the labels $-2, 2, -1, 3$ to the vertices u_1, u_2, u_3, u_4 respectively. If $m > 4$, Give the labels $\frac{m+2}{2}, 2-1, 3, -2$ to the vertices u_1, u_2, u_3, u_4, u_5 respectively. Then we give the labels $-3, -4, \ldots, \frac{-m}{2}$ respectively to the vertices $u_6, u_8, \ldots, u_m$ and give the labels $4, 5, \ldots, \frac{m}{2}$ to the vertices $u_7, u_9, \ldots, u_{m-1}$ respectively. Next we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and assign the labels $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Finally we assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively.

Subcase 2: n is even

Assign the labels to the vertices $u_i, 1 \leq i \leq m$ as in Subcase 1 of Case 2. Now we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and assign the labels $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally assign the label 1 to the vertex vn.

The table given below establish that this vertex labeling λ is a pair mean cordial of $Db(m, n)$ for all $m, n \geq 3$.

□

Nature of m and n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m and n are both odd	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$
m is odd and n is even	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
m and n are both even	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$
m is odd and n is even	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$

Table 4

Theorem 3.11. *The Umbrella graph $U_{n,m}, m > 2$ is pair mean cordial for all $n \geq 2$, m is odd and $m \neq 3$ and $n > 2$ and m is even .*

Proof. Let $V(U_{n,m}) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(U_{n,m}) = \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{u_i v_1 : 1 \leq i \leq m\} \cup \{v_j v_{j+1} : 1 \leq j \leq n-1\}$. Clearly $U_{n,m}$ has $m+n$ vertices and $2m+n-2$ edges. Then we have the following four cases:

Case 1: $m \equiv 0 (\bmod\ 4)$
There are two subcases arises:
Subcase 1: n is even
If $n = 2$, $U_{n,m}$ is not pair mean cordial. Suppose that $U_{n,m}$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $m - 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq m+n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m+n-1-(m-1) = n = 2 > 1$, a contradiction.
If $n > 2$, then we assign the labels $2, 3, \ldots, \frac{m+2}{2}$ to the vertices $u_1, u_3, \ldots, u_{m-1}$ respectively and assign the labels $-1, -2, \ldots, \frac{-m}{2}$ respectively to the vertices $u_2, u_4, \ldots, u_m$. Now we give the labels $\frac{m+4}{2}, \frac{-m-2}{2}$ to the vertices v_1, v_2 respectively and give the labels $\frac{-m-4}{2}, \frac{-m-6}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_3, v_5, \ldots, v_{n-1}$. Furthermore we give the labels $\frac{m+6}{2}, \frac{m+8}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_4, v_6, \ldots, v_{n-2}$ respectively and assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{2m+n-2}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.
Subcase 2: n is odd
As in Subcase 1, assign the labels to the vertices u_i, $1 \leq i \leq m$. Next we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_1, v_3, \ldots, v_{n-2}$ respectively and we give the labels $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{n-3}$. Finally assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively. Thus $\bar{\mathbb{S}}_{\lambda_1} = \frac{2m+n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{2m+n-1}{2}$
Case 2: $m \equiv 1 (\bmod\ 4)$
There are two subcases arises:
Subcase 1: n is even
If $n = 2$, let us assign the labels $2, 3, \ldots, \frac{m+1}{2}$ to the vertices $u_1, u_3, \ldots, u_{m-2}$ respectively and assign the labels $-1, -2, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_2, u_4, \ldots, u_{m-1}$ and assign the label $\frac{-m-1}{2}$ to the vertex u_m. Furthermore we give the labels $\frac{m+1}{2}, 1$ to the vertices v_1, v_2 respectively.
If $n > 2$, first assign the labels $2, 3, \ldots, \frac{m+3}{2}$ to the vertices $u_1, u_3, \ldots, u_m$ respectively. Next we assign the labels $-1, -2, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_2, u_4, \ldots, u_{m-1}$ and assign the label $\frac{-m-1}{2}$ to the vertex v_1 . Now we give the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively and give the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_3, v_5, \ldots, v_{n-3}$. Finally give the labels $1, 1$ to the vertices v_n, v_{n-1} respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{2m+n-2}{2} = \bar{\mathbb{S}}_{\lambda_1^c}$.
Subcase 2: n is odd
As in subcase 1 of Case 2, assign the labels to the vertices u_i, $1 \leq i \leq m$ and v_1. We now give the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Furthermore we give the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_3, v_5, \ldots, v_{n-2}$ and finally give the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{2m+n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{2m+n-1}{3}$.
Case 3: $m \equiv 2 (\bmod\ 4)$
As in Case 1, assign the labels to the vertices u_i,$1 \leq i \leq m$ and v_j,$1 \leq j \leq n$.

Case 4: $m \equiv 3 (\text{mod } 4)$
If $m = 3$ and $n = 2$, $U_{3,2}$ is not pair mean cordial. Suppose that $U_{3,2}$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 4$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4 - 2 = 2 > 1$, a contradiction.
If $m > 3$, As in Case 2, assign the labels to the vertices u_i,$1 \leq i \leq m$ and v_j,$1 \leq j \leq n$. □

Theorem 3.12. *The Butterfly graph $B(m, n)$ is pair mean cordial for all $m, n \geq 2$.*

Proof. Let $V(B(m, n)) = \{u_i, v_j, u, v, w : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(B(m, n)) = \{uw, vw, wu_i, wv_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_iu_{i+1}, v_jv_{j+1} : 1 \leq i \leq m - 1 and 1 \leq j \leq n - 1\}$. Then the Butterfly graph $B(m, n)$ has $m + n + 3$ vertices and $2(m + n)$ edges. We have the following two cases:
Case 1: m is odd
There are two subcases arises:
Subcase 1: n is odd
Let $\lambda(u) = 1$, $\lambda(v) = \frac{-m-n-2}{2}$ and $\lambda(w) = \frac{-m-1}{2}$. Now we assign the labels $2, 3, \ldots, \frac{m+3}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_m$ and assign the labels $-1, -2, \ldots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. Next we assign the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n+2}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally assign the label $\frac{m+n+2}{2}$ to the vertex v_n.
Subcase 2: n is even
Let $\lambda(v) = \frac{-m-n-3}{2}$. We give the labels to the vertices $u_j, u, w : 1 \leq j \leq m$ as in Subcase 1 of Case 1. Next we give the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n+3}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and give the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively.
Case 2: m is even
There are two subcases arises:
Subcase 1: n is odd
Let $\lambda(u) = 1$, $\lambda(v) = \frac{-m-n-3}{2}$ and $\lambda(w) = \frac{m+4}{2}$. Then we assign the labels $2, 3, \ldots, \frac{m+2}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and assign the labels $-1, -2, \ldots, \frac{-m}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. Finally we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and assign the labels $\frac{m+6}{2}, \frac{m+8}{2}, \ldots, \frac{m+n+3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively.
Subcase 2: n is even
Let $\lambda(v) = \frac{-m-n-2}{2}$. We give the labels to the vertices $u_j, u, w : 1 \leq j \leq m$ as in Subcase 1 of Case 2. Also we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and give the labels $\frac{m+6}{2}, \frac{m+8}{2}, \ldots, \frac{m+n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally assign the label $\frac{m+n+2}{2}$ to

the vertex v_n. In each cases $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = m + n$.

□

Theorem 3.13. *The Jelly fish graph $J(m, n)$ is pair mean cordial if and only if $m + n \leq 7$.*

Proof. Let $V(J(m, n)) = \{u_1, u_2, u_3, u_4, v_i, w_j : 1 \leq i \leq mand1 \leq j \leq n\}$ and $E(B(m, n)) = \{u_1u_3, u_1u_2, u_1u_4, u_2u3, u_3u_4, u_2v_i, u_4w_j : 1 \leq i \leq mand1 \leq j \leq n\}$. Clearly the Jelly fish graph $J(m, n)$ has $m + n + 4$ vertices and $m + n + 5$ edges. We have the following two cases:

Case 1: $m + n > 7$

There are two subcases arises:

Subcase 1: $m + n$ is even

Suppose that Jm, n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 5. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 5 = m + n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m + n - 5 \geq 3 > 1$, a contradiction.

Subcase 2: $m + n$ is odd

Suppose that Jm, n is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 6. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 6$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 6 = m + n - 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m + n - 7 \geq 2 > 1$, a contradiction.

Case 2: $m + n \leq 7$

There are six subcases arises:

Subcase 1: $m = 1$ Then $n \leq 6$. Let $\lambda(u_1) = 3$, $\lambda(u_2) = -1$, $\lambda(u_3) = -2$ and $\lambda(v_1) = 2$.

Nature of n	u_4	v_1	w_2	w_3	w_4	w_5	w_6
$n = 1$	-3	1					
$n = 2$	1	-3	1				
$n = 3$	-3	-4	4	1			
$n = 4$	4	-3	-4	1	1		
$n = 5$	4	-3	-4	-5	5	1	
$n = 6$	4	-3	-4	-5	5	-3	1

Table 5

Hence if n is odd, $\bar{\mathbb{S}}_{\lambda_1} = \frac{n+5}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+7}{2}$ and if n is even, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+6}{2}$

Subcase 2: $m = 2$

Then $n \leq 5$. Let $\lambda(u_1) = 3$, $\lambda(u_2) = -1$, $\lambda(v_1) = 1$ and $\lambda(v_2) = 2$.

Hence if n is odd, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+7}{2}$ and if n is even, $\bar{\mathbb{S}}_{\lambda_1} = \frac{n+6}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+8}{2}$.

Subcase 3: $m = 3$

Then $n \leq 4$. Let $\lambda(u_1) = -2$, $\lambda(u_2) = -1$,$\lambda(u_3) = 4$, $\lambda(u_4) = -3$, $\lambda(v_1) = 1$, $\lambda(v_2) = 2$ and $\lambda(v_3) = 3$.

Nature of n	u_3	u_4	w_1	w_2	w_3	w_4	w_5
$n=1$	-3	-2	3				
$n=2$	-2	-3	4	-4			
$n=3$	-2	4	-3	-4	1		
$n=4$	-2	4	-3	-4	-5	5	
$n=5$	-2	4	-3	-4	-5	5	-3

Table 6

Nature of n	w_1	w_2	w_3	w_4
$n=1$	-4			
$n=2$	-4	4		
$n=3$	-4	-5	5	
$n=4$	-4	-5	5	4

Table 7

Hence if n is odd, $\bar{\mathbb{S}}_{\lambda_1} = \frac{n+7}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+9}{2}$ and if n is even, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+8}{2}$.
Subcase 4: $m=4$
Then $n \leq 3$. Let $\lambda(u_1) = -2$, $\lambda(u_2) = -1$,$\lambda(u_3) = 4$, $\lambda(u_4) = -3$, $\lambda(v_1) = 1$, $\lambda(v_2) = 2$, $\lambda(v_3) = 3$ and $\lambda(v_4) = -4$.

Nature of n	w_1	w_2	w_3
$n=1$	4		
$n=2$	-5	5	
$n=3$	-5	5	5

Table 8

Hence if n is odd, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+9}{2}$ and if n is even,$\bar{\mathbb{S}}_{\lambda_1} = \frac{n+8}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+10}{2}$.
Subcase 5: $m=5$
Then $n \leq 2$. Let $\lambda(u_1) = -2$, $\lambda(u_2) = -1$,$\lambda(u_3) = 4$, $\lambda(u_4) = -3$, $\lambda(v_1) = 1$, $\lambda(v_2) = 2$, $\lambda(v_3) = 3$, $\lambda(v_4) = -4$ and $\lambda(v_5) = -5$.
If $n=1$, then $\lambda(w_1) = 5$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n+9}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+11}{2}$. If $n=2$, then $\lambda(w_1) = 5$, $\lambda(w_2) = 5$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+10}{2}$.
Subcase 6: $m=6$
Then $n=1$. Let $\lambda(u_1) = -2$, $\lambda(u_2) = -1$,$\lambda(u_3) = 4$, $\lambda(u_4) = -3$, $\lambda(v_1) = 1$, $\lambda(v_2) = 2$, $\lambda(v_3) = 3$, $\lambda(v_4) = -4$, $\lambda(v_5) = -5$, $\lambda(v_6) = 5$ and $\lambda(w_1) = 4$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n+10}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+12}{2}$. □

Theorem 3.14. *The triangular book graph $B(3,n)$ with n pages is not a pair mean cordial for all $n > 1$ and except for $n = 1$ and 5.*

Proof. Let $V(B(3,n)) = \{u, v, u_i : 1 \leq i \leq n\}$ and $E(B(3,n)) = \{uv, uu_i, vu_i : 1 \leq i \leq n\}$. Then it has $n+2$ vertices and $2n+1$ edges. We have the following

four cases:
Case 1: $n = 1$
Let $\lambda(u) = 1$, $\lambda(v) = 1$ and $\lambda(u_1) = -1$. Then $\bar{\mathbb{S}}_{\lambda_1} = 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2$.
Case 2: $2 \leq n \leq 4$
Suppose $B(3,n)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label 1 is $n-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(n-1) = n+2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+2-(n-1) = 3 > 1$, a contradiction.
Case 3: $n = 5$
Let $\lambda(u) = -1$, $\lambda(v) = -2$,$\lambda(u_1) = 2$, $\lambda(u_2) = 3$, $\lambda(u_3) = -3$, $\lambda(u_4) = 1$ and $\lambda(u_5) = 3$. Then $\bar{\mathbb{S}}_{\lambda_1} = 5$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 6$.
Case 4: $n > 5$
There are two subcases arises:
Subcase 1: n is even
Then $n \geq 6$. Suppose $B(3,n)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 4. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-4 = 2n-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n-3-4 = 2n-7 \geq 5 > 1$, a contradiction.
Subcase 2: n is odd
Then $n \geq 7$. Suppose $B(3,n)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 6. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 6$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-6 = 2n-5$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n-5-6 = 2n-11 \geq 3 > 1$, a contradiction.
□

Theorem 3.15. *The quadrilateral book graph $B(4,n)$ with n pages is not a pair mean cordial for all $n \geq 1$ and except for $n = 2, 3$ and 4.*

Proof. Let $V(B(4,n)) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B(4,n)) = \{uv, uu_i, vu_i, u_iv_i : 1 \leq i \leq n\}$. Then it has $2n+2$ vertices and $3n+1$ edges. We have the following three cases:
Case 1: $n = 1$
Suppose $B(4,n)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-1 = 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3-1 = 2 > 1$, a contradiction.
Case 2: $n = 2$
Let $\lambda(u) = -2$, $\lambda(v) = 3$,$\lambda(u_1) = 2$, $\lambda(u_2) = 1$, $\lambda(v_1) = -1$ and $\lambda(v_2) = -3$. Then $\bar{\mathbb{S}}_{\lambda_1} = 3$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 4$.
Case 3: $n = 3$
Let $\lambda(u) = -2$, $\lambda(v) = 3$,$\lambda(u_1) = 2$, $\lambda(u_2) = 4$, $\lambda(u_3) = 1$, $\lambda(v_1) = -1$, $\lambda(v_2) = -3$ and $\lambda(v_3) = -4$. Then $\bar{\mathbb{S}}_{\lambda_1} = 5$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 5$.
Case 4: $n = 4$
Let $\lambda(u) = -2$, $\lambda(v) = 3$,$\lambda(u_1) = 2$, $\lambda(u_2) = 4$, $\lambda(u_3) = 5$, $\lambda(u_4) = 1$,

$\lambda(v_1) = -1$, $\lambda(v_2) = -3$, $\lambda(v_3) = -4$ and $\lambda(v_4) = -5$. Then $\bar{\mathbb{S}}_{\lambda_1} = 6$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 7$.

Case 5: $n > 4$

Then $n \geq 5$. Suppose $B(4,n)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $n+2$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n+2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(n+2) = 2n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n-1-(n+2) = n-3 \geq 2 > 1$, a contradiction.

□

4. Conclusion

Cahit[2] first proposed the idea of cordial labeling of in 1987 and Cordial labeling has grown to be a popular research area in graph labeling. The various kinds of cordial labeling were studied by several authors[1,3,5,7,13,14,17-23]. The notion of mean labeling was introduced in [16] and the idea of pair difference cordial labeling was first discussed in [8]. These two ideas served as our inspiration for introducing the pair mean cordial labeling in [9]. The results of the pair mean cordial labeling of few graphs including the closed helm graph, web graph, jewel graph, sunflower graph, flower graph, tadpole graph, dumbbell graph, umbrella graph, butterfly graph, jelly fish, triangular book graph, quadrilateral book graph are presented in the current paper. Investigating the pair mean cordial labeling of scorpion graph, spider graph, generalized Peterson graph, generalized Heawood graph, cubic diamond k-chain graph is more challenging to study. Studying the pair mean cordial labeling behavior of some other special graphs such as olive tree, coconut tree, step graph, lotus graph, generalized web graph, slanting ladder graph, pappus graph, dyck graph, bloom graph, jahangir graph, shadow graph, shackle graph, tensor product graph and bull graph would be one of the open problem for future research.

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