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STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF CLASS r IN α-NORM UNDER THE LIGHT OF MEASURE THEORY

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Abstract. The aim of this work is to present some interesting results on weighted ergodic functions and prove the existence and uniqueness of Stepanov-like pseudo almost automorphic solutions using the spectral decomposition of the phase space developed by Adimy and co-authors. We also give the next challenge of this work.

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1. Introduction

In this work, we study the existence and uniqueness of Stepanov-like pseudo almost periodic solutions of class r for the following neutral partial functional differential equation

$$u'(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1}$$

where $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators on a Banach space X, $C_\alpha = C([-r, 0], D(A^\alpha))$, $0 < \alpha < 1$, denotes the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$, A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be describe later and

$$\|\varphi\|_{C_\alpha} = \|A^\alpha \varphi\|_{C([-r,0],X)}.$$

For $t \geq 0$, and $u \in C([-r, a], D(A^\alpha))$, $a > 0$ and u_t denotes the history function of C_α defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \; -r \leq \theta \leq 0.$$

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and L is a bounded linear operator from C_α into X and $f : \mathbb{R} \to X$ is a continuous function.

Some recent contributions concerning pseudo almost automorphic solutions for abstract differential equations similar to equation(1) have been made. Almost automorphic functions are more general than almost periodic functions and they were introduced by Bochner [4, 5], for more details about this topics we refer to the recents books [11, 13] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. In [12] the authors introduced and studied a new class of Stepanov-like almost automorphic functions with values in a Banach space. Almost automorphic solutions in the context of differential equations has been studied by many authors.

However, these results and many others obtain in literature are not correct. For example the decomposition result of weighted pseudo almost periodic functions in classical sens are not unique. The completeness based on the uniqueness decomposition result is not true. It follows that the uniqueness of existence weighted pseudo almost periodic solution based on the completeness is also not true.

The aims of this work is to correct many results obtained in the literature and also, we desire to generalize the results obtain in the classical results on weighted pseudo almost periodic functions .
Our approach is based on the spectral decomposition of the phase space developed in [2] and a new approach developped in [3].

This work is to generalise [17] by proving the existence of Stepanov-like pseudo almost automorphic solutions of equation (1) when the delay is distributed on $[-r, 0]$.

This work is organised as follow, in section 2 we recall some preliminary results about analytic semigroups and fractional power associated to its generator will be used throughout this work. In section 3, we recall some prelimary results on variation of constants formula and spectral decomposition. In section 4 and section 5, we recall some prelimary results on pseudo almost automorphic and Stepanov like pseudo almost automorphic functions that will be used in this work. In section 6, we prove some properties of S^p-pseudo almost automorphic function of class r. In section 7, we discuss the main result of this paper. Using the strict contraction principle we study the existence and uniqueness of Stepanov-like pseudo almost automorphic solution of class r for equation(1). Finally, for illustration, we propose to study the existence and uniqueness of S^p-pseudo almost automorphic solution for some model arising in the population dynamics.

2. Analytic semigroup

The purpose of this section is to collect some background materials required throughout the paper. These materials include, on the one hand, the fractional power A^{α} for $0 < \alpha < 1$ of A.

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. This allows us to define the fractional power A^{α} for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^{\alpha})$ dense in X. The closeness of A^{α} implies that $D(A^{\alpha})$, endowed with the graph norm of A^{α}, $|x| = \|x\| + \|A^{\alpha}x\|$, is a Banach space. Since A^{α} is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_{\alpha} = \|A^{\alpha}x\|$. Thus, $D(A^{\alpha})$ equipped with the norm $|.|_{\alpha}$, is a Banach space, which we denote by X_{α}. For $0 < \beta \leq \alpha < 1$, the imbedding $X_{\alpha} \hookrightarrow X_{\beta}$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. [14] *Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have*
i) $T(t) : X \to D(A^{\alpha})$ for every $t > 0$.
ii) $T(t)A^{\alpha}x = A^{\alpha}T(t)x$ for every $x \in D(A^{\alpha})$ and $t \geq 0$.
iii) for every $t > 0$, $A^{\alpha}T(t)$ is bounded on X and there exist $M_{\alpha} > 0$ and $\omega > 0$ such that

$$\|A^{\alpha}T(t)\| \leq M_{\alpha}e^{-\omega t}t^{-\alpha} \text{ for } t > 0.$$

iv) If $0 < \alpha \leq \beta < 1$, $D(A^{\beta}) \hookrightarrow D(A^{\alpha})$.
v) There exists $N_{\alpha} > 0$ such that

$$\|(T(t) - I)A^{-\alpha}\| \leq N_{\alpha}t^{\alpha} \text{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1}T(t)dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$ and Γ is the gamma function
Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

3. Spectral decomposition

The purpose of this section is to collect some background materials on the spectral decomposition of the phase space and variation of constants formula.

To equation (1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C_\alpha, \end{cases} \tag{2}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.
For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_α by

$$\mathcal{U}(t)\varphi = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in C_\alpha. \end{cases}$$

Proposition 3.1. [1]*Let $\mathcal{A}_\mathcal{U}$ defined on C_α by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) \\ = \Big\{ \varphi \in C_\alpha,\ \varphi' \in C_\alpha,\ \varphi(0) \in D(A),\ \varphi(0)' \in \overline{D(A)} \ \text{ and } \ \varphi(0)' = -A\varphi(0) + L(\varphi) \Big\} \\ \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \ \text{ for } \ \varphi \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_α.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : \ c \in X\}$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \ \theta \in [-r, 0[, \\ \\ c & \text{if } \ \theta = 0. \end{cases}$$

Consider the extension $\mathcal{A}_\mathcal{U}$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \Big\{ \varphi \in C^1([-r, 0]; X_\alpha) : \ \varphi(0) \in D(A) \ \text{ and } \varphi(0)' \in \overline{D(A)} \Big\} \\ \widetilde{\mathcal{A}_\mathcal{U}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

We make the following assertion:

($\mathbf{H_0}$) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X and satisfies $0 \in \rho(A)$.

Lemma 3.2. [2] *Assume that* $(\boldsymbol{H_0})$ *holds. Then,* $\widetilde{\mathcal{A}_{\mathcal{U}}}$ *satisfies the Hille-Yosida condition on* $C_\alpha \oplus \langle X_0 \rangle$ *there exist* $\widetilde{M} \geq 0,\ \widetilde{\omega} \in \mathbb{R}$ *such that* $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ *and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \ \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to equation (2).

Theorem 3.3. [1] *Assume that* $(\boldsymbol{H_0})$ *holds. Then for all* $\varphi \in C_\alpha$*, the solution* u *of equation* (2) *is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \ \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-1}$.

Definition 3.4. We say a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset.$$

For the sequel, we make the following assumption:

$(\mathbf{H_1})$ $T(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

We get the following result on the spectral decomposition of the phase space C_α.

Proposition 3.5. [1] *Assume that* $(\boldsymbol{H_0})$ *and* $(\boldsymbol{H_1})$ *hold. If the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic, then the space* C_α *is decomposed as a direct sum*

$$C_\alpha = S \oplus U$$

of two $\mathcal{U}(t)$ *invariant closed subspaces* S *and* U *such that the restricted semigroup on* $\mathcal{U}$ *is a group and there exist positive constants* $\overline{M}$ *and* ω *such that*

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\omega t}|\varphi| \ \text{ for } t \geq 0 \text{ and } \varphi \in S \\ \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\omega t}|\varphi| \quad \text{for } t \leq 0 \text{ and } \varphi \in U, \end{aligned}$$

where S *and* U *are called respectively the stable and unstable space,* Π^s *and* Π^u *denote respectively the projection operator on* S *and* U.

4. (μ,ν)-Pseudo Almost Automorphic Functions

Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

We recall some properties about pseudo almost automorphic functions. Let

$BC(\mathbb{R}, X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

Definition 4.1. A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to+\infty} \phi(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} g(t - s_n) = \phi(t)$$

for each $t \in \mathbb{R}$.

We denote by $AA(\mathbb{R}, X)$, the space of all such functions.

Proposition 4.2. [13] *$AA(\mathbb{R}, X)$ equipped with the sup norm is a Banach space.*

Definition 4.3. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called almost automorphic in $t \in \mathbb{R}$ uniformly for each x in X_1 if for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n\to+\infty} \phi(t + s_n, x) \text{ in } X_2$$

is well defined for each $t \in \mathbb{R}$ and each $x \in X_1$ and

$$\lim_{n\to+\infty} g(t - s_n, x) = \phi(t, x) \text{ in } X_2$$

for each $t \in \mathbb{R}$ and for every $x \in X_1$.

Denote by $AA(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

Definition 4.4. A bounded continuous function $\phi : \mathbb{R} \to X$ is called compact almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to+\infty} \phi(t + s_n) \text{ and } \lim_{n\to+\infty} g(t - s_n) = \phi(t)$$

uniformly on compact subsets of $\mathbb{R}$.

We denote by $AA_c(\mathbb{R}; X)$, the space of all such functions.

It is well known that $AA_c(\mathbb{R}; X)$ is closed subsets of $(BC(\mathbb{R}, X), |\,.|_\infty)$.

In view of the above, the proof of the next lemma is straightforward.

Lemma 4.5. [13] *$AA_c(\mathbb{R}; X)$ equipped with the sup norm is a Banach space.*

Definition 4.6. Let X_1 and X_2 be two Banach spaces. A continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called compact almost automorphic in $t \in \mathbb{R}$ if every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n\to+\infty} \phi(t + s_n, x) \text{ and } \lim_{n\to+\infty} g(t - s_n, x) = \phi(t, x) \text{ in } X_2$$

where the limits are uniform on compact subsets of $\mathbb{R}$ for each $x \in X_1$.

Denote by $AA_c(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

To study delayed differential equations for which the history belong to $C([-r, 0]; X)$, we introduce the space

$$\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r) = \Big\{u \in BC(\mathbb{R}, X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t]} |u(\theta)|_\alpha \Big) d\mu(t) = 0 \Big\}.$$

In addition to above-mentioned space, we consider the following spaces

$$\mathcal{E}(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu) = \Big\{u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} |u(t, x)|_\alpha d\mu(t) = 0 \Big\},$$

$$\begin{aligned} &\mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r) \\ &= \Big\{u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t]} |u(\theta, x)|_\alpha \Big) d\mu(t) = 0 \Big\}, \end{aligned}$$

where in both cases the limit (as $\tau \to +\infty$) is uniform in compact subset of X_α.

In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ and $\mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$ are continuously embedded in $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$ and $\mathcal{E}(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu)$, respectively.

On the other hand, one can observe that a ρ-weighted pseudo almost periodic functions is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu([-\tau, \tau]) = 2\tau$ for all $\tau \geq 0$.

Example 4.7. [3] Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt, \quad \text{for } A \in \mathcal{B}, \tag{3}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 4.8. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X_\alpha$ is called (μ, ν)-pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}, X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$.

We denote by $PAA(\mathbb{R}, X_\alpha, \mu, \nu)$ the space of all such functions.

Definition 4.9. A bounded continuous function $\phi : \mathbb{R} \to X$ is called compact $\alpha - (\mu, \nu)$-pseudo almost automorphic if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA_c(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$.

We denote by $PAA_c(\mathbb{R}; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.10. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \to X_\alpha$ is called uniformly compact $\alpha - (\mu, \nu)$-pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA_c(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$.

We denote by $PAA_c(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.11. A bounded continuous function $\phi : \mathbb{R} \to X$ is called $\alpha-(\mu,\nu)$-pseudo almost automorphic of class r (respectively compact $\alpha-(\mu,\nu)$-pseudo almost automorphic of class r) if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA_c(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

We denote by $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r)$) the space of all such functions.

Definition 4.12. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \to X_\alpha$ is called uniformly $\alpha-(\mu,\nu)$-pseudo almost automorphic of class r (respectively uniformly $\alpha-(\mu,\nu)$-pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}\times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}\times X_\alpha; X_\alpha, \mu, \nu, r)$ (respectively if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA_c(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$.

We denote by $PAA(\mathbb{R}\times X_\alpha; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c(\mathbb{R}\times X_\alpha; X_\alpha, \mu, \nu, r)$) the space of all such functions.

5. (μ,ν)-Stepanov-Like Pseudo Almost Automorphic Functions

Definition 5.1. The Bochner transform $f^b(t,s),\ t \in \mathbb{R},\ s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t,s) = f(t+s).$$

Remark 5.1. If $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 5.2. The Bochner transform $F^b(t,s,u),\ t \in \mathbb{R},\ s \in [0,1],\ u \in X$ of a function $F(t,u)$ on $\mathbb{R} \times X$, with values in X, is defined by

$$F^b(t,s,u) = F(t+s,u) \text{ for each } u \in X.$$

Definition 5.3. Let $p \in [1,+\infty[$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1], X))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |f(s)|^p ds \Big)^{\frac{1}{p}}.$$

Definition 5.4. A function $f \in BS^p(\mathbb{R}, X_\alpha)$ is called (μ,ν)-S^p pseudo-almost automorphic (or Stepanov-like pseudo-almost automorphic) if it can be expressed as $f = h+\varphi$, where $h^b \in AA(\mathbb{R}, L^p((0,1), X_\alpha))$ and $\varphi^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu)$. The collection of such functions will be denoted by $PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu)$.

In other words, a function $f \in L^p(\mathbb{R}, X_\alpha)$ is said to be S^p-pseudo-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1), X)$ is pseudo-almost automorphic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to X$ such that $f = h + \varphi$, where $h^b \in AA(\mathbb{R}, L^p((0,1), X))$ and $\varphi^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X), \mu, \nu)$, i.e., according to [12] (Definition 2.5 Page 2660.), for each real sequence (s_m), there exists a subsequence (s_n) and a function $g \in L^p_{loc}(\mathbb{R}; X)$ such that

$$\lim_{n\to+\infty}\Big(\int_t^{t+1}|g(s)-h(s+s_n)|^p ds\Big)^{\frac{1}{p}}=0$$

and

$$\lim_{n\to+\infty}\Big(\int_t^{t+1}|g(s-s_n)-h(s)|^p ds\Big)^{\frac{1}{p}}=0$$

pointwise on $\mathbb{R}$ and

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\Big(\int_t^{t+1}|\varphi(s)|^p_\alpha ds\Big)^{\frac{1}{p}}d\mu(t)=0.$$

Definition 5.5. A function $f \in BS^p(\mathbb{R}, X)$ is called (μ,ν)-S^p-pseudo-almost automorphic of infinite class (or Stepanov-like pseudo-almost automorphic of class r) if it can be expressed as $f = h+\varphi$, where $h^b \in AA(\mathbb{R}, L^p((0,1), X))$ and $\varphi^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X), \mu, \nu, r)$ i.e.,

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\varphi(s)|^p ds\Big)^{\frac{1}{p}}d\mu(t)=0.$$

The collection of such functions will be denoted by $PAAS^p(\mathbb{R}, X, \mu, \nu, r)$.

Definition 5.6. A function $f : \mathbb{R}\times X_1 \to X_2,\ (t,x)\mapsto f(t,x)$ with $f(.,x) \in L^p(\mathbb{R}, X_2)$ is called (μ,ν)-S^p-pseudo-almost automorphic of class r (or Stepanov-like pseudo-almost automorphic of class r) if it can be expressed as $f = h+\varphi$, where $h^b \in AA(\mathbb{R}\times X_1, L^p((0,1), X_2))$ and $\varphi^b \in \mathcal{E}[(\mathbb{R}\times X_1, L^p((0,1), X_2), \mu, \nu, r]$ i.e.,

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\varphi(s,x)|^p_{X_2} ds\Big)^{\frac{1}{p}}d\mu(t)=0.$$

The collection of such functions will be denoted by $PAAS^p[\mathbb{R}\times X_1, L^p((0,1), X_2), \mu, \nu, r]$.

Definition 5.7. A bounded continuous function $\phi : \mathbb{R} \to X$ is called (μ,ν)-S^p-pseudo compact almost automorphic of class r if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA_c(\mathbb{R}, L^p((0,1), X))$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; L^p((0,1), X), \mu, \nu, r)$.

We denote by $PAA_cS^p(\mathbb{R}, L^p((0,1), X), \mu, \nu, r)$, the space of all such functions.

Definition 5.8. Let X_1 and X_2 be two Banach spaces. A bounded continuous function
$\phi : \mathbb{R}\times X_1 \to X_2$ is called uniformly (μ,ν)-S^p-pseudo almost automorphic of class r (respectively uniformly pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}\times X_1, L^p((0,1), X_2))$ and $\phi_2 \in \mathcal{E}(\mathbb{R}\times X_1, L^p((0,1), X_2), \mu, \nu, r)$ (respectively if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA_c(\mathbb{R}\times X_1, L^p((0,1), X_2))$ and $\phi_2 \in \mathcal{E}(\mathbb{R}\times X_1, L^p((0,1), X_2), \mu, \nu, r))$.

We denote by $PAAS^p(\mathbb{R}\times X_1, L^p((0,1),X_2),\mu,\nu,\infty)$ (respectively $PAA_cS^p(\mathbb{R}\times X_1, L^p((0,1),X_2),\mu,\nu,r))$ the space of all such functions.

6. Properties of μ-Stepanov-Like Pseudo Almost Automorphic Functions of Class r

From $\mu,\nu\in\mathcal{M}$, we formulate the following hypothese.

($\mathbf{H_2}$) Let $\mu,\nu\in\mathcal{M}$ be such that $\limsup\limits_{\tau\to+\infty}\dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\delta<\infty.$

Lemma 6.1. *Assume ($\boldsymbol{H_2}$) holds. $PAAS^p(\mathbb{R},X_\alpha,\mu,\nu,r)$ is a closed subspace of $BS^p(\mathbb{R},X_\alpha)$.*

Proof. Let $(x_n)_n$ be a sequence in $PAAS^p(\mathbb{R};L^p((0,1),X),\mu,\nu,r)$ such that $\lim\limits_{n\to\infty} x_n=x$ in $BS^p(\mathbb{R};X_\alpha)$. For each n, let $x_n=y_n+z_n$ with $y_n^b\in AA(\mathbb{R};L^p((0,1),X_\alpha))$ and $z_n^b\in\mathcal{E}(\mathbb{R};L^p((0,1),X_\alpha),\mu,\nu,r)$. $(y_n)_n$ converges to some $y\in BS^p(\mathbb{R};X_\alpha)$ and $(z_n)_n$ also converges to some $z\in BS^p(\mathbb{R};X_\alpha)$. In[12] Theorem 2.3, $y\in AA(\mathbb{R};L^p((0,1),X_\alpha))$. It remains to show that $z\in\mathcal{E}(\mathbb{R};L^p((0,1),X_\alpha),\mu,\nu,r)$

Since

$$\begin{aligned}
&\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|z(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t)\\
=\;&\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|z(s)-z_n(s)+z_n(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t),
\end{aligned}$$

then by the Minkowski's inequality, we also have

$$\begin{aligned}
&\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|z(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t)\\
\le\;&\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_\theta^{\theta+1}|z(s)-z_n(s)|_\alpha^p ds\Big)^{\frac1p}+\Big(\int_\theta^{\theta+1}|z_n(s)|^p)ds\Big)^{\frac1p}\Big]d\mu(t)\\
\le\;&\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|z(s)-z_n(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t)\\
+\;&\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|z_n(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t)\\
\le\;&\|z-z_n\|_{S^p,\alpha}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|z_n(s)|_\alpha^p ds\Big)^{\frac1p}d\mu(t).
\end{aligned}$$

Then we get $z\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$, hence $x\in PAAS^p(\mathbb{R},X_\alpha,\mu,\nu,r)$. □

Consequently, we have the following lemma:

Lemma 6.2. *The space $PAAS^p(\mathbb{R};L^p((0,1),X),\mu,\nu,r)$ endowed with the $\|\,.\,\|_{S^p}$ norm is a Banach space.*

Next, we give a characterization of (μ,ν)-ergodic functions of class r.

Theorem 6.3. *($\boldsymbol{H_2}$) holds and let $\mu, \nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BS^p(\mathbb{R}, X)$. Then the following assertions are equivalent:*

i) $f \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$.

ii) $\displaystyle\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) = 0.$

iii) $\forall \varepsilon > 0$, $\displaystyle\lim_{\tau \to +\infty} \frac{\mu\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} > \varepsilon \Big\}\Big)}{\nu([-\tau,\tau] \setminus I)} = 0.$

Proof. *i)* $\Leftrightarrow$ *ii)* Let us pose $A = \nu(I)$, $B = \displaystyle\int_I \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t)$. Since the interval I is bounded and the function f is bounded and continuous then A and $B \in \mathbb{R}$. For $\tau > 0$ such that $I \subset [-\tau,\tau]$ and $\nu([-\tau,\tau] \setminus I) > 0$, we have

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) \\
= \quad & \frac{1}{\nu([-\tau,\tau]) - A} \Big[\int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) - B \Big] \\
= \quad & \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau]) - A} \Big[\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) - \frac{B}{\nu([-\tau,\tau])} \Big].
\end{aligned}$$

From above and the fact that $\nu(\mathbb{R}) = +\infty$, we conclude that *ii)* is equivalent to

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) = 0,$$

that is *i)*.

iii) $\Rightarrow$ *ii)* Let us pose A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} > \varepsilon \Big\}$$

and

$$B_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I) : \sup_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}} \le \varepsilon \Big\}.$$

Assume that *iii)* holds, that is

$$\lim_{\tau \to +\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)} = 0. \tag{4}$$

From the equality

$$\begin{aligned}&\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\ = &\int_{A_{\tau}^{\varepsilon}}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)+\int_{B_{\tau}^{\varepsilon}}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t).\end{aligned}$$

We deduce that for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}\mu(t)\leq\|f\|_{S^p}\frac{\mu(A_{\tau}^{\varepsilon})}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_{\tau}^{\varepsilon})}{\nu([-\tau,\tau]\setminus I)},$$

By using $(\mathbf{H_2})$, it follows that

$$\text{for all }\ \varepsilon>0,\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\leq\alpha\varepsilon,\ \text{ for any }\ \varepsilon>0,$$

consequently (ii) holds.

$ii)\Rightarrow iii)$ Assume that $ii)$ holds. From the following inequality

$$\begin{aligned}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)&\geq\int_{A_{\tau}^{\varepsilon}}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\ \frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)&\geq\varepsilon\frac{\mu(A_{\tau}^{\varepsilon})}{\nu([-\tau,\tau]\setminus I)}\\ \frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)&\geq\frac{\mu(A_{\tau}^{\varepsilon})}{\nu([-\tau,\tau]\setminus I)},\end{aligned}$$

for τ sufficiently large, we obtain equation (4), that is $iii)$. $\square$

From $\mu\in\mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H_3})$ For all $a,\ b$ and $c\in\mathbb{R}$, such that $0\leq a<b\leq c$, there exist δ_0 and $\alpha_0>0$ such that

$$|\delta|\geq\delta_0\Rightarrow\mu(a+\delta,b+\delta)\geq\alpha_0\mu(\delta,c+\delta).$$

$(\mathbf{H_4})$ For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau:\ a\in A\})\leq\beta\mu(A)\ \text{ when } A\in\mathcal{B}\text{ satisfies }\ A\cap I=\emptyset.$$

We have the following results due to [3]

Lemma 6.4. [3] *Hypothesis* $(\boldsymbol{H_4})$ *implies* $(\boldsymbol{H_3})$.

Lemma 6.5. [3, 6] $\mu,\nu\in\mathcal{M}$ *satisfy* $(\boldsymbol{H_4})$ *and* $f\in PAA(\mathbb{R},X,\mu,\nu)$ *be such that*

$$f=g+h$$

where $g\in AP(\mathbb{R},X)$ *and* $h\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$. *Then*

$$\{g(t),\ t\in\mathbb{R}\}\subset\overline{\{f(t),\ t\in\mathbb{R}\}}\ \textit{(the closure of the range of f)}.$$

Lemma 6.6. [17] *Let* $\mu,\nu\in\mathcal{M}$. *Assume* $(\boldsymbol{H_4})$ *holds. Then the decomposition of a* (μ,ν)*-pseudo-almost periodic function* $\phi=\phi_1+\phi_2$, *where* $\phi_1\in AA(\mathbb{R},X)$ *and* $\phi_2\in\mathcal{E}(\mathbb{R},X,\mu,\nu,r)$, *is unique.*

Definition 6.7. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

From [3] $\sim$ is a binary equivalence relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \{\varpi \in \mathcal{M} : \ \mu \sim \varpi\}.$$

Theorem 6.8. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAAS^p(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = PAAS^p(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \emptyset$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for τ sufficiently large, it follows that

$$\begin{aligned} & \frac{\alpha_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau] \setminus I)} \\ \leq \ & \frac{\mu_2\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau] \setminus I)} \\ \leq \ & \frac{\beta_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau] \setminus I)} \end{aligned}$$

By using Theorem 6.3 we deduce that $\mathcal{E}(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = \mathcal{E}(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$. From the definition of a (μ, ν)-pseudo almost periodic function, we deduce that $PAAS^p(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = PAAS^p(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$. □

Let $\mu, \nu \in \mathcal{M}$ we denote by

$$cl(\mu, \nu) = \{\varpi_1, \varpi_2 \in \mathcal{M} : \ \mu \sim \varpi_2 \text{ and } \ \nu \sim \varpi_2\}.$$

Lemma 6.9. [6] *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$). Then $PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ is invariant by translation, that is $f \in PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ implies $f_\gamma \in PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ for all $\gamma \in \mathbb{R}$.*

Corollary 6.10. *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_5}$). Then $PAAS^p(\mathbb{R}, X_\alpha, \mu, r)$ is invariant by translation, that is $f \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ implies $f_\gamma \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ for all $\gamma \in \mathbb{R}$.*

Proof. It suffices to prove that $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ is invariant by translation. Let $f \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ and $F^t(\theta) = \sup\limits_{\theta \in [t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}}$.

Then $F^t \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, but since $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ is invariant by translation, it follows that

$$\begin{aligned} & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} F^t(\theta + \gamma) d\mu(t) \\ = & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \Big(\int_{\theta}^{\theta+1} |f(s+\gamma)|_{\alpha}^{p} ds \Big)^{\frac{1}{p}} d\mu(t) = 0, \end{aligned}$$

which implies that $f(. + \gamma) \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. $\square$

In what follows, we prove some preliminary results concerning the composition of (μ, ν)-Stepanov-pseudo almost periodic functions of class r.

Theorem 6.11. *Let $\mu, \nu \in \mathcal{M}$, $\phi \in PAAS^p(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu, r)$ and $h \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Assume that there exists a function $L_\phi : \mathbb{R} \to [0, +\infty[$ sastisfies*

$$|\phi(t, x_1) - \phi(t, x_2)| \leq L_\phi(t) |x_1(t) - x_2(t)|_\alpha \quad \text{for } t \in \mathbb{R} \text{ and for } x_1, x_2 \in L^p((0,1), X_\alpha). \tag{5}$$

If

$$\begin{aligned} & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t+1]} L_\phi(\theta) \Big) d\mu(t) < \infty \quad \text{and} \\ & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t+1]} L_\phi(\theta) \Big) \Big(\int_{\theta}^{\theta+1} |\xi(s)|_{\alpha}^{p} ds \Big)^{\frac{1}{p}} d\mu(t) = 0 \end{aligned} \tag{6}$$

for each $\xi \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$, then the function $t \to \phi(t, h(t))$ belongs to $PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$ where $\phi_1^b \in AA(\mathbb{R} \times X, L^p((0,1), X))$, $\phi_2^b \in \mathcal{E}(\mathbb{R} \times X, L^p((0,1), X), \mu, \nu, \infty)$ and $h_1^b \in AA(\mathbb{R}; L^p((0,1), X))$, $h_2^b \in \mathcal{E}(\mathbb{R}; L^p((0,1), X), \mu, \nu, \infty)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

From [18], $\phi_1(., h_1(.)) \in AA(\mathbb{R}; L^p((0,1), X))$. Now, we need to prove that both $\phi^b(., h(.)) - \phi^b(., h_1^b(.))$ and $\phi_2^b(., h_1^b(.))$ belong to $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. By equation (5), it follows that

$$\begin{aligned} & \frac{\mu\Big(\Big\{t \in [-\tau, \tau] : \sup_{\theta \in [t-r, t]} \Big(\int_{\theta}^{\theta+1} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_{\alpha}^{p} ds \Big)^{\frac{1}{p}} > \varepsilon \Big\}\Big)}{\nu([-\tau, \tau])} \\ \leq & \frac{\mu\Big(\Big\{t \in [-\tau, \tau] : \sup_{\theta \in [t-r, t]} \Big(\int_{\theta}^{\theta+1} (L_\phi(\theta) |h_2(\theta)|_\alpha)^p ds \Big)^{\frac{1}{p}} > \varepsilon \Big\}\Big)}{\nu([-\tau, \tau])} \end{aligned}$$

$$\leq \quad \frac{\mu\Big(\Big\{t \in [-\tau,\tau] : \Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h_2(\theta)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.$$

Since h_2 is (μ,ν)-ergodic of class r, Theorem 6.3 and equation (6) yield that for the above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t \in [-\tau,\tau] : \Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h_2(\theta)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0,$$

and then we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t \in [-\tau,\tau] : \sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0, \qquad (7)$$

By Theorem 6.3, equation (7) shows that $t \mapsto \phi(t,h(t)) - \phi(t,h_1(t))$ belongs to $\mathcal{E}(\mathbb{R}, L^p((0,1),X_\alpha),\mu,\nu,r)$.

Since ϕ_2^b is uniformly continuous on the compact set $K = \overline{\{h_1^b(t):\ t\in\mathbb{R}\}}$ with respect to the second variable x, we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$, one has

$$\|\xi_1 - \xi_2\|_{L^p} \leq \delta \Rightarrow |\phi_2^b(t,\xi_1(t)) - \phi_2^b(t,\xi_2(t))| \leq \varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i,\delta)$$

and then by Minkowski's inequality we have

$$\begin{aligned}
&\Big(\int_t^{t+1}|\phi_2(t,h_1(t))|_\alpha^p ds\Big)^{\frac{1}{p}}\\
\leq\ &\Big(\int_t^{t+1}|\phi_2(t,h_1(t))-\phi_2(t,z_i)|_\alpha^p ds\Big)^{\frac{1}{p}} + \Big(\int_t^{t+1}|\phi_2(t,z_i))|_\alpha^p ds\Big)^{\frac{1}{p}}\\
\leq\ &\varepsilon + \sum_1^{n(\varepsilon)}\Big(\int_t^{t+1}|\phi_2(t,z_i)|_\alpha^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

Since

$$\begin{aligned}
&\forall i \in \{1,...,n(\varepsilon)\},\\
&\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\phi_2(\theta,z_i)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0,
\end{aligned}$$

we deduce that

$$\forall\varepsilon > 0, \quad \limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\phi_2(\theta,h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \leq \varepsilon,$$

that implies

$$\forall \varepsilon > 0, \quad \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\int_{\theta}^{\theta+1} |\phi_2(\theta, h_1(\theta))|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) = 0.$$

Consequently $t \mapsto \phi_2(t, h(t))$ belongs to $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. $\square$

For $\mu \in \mathcal{M}$ and $\delta \in \mathbb{R}$, we denote μ_δ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\delta(A) = \mu([a+\delta : \; a \in A]) \tag{8}$$

Lemma 6.12. [3] *Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H_4})$. Then the measures μ and μ_δ are equivalent for all $\delta \in \mathbb{R}$.*

Lemma 6.13. [3] $(\boldsymbol{H_4})$ *implies*

$$\textit{for all } \; \sigma > 0 \;\; \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma, \tau+\sigma])}{\mu([-\tau,\tau])} < +\infty.$$

We have the following result.

Theorem 6.14. *Let $u \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$, then the function $t \to u_t$ belongs to $PAA_cS^p(C([-r,0], X_\alpha), \mu, \nu, r)$.*

Proof. Assume that $u = g + h$ where $g^p \in AA(\mathbb{R}; L^p((0,1), X_\alpha))$ and $h^p \in \mathcal{E}(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$. We can see that $u_t = g_t + h_t$. We want to show that $g_t^p \in AA_c(\mathbb{R}; L^p((0,1), X_\alpha))$ and $h_t^p \in \mathcal{E}(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$.
Firstly for a given sequence $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v \in BS^p(\mathbb{R}; X_\alpha)$ such that $g(s+s_n) \to v(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and $L > 0$ such that $K \subset [-L, L]$. For $\varepsilon > 0$, fix $N_{\varepsilon,L} \in \mathbb{N}$ such that $\|g(s+s_n) - v(s)\|_{S^p,\alpha} \leq \varepsilon$ for $s \in [-L, L]$ whenever $n \geq N_{\varepsilon,L}$. For $t \in K$ and $n \geq N_{\varepsilon,L}$, we have

$$\begin{aligned} \|g_{t+s_n} - v_t\|_{S^p,\alpha} &\leq \sup_{\theta\in[-L,L]} \|g(\theta+s_n) - v(\theta)\|_{S^p,\alpha} \\ &\leq \varepsilon. \end{aligned}$$

In view of above, g_{t+s_n} converges to v_t uniformly on K. Similary, one can prove that v_{t-s_n} converges to u_t uniformly on K. Thus, the function $s \to g_s$ belongs to $AA_c(L^p(0,1), C_\alpha)$.
Let us denote by

$$M_\delta(\tau) = \frac{1}{\nu_\alpha([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta}^{\theta+1} |h(\theta)|_\alpha \Big)^{\frac{1}{p}} d\mu_\delta(t),$$

where μ_δ and ν_δ are the positive measure defined by equation (8). By using Lemma 6.12, it follows that μ_α and μ are equivalent and ν_δ and ν are also equivalent. Then by using Theorem 6.8 we have $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu_\alpha, \nu_\alpha, r) = \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$, therefore $h \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu_\alpha, \nu_\alpha, r)$, that is

$$\lim_{\tau\to+\infty} M_\delta(\tau) = 0, \;\; \text{for all } \; \alpha \in \mathbb{R}.$$

On the other hand, for $r > 0$ we have

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}\Big(\sup_{\xi\in[-r,0]}|h(s+\xi)|_{\alpha}\Big)^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\sup_{\xi\in[-r,0]}\int_{\theta+\xi}^{\theta+1+\xi}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-2r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-2r,t-r]}\Big(\int_{\theta}^{\theta+1}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
& +\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{+\tau+r}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t+r)\\
& +\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t),
\end{aligned}$$

it follows that

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}\Big(\sup_{\xi\in[-r,0]}|h(s+\xi)|\Big)^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\ & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|^{p}ds\Big)^{p}d\mu(t)\\
+\ & \frac{1}{\nu([-\tau-r,\tau+r]}\int_{-\tau-r}^{+\tau+r}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|^{p}ds\Big)^{p}d\mu(t+r)\times\Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]+.
\end{aligned}$$

Consequently

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}|h(\theta+\xi)|\Big]\Big)d\mu(t)\\
\leq\ & \Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]\times M_r(\tau+r)+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|h(\theta)|\Big)d\mu(t),
\end{aligned}$$

which shows using Lemma 6.12 and Lemma 6.13 that which shows that u_t belongs to $PAA_cS^p(C([-r,0],X_\alpha),\mu,\nu,r)$. Thus, we obtain the desired result. □

7. Weighted Stepanov Like pseudo Almost Automorphic Solutions of Class r

In what follows, we will be looking at the existence of bounded integral solution of equation(1).

Theorem 7.1. *Assume that ($\boldsymbol{H_1}$) hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If $f\in BS^p(\mathbb{R},X)$, then there exists a unique bounded solution u of equation (1) on $\mathbb{R}$, given by*

$$u_t$$

$$\begin{aligned}
&= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds \\
&+ \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \qquad (9)\\
&\quad \textit{for } t\in\mathbb{R},
\end{aligned}$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_\mathcal{U})^{-1}$ *for* $\lambda > \widetilde{\omega}$, Π^s *and* Π^u *are the projections of* C_α *onto the stable and unstable subspaces, respectively.*

Proof. Let us first prove that the limits in equation (9). For $t \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|_\alpha ds &\le \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|f(s)|ds \\
&\le \overline{M}\widetilde{M}|\Pi^s|\Big[\int_{t-1}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|f(s)|ds \\
&\quad +\sum_{n=2}^{\infty}\Big(\int_{t-n}^{t-n+1}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|f(s)|ds\Big)\Big].
\end{aligned}$$

Let q such that $\dfrac{1}{q}+\dfrac{1}{p}=1$. Using the Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|_\alpha ds \\
\le\ & \overline{M}\widetilde{M}|\Pi^s|\Big[\int_{t-1}^{t}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-1}^{t}|f(s)|^p ds\Big)^{\frac{1}{p}} \\
& +\sum_{n=2}^{+\infty}\Big[\int_{t-n}^{t-n+1}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
\le\ & \overline{M}\widetilde{M}|\Pi^s|\Big[\int_{-\infty}^{+\infty}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-1}^{t}|f(s)|^p ds\Big)^{\frac{1}{p}} \\
& +\sum_{n=2}^{+\infty}\Big[\int_{t-n}^{t-n+1}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
\le\ & \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega}e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}}+\sum_{n=2}^{\infty}\Big[\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}}\Big] \\
\le\ & \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega}e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}} \\
& +\sum_{n=2}^{\infty}\Big[\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}(n-1)^{-\alpha q}\times(\omega q)^{-\alpha q}ds\Big)^{\frac{1}{q}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega} e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}}\\
&+ \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}\sum_{n=2}^{\infty}\Big[(n-1)^{-\alpha}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}ds\Big)^{\frac{1}{q}}\Big]\\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega} e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}} + \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}-1)^{\frac{1}{q}}\sum_{n=2}^{\infty}(n-1)^{-\alpha}e^{-\omega n}\\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega} e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}} + \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=2}^{\infty}(n-1)^{-\alpha}e^{-\omega n}\\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big[\Big(\int_0^{q\omega} e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}} + \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=2}^{\infty}e^{-\omega n}
\end{aligned}
$$

Since the serie $\sum_{n=1}^{+\infty} e^{-\omega n}$ is convergent, it follows that

$$
\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|ds \quad < \quad K \tag{10}
$$

with

$$
K = \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\int_0^{q\omega} e^{-s}s^{-\alpha q}ds + \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=2}^{\infty}e^{-\omega n}
$$

Set

$$
F(n,s,t) = \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s)) \text{ for } n \in \mathbb{N} \text{ for } s \leq t.
$$

For n is sufficiently large and $\sigma \leq t$, we have

$$
\begin{aligned}
\Big|\int_{-\infty}^{\sigma}F(n,s,t)ds\Big|_\alpha \quad &\leq \quad \overline{M}\widetilde{M}|\Pi^s|\Big[\int_{\sigma-1}^{\sigma}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{\sigma-1}^{\sigma}|f(s)|^p ds\Big)^{\frac{1}{p}}\\
&+\sum_{n=2}^{\infty}\Big(\int_{\sigma-n}^{\sigma-n+1}\frac{e^{-q\omega(t-s)}}{(t-s)^{\alpha q}}ds\Big)^{\frac{1}{q}}\Big(\int_{\sigma-n}^{\sigma-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big]\\
&\leq \quad \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(t-\sigma)}^{q\omega(\sigma+1)} e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}}\\
&+\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=2}^{\infty}\Big[\Big(\int_{q\omega(t-\sigma+n-1)}^{q\omega(t-\sigma+n)} e^{-s}s^{-\alpha q}\Big)^{\frac{1}{q}}\Big]\\
&\leq \quad \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(t-\sigma)}^{q\omega(\sigma+1)} e^{-s}(q\omega(t-\sigma))^{-\alpha q}ds\Big)^{\frac{1}{q}}\\
&+\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\times e^{-\omega(t-\sigma)}\sum_{n=2}^{\infty}e^{-\omega n}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(t-\sigma)^{-\alpha}\Big(\int_{q\omega(t-\sigma)}^{q\omega(\sigma+1)} e^{-s}ds\Big)^{\frac{1}{q}}\\
&\quad+\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\times e^{-\omega(t-\sigma)}\sum_{n=2}^{\infty}e^{-\omega n}\\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(1-e^{-q\omega})^{\frac{1}{q}}\times(t-\sigma)^{-\alpha}e^{-\omega(t-\sigma)}\\
&\quad+\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\times e^{-\omega(t-\sigma)}\sum_{n=2}^{\infty}e^{-\omega n}\\
&\leq K_1(t-\sigma)^{-\alpha}e^{-\omega(t-\sigma)}+K_2e^{-\omega(t-\sigma)},
\end{aligned}
$$

where

$$
K_1=\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(1-e^{-q\omega})^{\frac{1}{q}}\ \text{et}\ K_2=\frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=2}^{+\infty}e^{-\omega n}
$$

It follow that for n and m sufficiently large and $\sigma\leq t$, we have

$$
\begin{aligned}
&\Big|\int_{-\infty}^{t}F(n,s,t)ds-\int_{-\infty}^{t}F(m,s,t)ds\Big|_\alpha\\
\leq\ &\Big|\int_{-\infty}^{\sigma}F(n,s,t)ds\Big|_\alpha+\Big|\int_{-\infty}^{\sigma}F(m,s,t)ds\Big|_\alpha\\
&+\Big|\int_{\sigma}^{t}F(n,s,t)ds-\int_{\sigma}^{t}F(m,s,t)ds\Big|_\alpha\\
\leq\ &2\big(K_1(t-\sigma)^{-\alpha}e^{-\omega(t-\sigma)}+K_2e^{-\omega(t-\sigma)}\big)+\Big|\int_{\sigma}^{t}F(n,s,t)ds-\int_{\sigma}^{t}F(m,s,t)ds\Big|_\alpha.
\end{aligned}
$$

Since $\displaystyle\lim_{n\to+\infty}\int_{\sigma}^{t}F(n,s,t)ds$ exists, then

$$
\limsup_{n,m\to+\infty}\Big|\int_{-\infty}^{t}F(n,s,t)ds-\int_{-\infty}^{t}F(m,s,t)ds\Big|_\alpha\leq 2\big(K_1(t-\sigma)^{-\alpha}e^{-\omega(t-\sigma)}+K_2e^{-\omega(t-\sigma)}\big).
$$

If $\sigma\to-\infty$, then

$$
\limsup_{n,m\to+\infty}\Big|\int_{-\infty}^{t}F(n,s,t)ds-\int_{-\infty}^{t}F(m,s,t)ds\Big|=0.
$$

Thus, by the completeness of the phase space $\mathcal{B}$, we deduce that the limit

$$
\lim_{n\to+\infty}\int_{-\infty}^{t}F(n,s,t)ds=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))ds
$$

exists. In addition, one can see from equation (10) that the function

$$
\eta_1:t\to\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0f(s))ds
$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$
\eta_2:t\to\lim_{n\to+\infty}\int_{t}^{\infty}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_nX_0f(s))ds
$$

is well defined and bounded on $\mathbb{R}$. Using the same argument as in [1, Theorem 5.9], the integral solution u given by the formula

$$u_t = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$$

for $t\in\mathbb{R}$ is the only bounded integral solution of equation (1) on $\mathbb{R}$.■

Theorem 7.2. *Let $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ and Γ be the mapping defined by*

$$\Gamma g(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))ds \text{ for } t\in\mathbb{R}.$$

If $p>1$, then $\Gamma g\in\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$.

Proof. For each $n=1,2,3,\dots$ and $t\in\mathbb{R}$, set

$$X_n(t) = \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))ds - \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))ds.$$

We have

$$\begin{aligned}
&|X_n(t)|_\alpha\\
\le\ & \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))|_\alpha ds\\
+\ & \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n}|\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))|_\alpha ds\Big](0)\\
\le\ & \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1}\|\mathcal{A}^\alpha_{\mathcal{U}}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\|ds\\
+\ & \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n}\|\mathcal{A}^\alpha_{\mathcal{U}}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\|ds\Big](0)\\
\le\ & \overline{M}\widetilde{M}\int_{t-n}^{t-n+1}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\ |g(s)|_\alpha ds + \overline{M}\widetilde{M}\int_{t+n-1}^{t+n}\frac{e^{\omega(t-s)}}{(s-t)^\alpha}|\Pi^u|\ \|g(s)\|ds\\
\le\ & K\Big[\int_{t-n}^{t-n+1}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\ |g(s)|_\alpha ds + \int_{t+n-1}^{t+n}\frac{e^{\omega(t-s)}}{(s-t)^\alpha}|\Pi^u|\ |g(s)|_\alpha\Big]ds.
\end{aligned}$$

Set

$$K=\max(\overline{M}\widetilde{M}|\Pi^s|,\overline{M}\widetilde{M}|\Pi^u|).$$

Case:1 $n=1$.

$$|X_1(t)|_\alpha \ \le\ K\Big[\int_{t-1}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\ |g(s)|_\alpha ds + \int_{t}^{t+1}\frac{e^{\omega(t-s)}}{(s-t)^\alpha}|\Pi^u|\ |g(s)|_\alpha\Big]ds$$

Let q such that $\dfrac{1}{q}+\dfrac{1}{p}=1$. Using the Hölder's inequality, we obtain

$$\begin{aligned}
&|X_1(t)|_\alpha\\
\le\ & K\Big(\int_{t-1}^{t}\frac{e^{-q\omega(t-s)}}{(t-s)^{q\alpha}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-1}^{t}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}} + K\Big(\int_{t-1}^{t}\frac{e^{q\omega(t-s)}}{(s-t)^{q\alpha}}ds\Big)^{\frac{1}{q}}\Big(\int_{t}^{t+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\\
\le\ & K\Big(\int_0^{q\omega}\frac{e^{-s}}{s^{q\alpha}}\times(\omega q)^{\alpha q-1}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-1}^{t}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t}^{t+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]
\end{aligned}$$

$$\leq \quad \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_0^{q\omega} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-1}^{t}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_t^{t+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]$$

Let us pose

$\gamma_q(K,\omega)=\dfrac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\displaystyle\int_0^{q\omega} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}$, then we have

$$|X_1(t)|_\alpha \leq \gamma_q(K,\omega)\Big[\Big(\int_{t-1}^{t}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_t^{t+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]$$

Since $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$, then $X_1(t)\in\mathcal{E}(\mathbb{R},X_\alpha),\mu,\nu,r)$

Case:2 $n\geq 2$

We use same reasonning like above

Let q such that $\dfrac{1}{q}+\dfrac{1}{p}=1$. Using the Hölder's inequality, we obtain

$$\begin{aligned}
& |X_n(t)|_\alpha \\
\leq\ & K\Big(\int_{t-n}^{t-n+1}\frac{e^{-q\omega(t-s)}}{(t-s)^{q\alpha}}ds\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}} \\
+\ & K\Big(\int_{t+n-1}^{t+n}\frac{e^{q\omega(t-s)}}{(s-t)^{q\alpha}}ds\Big)^{\frac{1}{q}}\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}} \\
\leq\ & K\Big(\int_{q\omega(n-1)}^{q\omega n}\frac{e^{-s}}{s^{q\alpha}}\times(\omega q)^{\alpha q-1}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}(q\omega(n-1))^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}} \\
+\ & \Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1}{q}}}(n-1)^{-\alpha}\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}-1)^{\frac{1}{q}}(n-1)^{-\alpha}e^{-\omega n}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}(n-1)^{-\alpha}e^{-\omega n}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big] \\
\leq\ & \frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}e^{-\omega n}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]
\end{aligned}$$

$$\leq \quad \frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}e^{-\omega n}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]$$

$$\leq \quad \frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]$$

Set

$$\delta_q(K,\omega)=\frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}},$$

it follows from the Weierstrass test that the sequence of functions $\sum_{n=1}^{N}X_n(t)$ is uniformly convergent on $\mathbb{R}$.

Since $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ and

$$|X_n(t)|_\alpha\leq S_q(K,\omega)\Big[\Big(\int_{t-n}^{t-n+1}\|g(s)\|^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big],$$

where

$$S_q(K,\omega)=\max(\gamma_q(K,\omega),\delta_q(K,\omega)),$$

we conclude that $X_n\in\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$. Thus $\sum_{n=1}^{N}X_n(t)\in\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$ by Lemma 6.1. Observing that

$$\Gamma g(t)=\sum_{n=1}^{+\infty}X_n(t),$$

it follows that $\Gamma g(t)\in\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$. □

Theorem 7.3. *Let $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$.*
If $p>1$, then $\Gamma g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$.

Proof. For each $n=1,2,3,...$, let be X_n defined as previously.

Case 1 $n=1$.
We have

$$|X_1(t)|_\alpha^p\leq\gamma_q^p(K,\omega)\Big[\Big(\int_{t-1}^{t}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t}^{t+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]^p.$$

Using the Minkowski's inequality, we obtain

$$\Big(\int_\theta^{\theta+1}|X_1(s)|_\alpha^p ds\Big)^{\frac{1}{p}}$$

$$\leq \quad \gamma_q(K,\omega)\Big[\Big(\int_\theta^{\theta+1}\Big[\Big(\int_{s-1}^{s}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}ds\Big]^p\Big)^{\frac{1}{p}}+\Big(\int_\theta^{\theta+1}\Big[\Big(\int_{s}^{s+1}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}ds\Big]$$

$$\leq \quad \gamma_q(K,\omega)\Big[\Big(\sup_{s\in[\theta,\theta+1]}\int_{s-1}^{s}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}+\Big(\sup_{s\in[\theta,\theta+1]}\int_{s}^{s+1}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}\Big]$$

$$\begin{aligned}
&\leq \quad \gamma_q(K,\omega)\Big[\Big(\int_{\theta-1}^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta}^{\theta+2}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]\\
&\leq \quad \gamma_q(K,\omega)\Big[\Big(\int_{\theta-1}^{\theta}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta}^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\\
&\quad +\Big(\int_{\theta}^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta+1}^{\theta+2}\|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|X_1(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq \quad &\gamma_q(K,\omega)\Big[\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-1}^{\theta}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)+\\
&+\frac{1}{\mu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+1}^{\theta+2}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\Big].
\end{aligned}$$

where

$$\gamma_q(K,\omega)=\frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_0^{q\omega}e^{-s}s^{-\alpha q}ds\Big)^{\frac{1}{q}}.$$

Case: 2 $n\geq 2$.

By use the same reasonning like above, we have

$$|X_n(t)|_\alpha^p\leq \delta_q^p(K,\omega)\Big[\Big(\int_{t-n}^{t-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{t+n-1}^{t+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]^p,$$

where

$$\delta_q(K,\omega)=\frac{K}{(\omega q)^{\frac{1}{q}}}(e^{\omega q}+1)^{\frac{1}{q}}.$$

Using the Minkowski's inequality, we obtain

$$\begin{aligned}
&\Big(\int_{\theta}^{\theta+1}|X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\\
\leq \quad &\delta_q(K,\omega)\Big[\Big(\int_{\theta}^{\theta+1}\Big[\Big(\int_{s-n}^{s-n+1}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}ds\Big]^p\Big)^{\frac{1}{p}}+\Big(\int_{\theta}^{\theta+1}\Big[\Big(\int_{s+n-1}^{s+n}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}ds\Big]\\
\leq \quad &\delta_q(K,\omega)\Big[\Big(\sup_{s\in[\theta,\theta+1]}\int_{s-n}^{s-n+1}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}+\Big(\sup_{s\in[\theta,\theta+1]}\int_{s+n-1}^{s+n}|g(\xi)|_\alpha^p d\xi\Big)^{\frac{1}{p}}\Big]\\
\leq \quad &\delta_q(K,\omega)\Big[\Big(\int_{\theta-n}^{\theta-n+2}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta+n-1}^{\theta+n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]\\
\leq \quad &\delta_q(K,\omega)\Big[\Big(\int_{\theta-n}^{\theta-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta-n+1}^{\theta-n+2}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

$$+\Big(\int_{\theta+n-1}^{\theta+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}+\Big(\int_{\theta+n}^{\theta+n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big].$$

It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\quad&\delta_q(K,\omega)\Big[\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n}^{\theta-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n+1}^{\theta-n+2}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n-1}^{\theta+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)+\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n}^{\theta+n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\Big].
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
\leq\quad&S_q(K,\omega)\Big[\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n}^{\theta-n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n+1}^{\theta-n+2}\|g(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n-1}^{\theta+n}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)+\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n}^{\theta+n+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\Big],
\end{aligned}$$

where

$$S_q(K,\omega)=\max(\gamma(K,\omega),\delta(K,\omega)).$$

We conclude that $X_n\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$.

Thus $\sum_{n=1}^{N}X_n(t)\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ by Lemma 6.1. Observing that

$$\Gamma g(t)=\sum_{n=1}^{+\infty}X_n(t),$$

it follows that $\Gamma g(t)\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$. □

Theorem 7.4. *Let $h\in AA_c(\mathbb{R};L^p((0,1),X))$, then $\Gamma h\in AA_c(\mathbb{R};L^p((0,1),X))$.*

Proof. For a given sequence $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v\in BS^p(\mathbb{R};X_\alpha)$ such that $h(t+s_n)$ converges to $v(t)$ and $v(t-s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. From [8], if

$$\begin{aligned} w(t+s_n) &= \lim_{\lambda\to\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s+s_n))ds \\ &+ \lim_{\lambda\to\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s+s_n))ds \text{ for } t\in\mathbb{R},\ n\in\mathbb{N}, \end{aligned}$$

then $w(t+s_n)$ converges to

$$z(t)=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 v(s))ds.$$

It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. We get the following estimates

$$\begin{aligned} w(s+s_n)-z(s) &= \lim_{\lambda\to+\infty}\int_{-\infty}^{s}\mathcal{U}^s(s-\theta)\Pi^s(\widetilde{B}_\lambda X_0[h(\theta+s_n)-v(\theta)]d\theta \\ &\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{s}\mathcal{U}^u(s-\theta)\Pi^u(\widetilde{B}_\lambda X_0[h(\theta+s_n)-v(\theta)]d\theta \\ |w(s+s_n)-z(s)|_\alpha &\leq K\int_{-\infty}^{s}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta \\ &+ K\int_{s}^{+\infty}\frac{e^{\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta. \end{aligned}$$

where

$$K=\max(\overline{M}\widetilde{M}|\Pi^s|,\overline{M}\widetilde{M}|\Pi^u|).$$

For each $k=1,2,3,...$, set

$$X_k(s)=K\int_{s-k}^{s-k+1}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta+K\int_{s+k-1}^{s+k}\frac{e^{\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|_\alpha d\theta.$$

Case: 1 $n=1$

$$X_1(s)=K\int_{s-1}^{s}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta+K\int_{s}^{s+1}\frac{e^{\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|_\alpha d\theta.$$

Using the Hölder's inequality, we obtain

$$\begin{aligned} & X_1(s) \\ \leq\ & \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_0^{q\omega n}e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{s-1}^{s}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}} \\ +\ & \Big(\int_{s}^{s+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big] \\ \leq\ & \gamma_q(K,\omega)\Big[\Big(\int_{s-1}^{s}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}+\Big(\int_{s}^{s+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big], \end{aligned}$$

where

$$\gamma_q(K,\omega)=\frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_0^{q\omega n}e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}},$$

and by the Minkowski's inequality, we have

$$\begin{aligned}\Big(\int_t^{t+1}|X_1(s)|_\alpha^p ds\Big)^{\frac{1}{p}} &\leq \gamma_q(K,\omega)\Big[\Big(\int_t^{t+1}\Big[\Big(\int_{s-1}^{s}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}\\
&\quad+\Big(\int_t^{t+1}\Big[\Big(\int_{s}^{s+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}\Big]\\
&\leq \gamma_q(K,\omega)\Big(\int_0^{q\omega n}e^{-s}s^{-q\alpha}ds-\int_0^{q\omega(n-1)}e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\\
&\quad\Big[\Big(\sup_{s\in[t,t+1]}\int_{s-1}^{s}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\\
&\quad+\Big(\sup_{s\in[t,t+1]}\int_{s}^{s+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]\\
&\leq 2\gamma_q(K,\omega)\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|v(\theta)-h(\theta+s_n)|_\alpha^p ds\Big)^{\frac{1}{p}}.\end{aligned}$$

Case:2 $n\geq 2$

$$X_k(s)=K\int_{s-k}^{s-k+1}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta+K\int_{s+k-1}^{s+k}\frac{e^{\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta.$$

Using the Hölder's inequality, we obtain

$$\begin{aligned}&X_k(s)\\
\leq\ & \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n}e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{s-k}^{s-k+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\\
+\ & \Big(\int_{s+k-1}^{s+k}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]\\
\leq\ & \delta_q(K,\omega)\Big[\Big(\int_{s-k}^{s-k+1}|v(\theta)-h(\theta+s_n)|^p d\theta\Big)^{\frac{1}{p}}+\Big(\int_{s+k-1}^{s+k}|v(\theta)-h(\theta+s_n)|^p d\theta\Big)^{\frac{1}{p}}\Big],\end{aligned}$$

where

$$\delta_q(K,\omega)=\frac{K}{(\omega q)^{\frac{1}{q}}}(e^{q\omega}+1)^{\frac{1}{q}},$$

and by the Minkowski's inequality, we have

$$\begin{aligned}\Big(\int_t^{t+1}|X_k(s)|^p ds\Big)^{\frac{1}{p}} &\leq \delta_q(K,\omega)\Big[\Big(\int_t^{t+1}\Big[\Big(\int_{s-k}^{s-k+1}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}\\
&\quad+\Big(\int_t^{t+1}\Big[\Big(\int_{s+k-1}^{s+k}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}\Big]\end{aligned}$$

$$\begin{aligned}
&\leq \delta_q(K,\omega)\Big[\Big(\sup_{s\in[t,t+1]}\int_{s-k}^{s-k+1}|v(\theta)-h(\theta+s_n)|^p d\theta\Big)^{\frac{1}{p}}\\
&\quad+\Big(\sup_{s\in[t,t+1]}\int_{s+k-1}^{s+k}|v(\theta)-h(\theta+s_n)|_\alpha^p d\theta\Big)^{\frac{1}{p}}\Big]\\
&\leq 2\delta_q(K,\omega)\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|v(\theta)-h(\theta+s_n)|_\alpha^p ds\Big)^{\frac{1}{p}},
\end{aligned}$$

Since

$$\sum_{k=1}^{+\infty}X_k(s)=K\int_{-\infty}^{s}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|d\theta+K\int_{+\infty}^{s}\frac{e^{\omega(s-\theta)}}{(s-\theta)^\alpha}|v(\theta)-h(\theta+s_n)|_\alpha d\theta,$$

it follows that

$$\Big(\int_t^{t+1}|w(s+s_n)-z(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\leq 2S_q(K,\omega)\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|v(\theta)-h(\theta+s_n)|_\alpha^p ds\Big)^{\frac{1}{p}},$$

where

$$S_q=\max(\gamma_q(K,\omega),\delta_q(K,\omega)).$$

Fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $\Gamma\subset[\frac{-L}{2},\frac{L}{2}]$ with

$$|h(s+s_n)-v(s)|_\alpha \quad\leq\quad \varepsilon \ \text{ for }\ n\geq N_\varepsilon \ \text{ and }\ s\in[-L,L].$$

Then, for each $t\in\Gamma$, ones has

$$\Big(\int_t^{t+1}|w(s+s_n)-z(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\leq 2S_q(K,\omega)\varepsilon,$$

which proves that the convergence is uniform on Γ, by the fact that the last estimate is independent of $t\in\Gamma$. Proceeding as previously, one can similarly prove that

$$\lim_{n\to+\infty}\Big(\int_t^{t+1}|w(s)-z(s-s_n)|_\alpha^p ds\Big)^{\frac{1}{p}}=0,$$

which implies that $\Gamma h\in AA_c(\mathbb{R};L^p((0,1),X_\alpha))$. □

For the existence of pseudo almost automorphic solution, we make the following assertion.

($\mathbf{H_5}$) $f:\mathbb{R}\to X$ is $cl(\mu,\nu)$-S^p-pseudo almost automorphic of class r.

Theorem 7.5. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_3}$) and ($\boldsymbol{H_5}$) hold. If $p>1$, then equation*(1) *has a unique $cl(\mu,\nu)$-S^p-pseudo almost automorphic solution of class r.*

Proof. Since f is S^p-pseudo almost automorphic function, f has a decomposition $f=f_1+f_2$ where $f_1^b\in AA(\mathbb{R},L^p((0,1),X_\alpha))$ and $f_2^b\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$. Using Theorem 7.1, Theorem 7.3 and Theorem 7.4, we get the desired result. □

Our next objective is to show the existence of pseudo almost automorphic solutions of class r for the following problem

$$u'(t) = -Au(t) + L(u_t) + f(t, u_t) \text{ for } t \in \mathbb{R} \tag{11}$$

where $f : \mathbb{R} \times C_\alpha \to X$ is a continuous.

For the sequel, we make the following assertion.

For the sequel, we make the following assertion.

($\mathbf{H_6}$) The instable space $U \equiv \{0\}$.

($\mathbf{H_7}$) $f : \mathbb{R} \times C_\alpha \to X$ is uniformly $cl(\mu,\nu)$-S^p pseudo compact almost automorphic of class r such that there exists a positive constant L_f such that

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| \le L_f \|\varphi_1 - \varphi_2\|_{C_\alpha} \quad \text{for all } \ t \in \mathbb{R} \quad \text{and} \quad \varphi_1, \varphi_2 \in C_\alpha.$$

where L_f satisfies conditions of Theorem 6.11.

Theorem 7.6. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_3}$), ($\boldsymbol{H_4}$), ($\boldsymbol{H_6}$) and ($\boldsymbol{H_7}$) hold. If*

$$S_q(\Delta,\omega) L_f < 1,$$

where $S_q(\Delta,\omega) = \max(\gamma_q(K,\omega), \delta_q(K,\omega))$, then equation (11) *has a unique $cl(\mu,\nu)$-S^p-pseudo compact almost automorphic mild solution of class r.*

Proof. Let x be a function in $PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$, from Theorem 6.14 the function $t \to x_t$ belongs to $PAA_c(C_\alpha, \mu, \nu, r)$. Hence Theorem 6.11 implies that the function $g(.) := f(., x_.)$ is in $PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, \infty)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider now the mapping

$$\mathcal{H} : PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, \infty) \to PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s (\widetilde{B}_\lambda X_0 f(s, x_s)) ds \Big](0).$$

From Theorem 7.1, Theorem 7.3 and Theorem 7.4, we can infer that $\mathcal{H}$ maps $PAA_c(\mathbb{R}; L^p((0,1), X), \mu, \nu, r)$ into $PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$. It suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$.

Let $x_1, x_2 \in PAA_c(\mathbb{R}; L^p((0,1), X_\alpha), \mu, \nu, r)$. Then we have

$$\begin{aligned} |\ \mathcal{H}x_1(t) - \mathcal{H}x_2(t)|_\alpha &\le \Big| \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s (\widetilde{B}_\lambda X_0 (f(s, x_{1s}) - f(s, x_{2s}))) ds \Big|_\alpha \\ &\le \overline{M}\widetilde{M} |\Pi^s| L_f \int_{-\infty}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |x_{1s} - x_{2s}|_\alpha ds. \end{aligned}$$

For each $n = 1, 2, 3, ...,$ set

$$X_n(t) = \Delta L_f \int_{t-n}^{t-n+1} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |x_{1s} - x_{2s}|_\alpha ds,$$

where

$$\Delta = \overline{M}\widetilde{M}|\Pi^s|$$

Then for each $n = 1, 2, 3, ...,$ using a same reasoning as in the proof of Theorem 7.2.

Case:1 $n = 1$.

$$X_1(t) \quad \leq \quad \Delta L_f \int_{t-1}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |x_{1s} - x_{2s}|_\alpha ds,$$

then, we have

$$X_1(t) \quad \leq \quad \frac{\Delta L_f}{(\omega q)^{\frac{1-\alpha q}{q}}} \Big(\int_0^{q\omega} e^{-s} s^{-q\alpha} ds \Big)^{\frac{1}{q}} \Big[\Big(\int_{t-1}^{t} |x_{1s} - x_{2s}|_\alpha^p ds \Big)^{\frac{1}{p}} \Big],$$

it follows that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |X_1(s)|^p ds \Big)^{\frac{1}{p}} \quad \leq \quad \gamma_q(\Delta, \omega) L_f \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |x_1(s) - x_2(s)|_\alpha^p ds \Big)^{\frac{1}{p}} \Big]$$

,

where

$$\gamma_q(\Delta, \omega) = \frac{\Delta L_f}{(\omega q)^{\frac{1-\alpha q}{q}}} \Big(\int_0^{q\omega} e^{-s} s^{-q\alpha} ds \Big)^{\frac{1}{q}}$$

Case:2 $n \geq 2$

$$\begin{aligned} X_n(t) \quad &\leq \quad \frac{\Delta L_f}{(\omega q)^{\frac{1-\alpha q}{q}}} \Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s} s^{-q\alpha} ds \Big)^{\frac{1}{q}} \Big[\Big(\int_{t-n}^{t-n+1} |x_{1s} - x_{2s}|_\alpha^p ds \Big)^{\frac{1}{p}} \Big] \\ &\leq \quad \frac{\Delta}{(\omega q)^{\frac{1}{q}}} (e^{q\omega} + 1)^{\frac{1}{q}} L_f \Big[\Big(\int_{t-1}^{t} |x_{1s} - x_{2s}|_\alpha^p ds \Big)^{\frac{1}{p}} \Big] \end{aligned}$$

and

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |X_n(s)|^p ds \Big)^{\frac{1}{p}} \quad \leq \quad \delta_q(K, \omega) L_f \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |x_1(s) - x_2(s)|_\alpha^p ds \Big)^{\frac{1}{p}} \Big],$$

where

$$\delta_q(\Delta, \omega) = \frac{\Delta}{(\omega q)^{\frac{1}{q}}} (e^{q\omega} + 1)^{\frac{1}{q}}.$$

Since

$$\sum_{n=1}^{+\infty} X_n(t) = \Delta L_f\Big(\int_{-\infty}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^{\alpha}} |x_{1s} - x_{2s}|_{\alpha} ds\Big),$$

we deduce that

$$\sup_{t\in\mathbb{R}} \Big(\int_{t}^{t+1} |\ \mathcal{H}x_1(s) - \mathcal{H}x_2(s)|_{\alpha}^{p} ds\Big)^{\frac{1}{p}} \leq S_q(\Delta,\omega) L_f \|x_1 - x_2\|_{S^p,\alpha}.$$

Thus $\mathcal{H}$ is a contractive mapping. We conclude that equation(11), has one and only one $cl(\mu,\nu)$-S^p-pseudo almost automorphic solution of class r and this fixed point satisfies the integral equation

$$u_t = \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_{\lambda} X_0 f(s,u_s))ds,$$

which ends the proof. □

8. Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_{-\infty}^{0} G(\theta) z(t+\theta,x) d\theta + \sin\Big(\dfrac{1}{2+\cos t + \cos\sqrt{2}t}\Big) + g(t) \\ \qquad\qquad +h\Big(t, \dfrac{\partial}{\partial x} z(t+\theta,x)\Big) \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi] \\ \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R}, \end{cases} \tag{12}$$

where $G : [-r,0]$ into $\mathbb{R}$ is a continuous function, $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and lipschitzian with respect to the second argument and $g : \mathbb{R}\times[0,\pi] \to \mathbb{R}$ is a bounded continuous function defined by

$$g(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ \\ -(t+1)e^{-t} \text{ for } t \geq 0. \end{cases}$$

To rewrite equation(12) in the abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x\in X$,

$$\|x\|_{L^2} = \Big(\int_{0}^{\pi} |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $A : X \to X$ be defined by

$$\begin{cases} D(A) = H^2(0,\pi)\cap H_0^1(0,\pi) \\ Ay = y''. \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 : \ n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), \ s \in [0, \pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2 (y, e_n) e_n, \ y \in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X : \ \sum_{n=1}^{+\infty} n(y, e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} y = \sum_{n=1}^{+\infty} n(y, e_n) e_n, \ y \in D(A).$$

Lemma 8.1. [15] *If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and* $|y'| = |A^{\frac{1}{2}} y|$.

It is well known that $-A$ is the generator of a compact analytic semigroup semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x, e_n)e_n, \ x \in X.$$

Then $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satified. Here we choose $\alpha = \dfrac{1}{2}$.
.
We define $f : \mathbb{R} \times C_{\frac{1}{2}} \to X$ and $L : C_{\frac{1}{2}} \to X$ as follows

$$\begin{aligned} & f(t,\varphi)(x) \\ = \ & \sin\Big(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\Big) + g(t) + h\Big(t, \frac{\partial}{\partial x}\varphi(\theta)(x)\Big) \ \text{ for } x \in [0,\pi] \ \text{ and } \ t \in \mathbb{R}, \\ & L(\varphi)(x) \\ = \ & \int_{-r}^{0} G(\theta)\varphi(\theta)(x)d\theta \ \text{ for } \ -r \leq \theta \leq 0 \ \text{ and } \ x \in [0,\pi]. \end{aligned}$$

Let us pose $v(t) = z(t, x)$. Then equation(12) takes the following abstract form

$$v'(t) = Av(t) + L(v_t) + f(t, v_t) \text{ for } t \in \mathbb{R}. \tag{13}$$

Consider the measures μ and ν where its Radon-Nikodym derivative are respectively $\rho_1, \rho_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 \text{ for } t > 0 \\ e^t \text{ for } t \leq 0. \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [3] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy Hypothesis $(\mathbf{H_4})$ and from [9], $\sin\Big(\dfrac{1}{2+\cos t + \cos\sqrt{2}t}\Big)$ is compact almost automorphic. We also have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-\tau}^0 e^t dt + \int_0^\tau dt}{\displaystyle 2\int_0^\tau t dt} = \limsup_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.

Let $p \geq 1$, since r is given then we have

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |g(s)|^p_{\frac{1}{2}} ds\Big)^{\frac{1}{p}} d\mu(t) \\
= \; & \frac{1}{\mu([-\tau,\tau])} \int_0^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|A^{\frac{1}{2}} g(s)\|^p ds\Big)^{\frac{1}{p}} dt \\
= \; & \frac{1}{\mu([-\tau,\tau])} \int_0^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|g'(s)\|^p ds\Big)^{\frac{1}{p}} dt \\
\leq \; & \frac{1}{\nu([-\tau,\tau])} \int_0^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_{t-r}^{t+1} s^p e^{-ps} ds\Big)^{\frac{1}{p}} dt \\
\leq \; & \frac{1}{\nu([-\tau,\tau])} \int_0^{+\tau} \Big(\int_{t-r}^{t+1} s^p e^{-s} ds\Big)^{\frac{1}{p}} dt \\
\leq \; & \frac{1}{\nu([-\tau,\tau])} \int_0^{+\tau} \Big[\Big(\int_{t-r}^{t+1} s^{p^2}\Big)^{\frac{1}{p}} \Big(\int_{t-r}^{t+1} e^{-qs} ds\Big)^{\frac{1}{q}}\Big]^{\frac{1}{p}} dt \\
\leq \; & \frac{1}{\mu([-\tau,\tau])} \int_0^{+\tau} \Big[[(t+1)^{p^2+1}]^{\frac{1}{p}} e^{-(t-r)}\Big]^{\frac{1}{p}} dt \\
\leq \; & \frac{1}{\nu([-\tau,\tau])} \int_0^{+\infty} \Big[(t+1)^{p+1} e^{-(t-r)}\Big]^{\frac{1}{p}} dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\nu([-\tau,\tau])}\int_0^{+\infty}(t+1)^2 e^{-\frac{(t-r)}{p}}dt\\
&\leq \frac{p(2p^2+2p+1)e^{\frac{r}{p}}}{\tau^2}.
\end{aligned}
$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}dt=0.$$

It follows that $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$, consequently, f is uniformly μ-S^p-pseudo almost periodic of class r. Moreover, L is a bounded linear operator from $C_{\frac{1}{2}}$ to X.
Let k be the lipschiz constant of h, then for every $\varphi_1,\varphi_2\in C_{\frac{1}{2}}$ and $t\geq 0$, we have

$$
\begin{aligned}
\|f(t,\varphi_1)(x)-f(t,\varphi_2)(x)\| &= \Big(\int_0^\pi\Big[h\Big(\theta,\frac{\partial}{\partial x}\varphi_1(\theta,x)\Big)-h\Big(t,\frac{\partial}{\partial x}\varphi_1(t,x)\Big)\Big]^2dx\Big)^{\frac{1}{2}}\\
&\leq L_h\Big[\int_0^\pi\Big(\frac{\partial}{\partial x}\varphi_1(\theta,x)-\frac{\partial}{\partial x}\varphi_2(\theta,x)\Big)^2dx\Big]^{\frac{1}{2}}\\
&\leq L_h\sup_{-r\leq\theta\leq 0}\Big[\int_0^\pi\Big(\frac{\partial}{\partial x}\varphi_1(\theta,x)-\frac{\partial}{\partial x}\varphi_2(\theta,x)\Big)^2dx\Big]^{\frac{1}{2}}\\
&\leq L_h\|\varphi_1-\varphi_2\|_{C_\alpha}
\end{aligned}
$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of class r.

Lemma 8.2. [10] *If* $\int_{-r}^0|G(\theta)|d\theta<1$, *then the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic.*

For example, let us pose $G(\theta)=\dfrac{\theta^2-1}{(\theta^2+1)^2}$ for $\theta\in[-r,0]$. Then we can see that

$$\int_{-r}^0|G(\theta)|d\theta=\int_{-r}^0\Big|\frac{\theta^2-1}{(\theta^2+1)^2}\Big|d\theta=\Big[\frac{\theta}{\theta^2+1}\Big]_{-r}^0=\frac{r}{r^2+1}<1 \text{ if } r<1$$

and

$$
\begin{aligned}
&\int_{-r}^0|G(\theta)|d\theta=\int_{-r}^0\Big|\frac{\theta^2-1}{(\theta^2+1)^2}\Big|d\theta\\
&=\int_{-r}^{-1}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta+\int_{-1}^0\frac{-\theta^2+1}{(\theta^2+1)^2}d\theta=1-\frac{r}{r^2+1}<1 \text{ if } r\geq 1,
\end{aligned}
$$

Lemma 8.3. *Under the above assumptions, if Lip*(h) *is small enough, then equation*(12) *has a unique* $cl(\mu,\nu)$-S^p*-pseudo almost automorphic solution* v *of class* r.

9. Discussion

In this paper we give a new approach to study weighted Stepanov-like pseudo almost automorphic functions using the measure theory and study the existence and uniqueness of (μ,ν)-weighted Stepanov-like pseudo almost automorphic solutions of class r for some partial functional differential equations in a Banach space. This study uses the α-norm which is more general than a classical norm. Then, we obtain some more results which are more general than the classical ones (see [8] and [17] for example).
However, if we obtain the existence of a unique $cl(\mu,\nu)$-S^p-pseudo almost automorphic solution of class r, we cannot say that we have a unique S^p-pseudo almost automorphic solution of class r. The next challenge, is to find under which assumptions, we have a unique S^p-pseudo almost automorphic solution of class r.

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