J. Appl. Math. & Informatics Vol. **41***(2023), No. 4, pp. 821 - 829*
https://doi.org/10.14317/jami.2023.821

ON CHARACTERIZATIONS OF THE CONTINUOUS DISTRIBUTIONS BY INDEPENDENT PROPERTY OF THE DIFFERENCE-TYPE k-TH LOWER RECORD VALUES

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ABSTRACT. In this paper, we obtain characterizations of continuous distributions based on the independent property of generalized record values extending the characterization results reported by Jin and Lee [4], Skřivánková and Juhás [8]. Also, example of special cases of general classes as Bur types, Pareto, power and Weibull distribution are discussed.

AMS Mathematics Subject Classification : 60E05, 62E10, 62H10.
Key words and phrases : Independent property, generalized lower record values, continuous distributions.

1. Introduction

Record values are found in many situations of daily life as well as in many statistical applications. Also, we are often interested in observing new records and in recording them. For this reason, several continuous distributions have been characterized through the some properties of record values and generalized record values in literature. For example, several continuous distributions based on record values were extensively studied by many authors and such results are available in Skřivánková and Juhás [8], Khan and Faizan [6], Paul [7] and Jin and Lee [5].

Now some notations and definitions. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The j-th order statistic of a sample $(X_1, X_2, \cdots, X_n)$ is denoted by $X_{j:n}$. For a fixed positive integer k, Dziubdziela and Kopociński [3] defined the sequences $\{L^{(k)}(n), n \geq 1\}$ of k-th lower record times of $\{X_n, n \geq 1\}$ as follows :

$$L^{(k)}(1) = 1,$$

Received December 15, 2022. Revised April 26, 2023. Accepted April 28, 2023.

$$L^{(k)}(n+1) = min\{j > L^{(k)}(n) : X_{k:L^{(k)}(n)+k-1} > X_{k:j+k-1}\}.$$

The sequences $\{Y_n^{(k)}, n \geq 1\}$ with $Y_n^{(k)} = X_{k:L^{(k)}(n)+k-1}$, $n = 1, 2, \cdots$, are called the sequences of k-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L(n)}$, $n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = X_{k:k} = max\{X_1, X_2, \cdots, X_k\}$. For more details and references, see Ahsanullah et al. [2].

Let $\{Y_n^{(k)}, n \geq 1\}$ be the sequence of k-th lower record values. Then the probability density function of $Y_n^{(k)}, n \geq 1$ is given by

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}(H(x))^{n-1}(F(x))^{k-1}f(x),\ x > 0, \tag{1}$$

and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ for $1 \leq m < n$ and $n > 2$ is given by

$$\begin{aligned} f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = &\frac{k^n}{\Gamma(m)\Gamma(n-m)}(H(x))^{m-1} \\ &\cdot (H(y) - H(x))^{n-m-1}(F(y))^{k-1}\frac{f(x)}{F(x)}f(y),\ y < x, \end{aligned} \tag{2}$$

where $H(x) = -ln(F(x))$.

In this paper we have extended the results of Jin and Lee [4], Skřivánková and Juhás [8] and characterized the various continuous distributions using independent property of generalized lower record values.

2. Characterization of distributions $F(x) = exp\big(-e^{-g(x)}\big)$

For proving our Theorem 2.2, we need the following Lemma 2.1.

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to 0$ as $x \to 0$, $g(x) \to -\infty$ as $x \to \alpha^+$ and $g(x) \to \infty$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Suppose that $\frac{H\left(g^{-1}(u+v)\right)}{H\left(g^{-1}(v)\right)} = q(u,v)$ and $h(u,v) = \big(q(u,v)\big)^{m-1}\big(1 - q(u,v)\big)^{n-m-1} \times \frac{\partial}{\partial u}\big(q(u,v)\big)$, for $1 \leq m < n$, where $h(u,v) \neq 0$ and $\frac{\partial}{\partial u}\big(q(u,v)\big) \neq 0$ for any positive u and v. If $h(u,v)$ is independent of v, then $q(u,v)$ is a function of u only.*

Proof. Let

$$\begin{aligned} h(u,v) &= \big(q(u,v)\big)^{m-1}\big(1 - q(u,v)\big)^{n-m-1} \times \frac{\partial}{\partial u}\big(q(u,v)\big) \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{n-m-1}{j}\big(q(u,v)\big)^{j+m-1} \times \frac{\partial}{\partial u}\big(q(u,v)\big). \end{aligned} \tag{3}$$

Since $h(u,v)$ is independent of v, we obtain

$$h(u,v)=\sum_{j=0}^{\infty}(-1)^j\binom{n-m-1}{j}\big(q(u,v)\big)^{j+m-1}\times\frac{\partial}{\partial u}\big(q(u,v)\big)=t(u). \tag{4}$$

Integrating (4) with respect to u, we get

$$\begin{aligned}\int h(u,v)du&=\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+m)}\binom{n-m-1}{j}\big(q(u,v)\big)^{j+m}\frac{\partial}{\partial u}\big(q(u,v)\big)\\&=\int t(u)du+c=T(u).\end{aligned} \tag{5}$$

Here T is a function of u only and c is independent of u but may depend on v.

Now taking $u\to 0$, $q(u,v)\to 1$, we have c independently of v from (5). Differentiating (5) with respect to v, we get

$$\begin{aligned}\frac{\partial}{\partial v}\big(T(u)\big)&=\sum_{j=0}^{\infty}(-1)^j\binom{n-m-1}{j}\big(q(u,v)\big)^{j+m-1}\times\frac{\partial}{\partial v}\big(q(u,v)\big)\\&=t(u)\left(\frac{\partial}{\partial u}\big(q(u,v)\big)\right)^{-1}\left(\frac{\partial}{\partial v}\big(q(u,v)\big)\right)=0.\end{aligned} \tag{6}$$

Now we know $t(u)\neq 0$ and $\frac{\partial}{\partial u}q(u,v)\neq 0$, so we must have

$$\frac{\partial}{\partial v}\big(q(u,v)\big)=0. \tag{7}$$

Hence $q(u,v)$ is a function of u only.

This completes the proof. □

Theorem 2.2. *Let $\{X_n, n\geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α,β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x)\to 0$ as $x\to 0$, $g(x)\to-\infty$ as $x\to\alpha^+$ and $g(x)\to\infty$ as $x\to\beta^-$ for all $x\in(\alpha,\beta)$. Then $F(x)=exp\big(-e^{-g(x)}\big)$ for all $-\infty<g(x)<\infty$, if and only if $g(X_{L(m)}^{(k)})-g(X_{L(n)}^{(k)})$ and $g(X_{L(n)}^{(k)})$ are independent for $1\leq m<n$.*

Proof. If $F(x)=exp\big(-e^{-g(x)}\big)$, then it is easy to see that $g(X_{L(m)}^{(k)})-g(X_{L(n)}^{(k)})$ and $g(X_{L(n)}^{(k)})$ are independent for $1\leq m<n$. So we have to prove the converse.

Let us use the transformation $U=g(X_{L(m)}^{(k)})-g(X_{L(n)}^{(k)})$ and $V=g(X_{L(n)}^{(k)})$. The Jacobian of the transformation is $J=\frac{\partial}{\partial u}\big(g^{-1}(u)\big)\cdot\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big)$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}\big(g^{-1}(u)\big)$ and $\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big)$ are positive. Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$\begin{aligned} f_{U,V}(u,v) =& \frac{k^n}{\Gamma(m)\Gamma(n-m)} \big(H(g^{-1}(u+v))\big)^{m-1} h\big(g^{-1}(u+v)\big) f\big(g^{-1}(v)\big) \\ &\cdot [F\big(g^{-1}(v)\big)]^{k-1} \big(H(g^{-1}(v)) - H(g^{-1}(u+v))\big)^{n-m-1} \\ &\cdot \frac{\partial}{\partial v}(g^{-1}(v)) \frac{\partial}{\partial u}(g^{-1}(u+v)) \end{aligned} \tag{8}$$

for all $-\infty < v < \infty$ and $u > 0$, where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

The pdf $f_V(v)$ of V is given by

$$f_V(v) = \frac{k^n H\big(g^{-1}(v)\big)^{n-1}}{\Gamma(n)} f(g^{-1}(v)) \frac{\partial}{\partial v}(g^{-1}(v))[F\big(g^{-1}(v)\big)]^{k-1} \tag{9}$$

for all $-\infty < v < \infty$ and $m \geq 1$.

From (8) and (9), we can get the conditional pdf of $f_U(u|v)$ as

$$\begin{aligned} f_U(u|v) =& c \cdot \frac{\big(H(g^{-1}(u+v))\big)^{m-1} \big(H(g^{-1}(v)) - H(g^{-1}(u+v))\big)^{n-m-1}}{H\big(g^{-1}(v)\big)^{n-1}} \\ &\cdot h\big(g^{-1}(u+v)\big) \frac{\partial}{\partial u}(g^{-1}(u+v)) \end{aligned} \tag{10}$$

for all $-\infty < v < \infty$ and $u > 0$, where $c = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)}$.

Now, simplifying (10), we have

$$\begin{aligned} f_U(u|v) =& c \cdot \left(\frac{H\big(g^{-1}(u+v)\big)}{H\big(g^{-1}(v)\big)} \right)^{m-1} \left(1 - \frac{H\big(g^{-1}(u+v)\big)}{H\big(g^{-1}(v)\big)} \right)^{n-m-1} \\ &\cdot \frac{\partial}{\partial u} \left(\frac{H\big(g^{-1}(u+v)\big)}{H\big(g^{-1}(v)\big)} \right) \end{aligned} \tag{11}$$

for all $-\infty < v < \infty$ and $u > 0$, where $c = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)}$. Since U and V are independent and using Lemma 2.1, we see that

$$q(u,v) = \frac{H\big(g^{-1}(u+v)\big)}{H\big(g^{-1}(v)\big)}$$

is a function of u only. Thus we get

$$\frac{H\big(g^{-1}(u+v)\big)}{H\big(g^{-1}(v)\big)} = T(u) \tag{12}$$

where $T(u)$ is a function of u only. Letting $v \to 0$, so $g^{-1}(v) \to 0$ and $H\big(g^{-1}(v)\big) \to 1$, we get $T(u) = H\big(g^{-1}(u)\big)$ from $H(0) = 1$. Then, we obtain

$$H\big(g^{-1}(u+v)\big) = H\big(g^{-1}(u)\big) H\big(g^{-1}(v)\big) \tag{13}$$

for all $u > 0$ and $-\infty < v < \infty$.

By the theory of functional equation (see Aczel [1]), the only continuous solution of (13) with boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$ is

$$F(x) = exp\big(- e^{-g(x)}\big) \tag{14}$$

for all $-\infty < g(x) < \infty$.

This completes the proof. □

Applying with appropriate choice of $g(x)$ in Theorem 2.2, we have the following remark 2.1.

Remark 2.1. If we set $g(x) = x, -\infty < x < \infty$ and $k = 1$, then independent of $X_{L(m)} - X_{L(n)}$ and $X_{L(n)}$ characterizes Gumbel distribution in Jin and Lee [4].

3. Characterization of distributions $F(x) = e^{g(x)}$

For proving our Theorem 3.2, we need the following Lemma 3.1.

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to -\infty$ as $x \to \alpha^+$ and $g(x) \to 0$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Suppose that $\left(\frac{F(g^{-1}(u+v))}{F(g^{-1}(u))}\right)^k = e^{-k\cdot q(u,v)}$ and $h(u,v) = \big(k\cdot q(u,v)\big)^r e^{-k\cdot q(u,v)} \times \big(\frac{\partial}{\partial v} k\cdot q(u,v)\big)$, for $r \geq 0$ and $k \geq 1$, where $h(u,v) \neq 0$ and $\frac{\partial}{\partial v} k \cdot q(u,v) \neq 0$ for any positive u and v. If $h(u,v)$ is independent of u, then $k \cdot q(u,v)$ is a function of v only.*

Proof. Let

$$\begin{aligned} h(u,v) &= \big(k \cdot q(u,v)\big)^r e^{-k\cdot q(u,v)} \big(\frac{\partial}{\partial v} k \cdot q(u,v)\big) \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \big(k \cdot q(u,v)\big)^{j+r} \big(\frac{\partial}{\partial v} k \cdot q(u,v)\big). \end{aligned} \tag{15}$$

Since $h(u,v)$ is independent of u, we obtain

$$h(u,v) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \big(k \cdot q(u,v)\big)^{j+r} \big(\frac{\partial}{\partial v} k \cdot q(u,v)\big) = t(v). \tag{16}$$

Integrating (16) with respect to v, we get

$$\begin{aligned}\int h(u,v)dv &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{1}{(j+r+1)}\big(k\cdot q(u,v)\big)^{j+r+1}\\ &= \int t(v)dv + c = T(v).\end{aligned}\tag{17}$$

Here T is a function of v only and c is independent of v but may depend on u.

Now taking $v \to 0$, $k \cdot q(u,v) \to 0$, we have c independently of u from (17). Differentiating (17) with respect to u, we get

$$\begin{aligned}\frac{\partial}{\partial u}T(v) &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\big(k\cdot q(u,v)\big)^{j+r}\big(\frac{\partial}{\partial u}k\cdot q(u,v)\big)\\ &= t(v)\big(\frac{\partial}{\partial v}k\cdot q(u,v)\big)^{-1}\big(\frac{\partial}{\partial u}k\cdot q(u,v)\big) = 0.\end{aligned}\tag{18}$$

Now we know $t(v) \neq 0$ and $\frac{\partial}{\partial v}k \cdot q(u,v) \neq 0$, so we must have

$$\frac{\partial}{\partial u}k\cdot q(u,v) = 0.\tag{19}$$

Hence $k \cdot q(u,v)$ is a function of v only.

This completes the proof. □

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α,β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to -\infty$ as $x \to \alpha^+$ and $g(x) \to 0$ as $x \to \beta^-$ for all $x \in (\alpha,\beta)$. Then $F(x) = e^{g(x)}$ for all $-\infty < g(x) < 0$, if and only if $g(X_{L(n)}^{(k)}) - g(X_{L(m)}^{(k)})$ and $g(X_{L(m)}^{(k)})$ are independent for $1 \leq m < n$.*

Proof. If $F(x) = e^{g(x)}$, then it is easy to see that $g(X_{L(n)}^{(k)}) - g(X_{L(m)}^{(k)})$ and $g(X_{L(m)}^{(k)})$ are independent for $1 \leq m < n$.

Let us use the transformation $U = g(X_{L(m)}^{(k)})$ and $V = g(X_{L(n)}^{(k)}) - g(X_{L(m)}^{(k)})$. The Jacobian of the transformation is $J = \frac{\partial}{\partial u}\big(g^{-1}(u)\big)\cdot\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big)$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}\big(g^{-1}(u)\big)$ and $\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big)$ are positive. Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$\begin{aligned}f_{U,V}(u,v) =&\frac{k^n}{\Gamma(m)\Gamma(n-m)}\big(H(g^{-1}(u))\big)^{m-1}h\big(g^{-1}(u)\big)f\big(g^{-1}(u+v)\big)\\ &\cdot[F\big(g^{-1}(u+v)\big)]^{k-1}\big(H(g^{-1}(u+v)) - H(g^{-1}(u))\big)^{n-m-1}\\ &\cdot\frac{\partial}{\partial u}\big(g^{-1}(u)\big)\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big)\end{aligned}\tag{20}$$

for all $-\infty < u < 0$ and $v < 0$, where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{k^m H\big(g^{-1}(u)\big)^{m-1}}{\Gamma(m)} f\big(g^{-1}(u)\big)\frac{\partial}{\partial u}\big(g^{-1}(u)\big)[F\big(g^{-1}(u)\big)]^{k-1} \tag{21}$$

for all $-\infty < u < 0$ and $m \geq 1$.

From (20) and (21), we can get the conditional pdf of $f_V(v|u)$ as

$$\begin{aligned} f_V(v|u) &= \frac{k^{n-m}\big(H(g^{-1}(u+v)) - H(g^{-1}(u))\big)^{n-m-1}}{\Gamma(n-m)[F\big(g^{-1}(u)\big)]^k[F\big(g^{-1}(u+v)\big)]^{1-k}} \\ &\cdot f\big(g^{-1}(u+v)\big)\frac{\partial}{\partial v}\big(g^{-1}(u+v)\big) \\ &= c\Big(-k\cdot ln\frac{F\big(g^{-1}(u+v)\big)}{F\big(g^{-1}(u)\big)}\Big)^{n-m-1}\Big(\frac{F\big(g^{-1}(u+v)\big)}{F\big(g^{-1}(u)\big)}\Big)^k \\ &\cdot\Big(-\frac{\partial}{\partial v}\big(-k\cdot ln\frac{F\big(g^{-1}(u+v)\big)}{F\big(g^{-1}(u)\big)}\big)\Big), \end{aligned} \tag{22}$$

for all $-\infty < u < 0$ and $v < 0$, where $c = \frac{1}{\Gamma(n-m)}$.

Since U and V are independent and using Lemma 3.1, we see that $k\cdot q(u,v) = -ln\Big(\frac{F\big(g^{-1}(u+v)\big)}{F\big(g^{-1}(u)\big)}\Big)^k$ is a function of v only. Thus

$$\Big(\frac{F\big(g^{-1}(u+v)\big)}{F\big(g^{-1}(u)\big)}\Big)^k = T(v) \tag{23}$$

where $T(v)$ is a function of v only. Letting $u \to 0^-$, so $g^{-1}(u) \to \beta$ and $F(g^{-1}(u)) \to 1$, we get $T(v) = \big(F(g^{-1}(v))\big)^k$ from $F(\beta) = 1$. Then, we get

$$\big(F(g^{-1}(u+v))\big)^k = \big(F(g^{-1}(u))\big)^k\big(F(g^{-1}(v))\big)^k \tag{24}$$

for all $-\infty < u < 0$ and $v < 0$.

The equation (24) is equivalent to

$$F\big(g^{-1}(u+v)\big) = F\big(g^{-1}(u)\big)F\big(g^{-1}(v)\big) \tag{25}$$

for all $-\infty < u < 0$ and $v < 0$. □

Applying with appropriate choice of $g(x)$ in Theorem 3.2, we have the following remarks.

Remark 3.1. If we set $g(x) = -e^{-x}, -\infty < x < \infty$ and $k = 1$, then independent of $e^{-X_{L(n)}} - e^{-X_{L(m)}}$ and $e^{-X_{L(m)}}$ characterizes Gumbel distribution in Skřivánková and Juhás [8].

Remark 3.2. If we set $g(x) = -x^{-\alpha}, 0 < x < \infty$ and $k = 1$, then independent of $(X_{L(n)})^{-\alpha} - (X_{L(m)})^{-\alpha}$ and $(X_{L(m)})^{-\alpha}$ characterizes Fréchet distribution in Skřivánková and Juhás [8].

Remark 3.3. With suitable choice of $g(x)$, various distributions may be characterized as given in Table 1.

Table 1. Examples based on the distribution function $F(x) = e^{g(x)}$

Distribution	**g(x)**	**F(x)**
Burr type II	$ln\big(1+e^{-x}\big)^{-\theta}$	$(1+e^{-x})^{-\theta},\ -\infty < x < \infty,\ \theta > 0$
Burr type III	$ln\big(1+x^{-\theta}\big)^{-\theta}$	$(1+x^{-\theta})^{-\lambda},\ 0 < x < \infty,\ \theta > 0$
Pareto	$ln\big(1-x^{-\theta}\big)$	$1-x^{-\theta},\ 1 < x < \infty,\ \theta > 0$
power	$ln\big(x^{\theta}\big)$	$x^{\theta},\ 0 < x < \infty,\ \theta > 0$
Weibull	$ln\big(1-e^{-x^{\theta}}\big)$	$1-e^{-x^{\theta}},\ 0 < x < \infty,\ \theta > 0$

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

Acknowledgments : The author would like to thank the referees for their valuable comments which helped to improve the manuscript.

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