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SUBORDINATION AND SUPERORDINATION IMPLICATIONS ASSOCIATED WITH A CLASS OF NONLINEAR INTEGRAL OPERATORS†

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Abstract. In the present paper, we investigate the subordination and superordination implications for a class of certain nonlinear integral operators defined on the space of normalized analytic functions in the open unit disk. The sandwich-type theorem for these integral operators is also presented. Further, we extend some results given earlier as special cases of the main results presented here.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and a nonnegative integer n, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = F(w(z)) \quad (z \in \mathbb{U})$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then we have (cf. [18])

$$f \prec F \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

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Definition 1.1 (Miller and Mocanu [18])**.** Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If κ is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(\kappa(z), z\kappa'(z)) \prec h(z), \tag{1}$$

then κ is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $\kappa \prec q$ for all κ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant.

Recently, Miller and Mocanu [19] introduced the following differential superordinations, as the dual concept of differential subordinations.

Definition 1.2 (Miller and Mocanu [19])**.** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If κ and $\varphi(\kappa(z), z\kappa'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(\kappa(z), z\kappa'(z)), \tag{2}$$

then κ is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec \kappa$ for all κ satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Definition 1.3 (Miller and Mocanu [19])**.** We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ $(\zeta \in \partial\mathbb{U} \backslash E(f))$.

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

which are analytic in $\mathbb{U}$ and let $\mathcal{A}_1 = \mathcal{A}$. We also denote the class $\mathcal{D}$ by

$$\mathcal{D} := \{\varphi \in \mathcal{H} : \varphi(0) = 1 \quad \text{and} \quad \varphi(z) \neq 0 \quad (z \in \mathbb{U})\}.$$

Let $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ consisting of all functions which are, respectively, starlike in $\mathbb{U}$ and convex in $\mathbb{U}$(see, for details, [18]).

Now we introduce the following integral operator $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ defined by

$$I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z) := \left(\frac{\sum_{i=1}^{n} \alpha_i}{z^{\sum_{i=1}^{n} \alpha_i - p\beta} \phi(z)} \int_0^z \prod_{i=1}^{n} f_i^{\alpha_i}(t) t^{(1-p)\Sigma_{i=1}^{n} \alpha_i - 1} \varphi(t) dt \right)^{1/\beta} \tag{3}$$

$(f_i \in \mathcal{A}_p;\ \alpha_i \in \mathbb{C}(i = 1, 2, \cdots, n);\ \beta \in \mathbb{C}\backslash\{0\};\ \Re\{\Sigma_{i=1}^{n} \alpha_i\} > 0;\ \phi, \varphi \in \mathcal{D})$.

The integral operator defined by (3) have been extensively studied by many authors [[2]-[9], [13], [14], [17], [20], [21], [24]] with suitable restriction on the parameters α_i, γ_i, and β, and for f_i belonging to some favored classes of analytic functions. In particular, we introduce the following integral operators, which have been extensively studied by many authors as special cases of the integral operator defined by (3).

(i) For $p = 1,\ n = 2,\ \alpha_1 = \beta,\ \alpha_2 = \gamma,\ f_1(z) = f(z),\ f_2(z) = z$ and $\phi(z) = \varphi(z) = 1$, the integral operator

$$I_{\beta,\gamma}(f)(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right)^{1/\beta}$$

was introduced and studied by Bulboaca [3]-[5] and Miller and Mocanu [18].

(ii) For $p = 1,\ n = 2,\ \alpha_1 = \beta,\ \alpha_2 = \gamma,\ f_1(z) = f(z),\ f_2(z) = h(z),\ \phi(z) = 1$ and $\varphi(z) = zh'(z)/h(z)$, the integral operator

$$I_{h;\beta,\gamma}(f)(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)h^{\gamma-1}(t)h'(t)dt\right)^{1/\beta}$$

was studied by Cho and Bulboaca [8] (see, also [6]).

(iii) More recently, for $p = 1$, a general form of integral operators mentioned above are considered by Bulboaca [6, 7] (see, also [1]) and Cho *et al.* [9].

In the present paper, we obtain the subordination- and superordination-preserving properties of the integral operator $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ defined by (3) with the sandwich-type theorem. Furthermore, we extend the results given by Cho and Srivastava [10] and Owa and Srivastava [22] as some special cases of main results presented here.

The following lemmas will be required in our present investigation.

Lemma 1.4 (Miller and Mocanu [15]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition $\Re\{H(is,t)\} \le 0$, for all real s and $t \le -n(1+s^2)/2$, where n is a positive integer. If the function $\kappa(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and $\Re\{H(\kappa(z), z\kappa'(z))\} > 0$ $(z \in \mathbb{U})$, then $\Re\{\kappa(z)\} > 0$ $(z \in \mathbb{U})$.*

Lemma 1.5 (Miller and Mocanu [16]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\Re\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies the inequality $\Re\{\beta q(z) + \gamma\} > 0$ $(z \in \mathbb{U})$.

Lemma 1.6 (Miller and Mocanu [18]). *Let $\kappa \in \mathcal{Q}$ with $\kappa(0) = a$ and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \not\equiv a$ and $n \ge 1$. If q is not subordinate to κ, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(f)$, for which*

$$q(\mathbb{U}_{r_0}) \subset \kappa(\mathbb{U}),\ q(z_0) = \kappa(\zeta_0) \quad \textit{and} \quad z_0 q'(z_0) = m\zeta_0\kappa'(\zeta_0) \quad (m \ge n).$$

Let $c \in \mathbb{C}$ with $\Re\{c\} > 0$ and let

$$N := N(c) = \frac{|c|\sqrt{1+2\Re\{c\}} + \Im\{c\}}{\Re\{c\}}.$$

If $R(z)$ is the univalent function defined in $\mathbb{U}$ by $R(z) = 2Nz/(1-z^2)$, then the open door function defined by

$$R_c(z) := R\left(\frac{z+b}{1+\bar{b}z}\right) \quad (z \in \mathbb{U}), \tag{4}$$

where $b = R^{-1}(c)$ [18].

Remark 1.1. The function R_c defined by (4) is univalent in $\mathbb{U}$, $R_c(0) = c$ and $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane with slits along the half-lines $\Re\{w\} = 0$ and $|\Im\{w\}| \geq N$.

Lemma 1.7. *Let $\alpha_i (i = 1, 2, \cdots, n), \beta \in \mathbb{C}$ with $\beta \neq 0$, $\Re\{\Sigma_{i=1}^n \alpha_i\} > 0$ and $\phi, \varphi \in \mathcal{D}$. If $f_i \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$, where*

$$\mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n} := \left\{ f_i \in \mathcal{A}_p : \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} + (1-p)\Sigma_{i=1}^n \alpha_i + \frac{z\varphi'(z)}{\varphi(z)} \prec R_{\Sigma_{i=1}^n \alpha_i}(z) \right\} \tag{5}$$

and $R_{\Sigma_{i=1}^n \alpha_i}(z)$ is defined by (4) with $c = \Sigma_{i=1}^n \alpha_i$, then $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i) \in \mathcal{A}_p$, $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)/z^p \neq 0$ and

$$\Re\left\{ \beta \frac{z(I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z))'}{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)} + \frac{z\phi'(z)}{\phi(z)} + \sum_{i=1}^n \alpha_i - p\beta \right\} > 0 \quad (z \in \mathbb{U}),$$

where $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ is the integral operator defined by (3).

Proof. If we take $\alpha = 1,\ \gamma = \sum_{i=1}^n \alpha_i - p\beta$ and $\delta = \sum_{i=1}^n \alpha_i - p$ in Miller and Mocanu [[18], Theorem 2.5c], then we obtain that

$$p\alpha + \delta = \sum_{i=1}^n \alpha_i = p\beta + \gamma \text{ and } \Re\{p\alpha + \delta\} = \Re\left\{\sum_{i=1}^n \alpha_i\right\} > 0.$$

Let us consider

$$f(z) = \frac{\prod_{i=1}^n f_i^{\alpha_i}(z)}{z^{p(\sum_{i=1}^n \alpha_i - 1)}} = z^p \prod_{i=1}^n \left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \quad (z \in \mathbb{U}).$$

Since $f_i \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ for $i = 1, 2, \cdots, n$, we see that from (5), $f_i(z)/z^p \neq 0$ $(i = 1, 2, \cdots, n;\ z \in \mathbb{U})$. Hence $f \in \mathcal{A}_p$. Therefore Lemma 1.7 follows immediately from Miller and Mocanu [[18], Theorem 2.5c]. □

A function $L(z,t)$ defined on $\mathbb{U} \times [0, \infty)$ is the subordination chain (or Löwner chain) if $L(\cdot, t)$ is said to be analytic and univalent in $\mathbb{U}$ for all $t \in [0, \infty)$, $L(z, \cdot)$

is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ when $0 \leq s < t$.

Lemma 1.8 (Miller and Mocanu [18]). *Let $q \in \mathcal{H}[a,1]$ and let $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi(q(z), zq'(z)) \equiv h(z)$ $(z \in \mathbb{U})$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $\kappa \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then $h(z) \prec \varphi(\kappa(z), z\kappa'(z))$. implies that $q(z) \prec \kappa(z)$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.*

Lemma 1.9 (Pommerenke [23]). *The function $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. Suppose that $L(\cdot;t)$ ia analytic in $\mathbb{U}$ for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. If $L(z;t)$ satisfies*

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty)$$

and $|L(z;t)| \leq K_0|a_1(t)|$ $(|z| < r_0 < 1;\ 0 \geq t < \infty))$ for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

2. Main results

Unless otherwise mentioned, we assume throughout the sequel that the condition

$$\sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{n} \alpha_j f_j^{p+1}(0) + (p+1)!(\varphi'(0) - \phi'(0))\right] \neq (p+1)!\phi'(0)$$

is satisfied for the integral operator $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ defined by (3).

Subordination theorem involving the integral operator $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ defined by (3) is contained in Theorem 2.1 below.

Theorem 2.1. *Let $f_i, g_i \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ $(i = 1,2,\cdots,n)$, where $\mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ is defined by (5). Suppose also that*

$$\Re\left\{1 + \frac{z\nu''(z)}{\nu'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U};\ \nu(z) := \prod_{i=1}^{n}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\varphi(z)\right), \tag{6}$$

where

$$\rho = \frac{1 + |\sum_{i=1}^{n}\alpha_i|^2 - |1 - (\sum_{i=1}^{n}\alpha_i)^2|}{4\Re\{\sum_{i=1}^{n}\alpha_i\}} \quad (\Re\{\Sigma_{i=1}^{n}\alpha_i\} > 0). \tag{7}$$

Then the subordination:

$$\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\varphi(z) \prec \prod_{i=1}^{n}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\varphi(z), \tag{8}$$

implies that

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta}\phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z), \tag{9}$$

where $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ *is the integral operator defined by* (3)*. Moreover, the function*

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z)$$

is the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta}\phi(z) \ \text{ and } \ G(z) := \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z), \tag{10}$$

respectively. We note that F and G are well defined by Lemma 1.7. Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ $(|\zeta| = 1)$. Otherwise, we replace F and G by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$ $(0 < r < 1)$, respectively. Then these functions satisfy the conditions of Theorem 2.1 on $\overline{\mathbb{U}}$. We can prove that $F_r(z) \prec G_r(z)$, which enables us to obtain (9) on letting $r \to 1$.

We first show that, if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{11}$$

then $\Re\{q(z)\} > 0$ $(z \in \mathbb{U})$. From the definition of (3), we obtain

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z)\left(\beta\frac{z(I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z))'}{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)} + \frac{z\phi'(z)}{\phi(z)} + \Sigma_{i=1}^{n}\alpha_i - p\beta\right)\frac{1}{\Sigma_{i=1}^{n}\alpha_i} = \prod_{i=1}^{n}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\varphi(z). \tag{12}$$

We also have

$$\beta\frac{z(I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z))'}{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)} + \frac{z\phi'(z)}{\phi(z)} = p\beta + \frac{zG'(z)}{G(z)}. \tag{13}$$

It follows from (12) and (13) that

$$\sum_{i=1}^{n}\alpha_i\nu(z) = \sum_{i=1}^{n}\alpha_i G(z) + zG'(z). \tag{14}$$

Now, by a simple calculation with (14), we obtain

$$\begin{aligned} 1+\frac{z\nu''(z)}{\nu'(z)} &= 1+\frac{zG''(z)}{G'(z)}+\frac{zq'(z)}{q(z)+\Sigma_{i=1}^{n}\alpha_i} \\ &= q(z)+\frac{zq'(z)}{q(z)+\Sigma_{i=1}^{n}\alpha_i} \equiv h(z). \end{aligned} \tag{15}$$

From (6), we have

$$\Re\left\{h(z)+\sum_{i=1}^{n}\alpha_i\right\}>0 \quad (z\in\mathbb{U}),$$

and hence by using Lemma 1.5, we conclude that the differential equation (15) has a solution $q\in\mathcal{H}(\mathbb{U})$ with $q(0)=h(0)=1$. Let us put

$$H(u,v)=u+\frac{v}{u+\Sigma_{i=1}^{n}\alpha_i}+\rho, \tag{16}$$

where ρ is given by (7). From (6), (15) and (16), we obtain

$$\Re\{H(q(z),zq'(z))\}>0 \quad (z\in\mathbb{U}).$$

Now we proceed to show that $\Re\{H(is,t)\}\le 0$ for all real s and $t\le -(1+s^2)/2$. From (16), we have

$$\begin{aligned} \Re\{H(is,t)\} &= \Re\left\{is+\frac{t}{is+\Sigma_{i=1}^{n}\alpha_i}+\rho\right\} \\ &= \frac{t\Re\{\Sigma_{i=1}^{n}\alpha_i\}}{|\Sigma_{i=1}^{n}\alpha_i+is|^2}+\rho \\ &\le -\frac{E_\rho(s)}{2|\Sigma_{i=1}^{n}\alpha_i+is|^2}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} E_\rho(s) :=&(\Re\{\Sigma_{i=1}^{n}\alpha_i\}-2\rho)s^2-4\rho(\Im\{\Sigma_{i=1}^{n}\alpha_i\})s \\ &-2\rho|\Sigma_{i=1}^{n}\alpha_i|^2+\Re\{\Sigma_{i=1}^{n}\alpha_i\}. \end{aligned} \tag{18}$$

For ρ given by (7), we note that the coefficient of s^2 in the quadratic expression $E_\rho(s)$ given by (18) is positive or equal to zero. To check this, put $\Sigma_{i=1}^{n}\alpha_i=c$ so that $\Re\{\Sigma_{i=1}^{n}\alpha_i\}=c_1$ and $\Im\{\Sigma_{i=1}^{n}\alpha_i\}=c_2$. We have to verify that

$$c_1\ge 2\rho=\frac{1+|c|^2-|1-c^2|}{2c_1}. \tag{19}$$

The inequality (19) will hold if

$$2c_1^2+|1-c^2|\ge 1+|c|^2=1+c_1^2+c_2^2,$$

that is, if

$$|1-c^2|\ge 1-\Re\{c^2\},$$

which is obviously true. Moreover, the discriminant Δ of $E_\rho(s)$ is represented by

$$\frac{1}{4}\Delta = -4[\Re\{\Sigma_{i=1}^n \alpha_i\}]^2\rho^2 + 2\rho(1+|\Sigma_{i=1}^n \alpha_i|^2)\Re\{\Sigma_{i=1}^n \alpha_i\} - [\Re\{\Sigma_{i=1}^n \alpha_i\}]^2.$$

Then for the assumed value of ρ given by (7), we have $\Delta = 0$ and so the quadratic expression $E_\rho(s)$ by s in (18) is a perfect square.. Hence from (17), we see that $\Re\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. Thus, by using Lemma 1.4, we conclude that $\Re\{q(z)\} > 0$ $(z \in \mathbb{U})$, that is, G is convex in $\mathbb{U}$. Next, we prove that the subordination condition (8) implies that

$$F(z) \prec G(z) \tag{20}$$

for the functions F and G defined by (10). For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{1+t}{\Sigma_{i=1}^n \alpha_i} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{1+t}{\Sigma_{i=1}^n \alpha_i}\right) \neq 0 \quad (0 \leq t < \infty;\ \Re\{\Sigma_{i=1}^n \alpha_i\} > 0).$$

This shows that the function $L(z,t) = a_1(t)z + \cdots$ $(z \in \mathbb{U};\ 0 \leq t < \infty)$ satisfies the condition $a_1(t) \neq 0$ $\quad (0 \leq t < \infty)$. Since G is convex and $\Re\{\Sigma_{i=1}^n \alpha_i\} > 0$, we have

$$\Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \Re\left\{\sum_{i=1}^n \alpha_i + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0.$$

Furthermore, since G is convex, the following well-known growth and distortion sharp inequalities (see [11]) hold true:

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r} \quad (|z| \leq r < 1) \tag{21}$$

and

$$\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2} \quad (|z| \leq r < 1). \tag{22}$$

Hence, by using (21) and (22), we can see easily that the second assumption of Lemma 1.9 is satisfied. Therefore the function $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\nu(z) = G(z) + \frac{1}{\Sigma_{i=1}^n \alpha_i} zG_i'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \ (0 \leq t < \infty).$$

This implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \nu(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty).$$

Now suppose that F is not subordinate to G, then by Lemma 1.6, there exists point $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \le t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{1+t}{\sum_{i=1}^{n} \alpha_i} \zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{\Sigma_{i=1}^{n} \alpha_i} z_0 F'(z_0) \\ &= \prod_{i=1}^{n} \left(\frac{f_i(z_0)}{z_0^p} \right)^{\alpha_i} \varphi(z_0) \in \nu(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (8). This contradicts the above observation that $L(\zeta_0, t) \notin \nu(\mathbb{U})$. Therefore, the subordination condition (8) must imply the subordination given by (20). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 2.1.

□

Remark 2.1. We note that ρ given by (7) in Theorem 2.1 satisfies the inequality $0 < \rho \le 1/2$.

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let* $f_i, g_i \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n} (1 = 1, 2, \cdots, n)$*, where* $\mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ *is defined by* (5)*. Suppose also that*

$$\Re\left\{1 + \frac{z\nu''(z)}{\nu'(z)}\right\} > -\rho \qquad \left(z \in \mathbb{U};\ \nu(z) := \prod_{i=1}^{n} \left(\frac{g_i(z)}{z^p}\right)^{\alpha_i} \varphi(z)\right),$$

where ρ *is given by* (7)*. If the function*

$$\prod_{i=1}^{n} \left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \varphi(z)$$

is univalent in $\mathbb{U}$ *and*

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta} \phi(z) \in \mathcal{Q},$$

where $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ *is the integral operator defined by* (3)*, then the superordination*:

$$\prod_{i=1}^{n} \left(\frac{g_i(z)}{z^p}\right)^{\alpha_i} \varphi(z) \prec \prod_{i=1}^{n} \left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \varphi(z) \tag{23}$$

implies that

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta}\phi(z).$$

Moreover, the function

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_i)(z)}{z^p}\right)^{\beta}\phi(z)$$

is the best subordinant.

Proof. Now let us define the functions F and G, respectively, by (10). We first note that from using (12) and (13), we obtain

$$\begin{aligned}\nu(z) &= G(z) + \frac{1}{\sum_{i=1}^{n}\alpha_i} zG'(z)\\ &=: \varphi(G(z), zG'(z)).\end{aligned} \tag{24}$$

After a simple calculation, the equation (24) yields the relationship:

$$1 + \frac{z\nu''(z)}{\nu'(z)} = q(z) + \frac{zq'(z)}{q(z) + \sum_{i=1}^{n}\alpha_i},$$

where the function q is defined by (11). Then by using the same method as in the proof of Theorem 2.1, we can prove that $\Re\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (10) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the superordination condition (23) implies that

$$F(z) \prec G(z). \tag{25}$$

Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{1+t}{\sum_{i=1}^{n}\alpha_i} zG_i'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

Since G is convex and $\Re\{\sum_{i=1}^{n}\alpha_i\} > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.8, we conclude that the superordination condition (23) must imply the superordination given by (25). Furthermore, since the differential equation (24) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2. □

If we combine Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type theorem.

Theorem 2.3. *Let* $f_i, g_{i,k} \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n} (i = 1, 2, \cdots, n;\ k = 1, 2)$*, where* $\mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ *is defined by* (5)*. Suppose also that*

$$\Re\left\{1 + \frac{z\nu_k''(z)}{\nu_k'(z)}\right\} > -\rho \qquad \left(z \in \mathbb{U};\ \nu_k(z) := \prod_{i=1}^{n}\left(\frac{g_{i,k}(z)}{z^p}\right)^{\alpha_i} \varphi(z);\ k = 1, 2\right), \tag{26}$$

where ρ *is given by* (7)*. If the function*

$$\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \varphi(z)$$

is univalent in $\mathbb{U}$ *and*

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta} \phi(z) \in \mathcal{Q},$$

where $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ *is the integral operator defined by* (3)*, then the subordination relation*:

$$\prod_{i=1}^{n}\left(\frac{g_{i,1}(z)}{z^p}\right)^{\alpha_i} \varphi(z) \prec \prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \varphi(z) \prec \prod_{i=1}^{n}\left(\frac{g_{i,2}(z)}{z^p}\right)^{\alpha_i} \varphi(z)$$

implies that

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,1})(z)}{z^p}\right)^{\beta} \phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta} \phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,2})(z)}{z^p}\right)^{\beta} \phi(z).$$

Moreover, the functions

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,1})(z)}{z^p}\right)^{\beta} \phi(z) \quad \text{and} \quad \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,2})(z)}{z^p}\right)^{\beta} \phi(z)$$

are the best subordinant and the best dominant, respectively.

Remark 2.2. If we take $p = 1,\ n = 2,\ \alpha_1 = \beta,\ \alpha_2 = \gamma,\ \gamma_1 = \gamma_2 = 0,\ f_1 = f$, $f_2(z) = z$ in Theorems 2.1-2.3, then we have the results obtained by Owa and Srivastava [22] and Cho and Srivastava [10].

The the assumption of Theorem 2.3, that the functions

$$\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i} \varphi(z) \quad \text{and} \quad \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta} \phi(z)$$

need to be univalent in $\mathbb{U}$, will be replaced by another conditions in the following result.

Corollary 2.4. *Let* $f, g_k \in \mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}(k=1,2)$*, where* $\mathcal{A}_{\varphi;\alpha_1,\cdots,\alpha_n}$ *is defined by* (5)*. Suppose also that the condition* (26) *is satisfied and*

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\rho \quad \left(z\in\mathbb{U};\ \psi(z):=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\varphi(z);\ f_i\in\mathcal{Q}\right), \tag{27}$$

where ρ *is given by* (27)*. Then the subordination relation:*

$$\prod_{i=1}^{n}\left(\frac{g_{i,1}(z)}{z^p}\right)^{\alpha_i}\varphi(z) \prec \prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\varphi(z) \prec \prod_{i=1}^{n}\left(\frac{g_{i,2}(z)}{z^p}\right)^{\alpha_i}\varphi(z)$$

implies that

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,1})(z)}{z^p}\right)^{\beta}\phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta}\phi(z) \prec \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,2})(z)}{z^p}\right)^{\beta}\phi(z),$$

where $I^{\phi,\varphi}_{\alpha_i,\beta;n;p}$ *is the integral operator defined by* (3)*. Moreover, the functions*

$$\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,1})(z)}{z^p}\right)^{\beta}\phi(z) \quad \text{and} \quad \left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(g_{i,2})(z)}{z^p}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

Proof. In order to prove Corollary 2.4, we have to show that the condition (26) implies the univalence of $\psi(z)$ and

$$F(z):=\left(\frac{I^{\phi,\varphi}_{\alpha_i,\beta;n;p}(f_i)(z)}{z^p}\right)^{\beta}\phi(z).$$

Since $0<\rho\leq 1/2$ from Remark 2.1, the condition (27) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [12]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 2.3, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, by applying Theorem 2.3, we obtain Corollary 2.4. □

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