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# RESULTS IN $b$-METRIC SPACES ENDOWED WITH THE GRAPH AND APPLICATION TO DIFFERENTIAL EQUATIONS

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Abstract. In this research, under some specific situations, we precisely derive new coupled fixed point theorems in a complete b-metric space endowed with the graph. We also use the concept of coupled fixed points to ensure the solution of differential equations for the system of impulse effects.

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## 1. Introduction

Fixed point analysis is most useful tools in applied sciences. It is also applicable to show the existence of the solution of differential or integral equations. Bhaskar and Lakshmikantham [4] used coupled fixed point to show the existence of solution to differential equations. This result motivates many scholars to this subject. Graphs have been used by some authors in recent years to develop new varieties of fixed point theory. Jachymski's paper [13] is one of the best research on fixed point with graphs. For the more detail study in this emerging field, we can go through the papers [1, 2, 5, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18]. Alfuraidan and Khamsi [4] recently employed some coupled fixed point results in the directed graph.

**Definition 1.1.** [3] A map $\sigma : P \times P \to R^+$ on the set P is such that

**(i):** $\sigma(\mu,\nu) = 0$ if and only if $\mu = \nu$,
**(ii):** $\sigma(\mu,\nu) = \sigma(\nu,\mu)$,
**(iii):** $\sigma(\mu,\nu) \leq j[\sigma(\mu,w) + \sigma(w,\nu)]$ for all $\mu,\nu,w \in P$,

where $j \geq 1$, then $(P,\sigma)$ be a b-metric space.

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On P, consider $\Delta = \{(r,r) : r \in P\}$ and on the directed graph $G = (V(G), E(G))$, consider that all loops are in E(G) and G has no parallel edges. A sequence $\{t_i\}_{i=0}^{r}$ in V(G) with $t_0 = t, t_r = \mu$ and $(t_{i-1}, t_i) \in E(G)$ for all $i = 1, 2, ..., r$ is called a path from the vertex t to the vertex $\mu$.
For the vertex $\mu$, let $[\mu]_G = \{t \in P : there\ exists\ path\ from\ \mu\ to\ t\}$.
If every two vertices of G can be connected by a path, then G is called connected, i.e., $V(G) = [\mu]_G\ for\ all\ \mu \in P$.
By reversing the direction of each edge of the directed graph G, we obtained a directed graph, which is denoted by $G^{-1}$ with $V(G^{-1}) = V(G)$.
We get the undirected graph $\tilde{G}$ by neglecting the directions of the edges in the directed graph G with $V(\tilde{G}) = V(G)$, also

$$E(\tilde{G}) = E(G^{-1}) \cup E(G).$$

In this paper, the term $(P, \sigma)$ refers to a b-metric space with a directed graph G such that $V(G) = P$ and $\Delta \subseteq E(G)$. Additionally, we consider the product space $P \times P$ with a different graph defined by G, so that

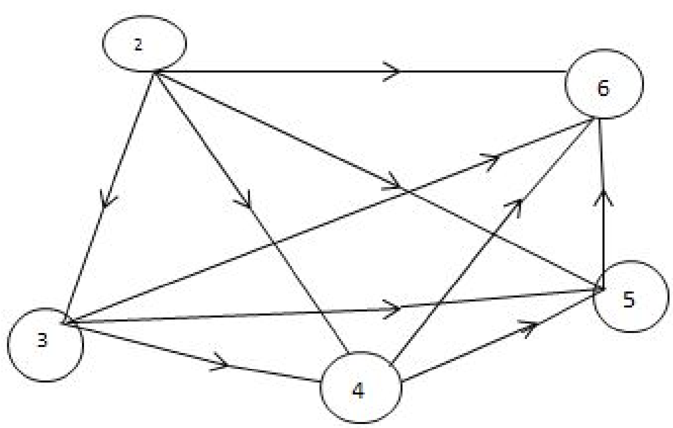
$$(\mu, \imath), (\nu, t) \in E(G) \quad if\ and\ only\ if\ (\mu, \nu) \in E(G)\ and\ (\imath, t) \in E(G)$$

for $(\mu, \nu), (\imath, t) \in P \times P$.

**Example 1.2.** Let $P = \{2, 3, 4, 5, 6\}$ be a set with the $\sigma : P \times P \to R^+$ defined by

$$\sigma(\mu, \nu) = \begin{cases} 1, & \mu\ or\ \nu \notin \{2,5\}\ and\ \mu \neq \nu, \\ 5, & \mu, \nu \in \{2,5\}\ and\ \mu \neq \nu, \\ 0, & \mu = \nu. \end{cases}$$

It is easy to show that $(P, \sigma)$ is a b metric space with graph for the coefficient $j = \frac{5}{2} > 1$.
The graph $G = (V(G), E(G))$ equipped with $V(G) = P$ and E(G) is represented by:



**Definition 1.3.** [11] A point $(\mu, \imath) \in P \times P$ is known as coupled fixed point of S if

$$S(\mu, \imath) = \mu \textit{ and } S(\imath, \mu) = \imath.$$

**Definition 1.4.** [4] Let the complete metric space $(P, \sigma)$ endowed with the direct graph G. The mapping $S : P \times P \to P$ has the mixed G-monotone condition if

$$(\mu_1, \mu_2) \in E(G) \Rightarrow (S(\mu_1, \imath), S(\mu_2, \imath)) \in E(G),$$

for each $\mu_1, \mu_2, \imath \in P$ and

$$(\nu_1, \nu_2) \in E(G) \Rightarrow (S(\imath, \nu_2), S(\imath, \nu_1)) \in E(G),$$

for each $\nu_1, \nu_2, \imath \in P$.

In this research, we use the concept of coupled fixed points on a complete b-metric space endowed with a graph. Also give sufficient conditions to ensure the solution of differential equations for the system with impulse effects. The result given here is an extension of Chandok et.al [11].

## 2. Main results

Consider, $(P, \sigma, G)$ represents the complete b-metric space endowed with directed graph G. Also, $S : P \times P \to P$ has mixed G-monotone property.

**Theorem 2.1.** *On $(P, \sigma, G)$, assume $S$ is continuous. Let, there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with*

$$\sum_{i=0}^{+\infty} s^i \left(\frac{(\beta + \gamma + \delta)}{(1 - \alpha - \gamma - \delta)}\right)^i < +\infty$$

*such that*

$$\begin{aligned} \sigma(S(\mu, \imath), S(l, v)) \leq \alpha & \frac{\sigma(\mu, S(\mu, \imath))\sigma(l, S(l, v)}{\sigma(l, v)} \\ & + \beta\sigma(\mu, l) + \gamma[\sigma(\mu, S(\mu, \imath)) + \sigma(l, S(l, v))] \\ & + \delta[\sigma(\mu, S(l, v)) + \sigma(l, S(\mu, \imath))] \end{aligned} \tag{1}$$

*satisfies for all $(\mu, \imath), (l, v) \in P \times P$ with $((\mu, \imath), (l, v)) \in E(G)$. If there exists $\mu_0, \imath_0 \in P$ such that $((\mu_0, \imath_0), (S(\mu_0, \imath_0), S(\imath_0, \mu_0))) \in E(G)$, then $S$ admits a coupled fixed point $(\mu^*, \imath^*) \in P \times P$.*

*Proof.* Let $\mu_1 = S(\mu_0, \imath_0)$ and $\imath_1 = S(\imath_0, \mu_0)$. Then we get

$$((\mu_0, \imath_0), (\mu_1, \imath_1) \in E(G)).$$

Hence,

$$\begin{aligned} \sigma(\mu_2, \mu_1) &= \sigma(S(\mu_1, \imath_1), S(\mu_0, \imath_0)) \\ &\leq \alpha \frac{\sigma(\mu_1, S(\mu_1, \imath_1))\sigma(\mu_0, S(\mu_0, \imath_0))}{\sigma(\mu_1, \mu_0)} \\ &+ \beta\sigma(\mu_1, \mu_0) + \gamma[\sigma(\mu_1, S(\mu_1, \imath_1)) + \sigma(\mu_0, S(\mu_0, \imath_0))] \end{aligned}$$

$$
\begin{aligned}
& + \quad \delta[\sigma(\mu_1, S(\mu_0, \imath_0)) + \sigma(\mu_0, S(\mu_1, \imath_1))] \\
\sigma(\mu_2, \mu_1) \quad & \leq \quad \alpha \frac{\sigma(\mu_1, \mu_2)\sigma(\mu_0, \mu_1)}{\sigma(\mu_1, \mu_0)} \\
& + \quad \beta\sigma(\mu_1, \mu_0) + \gamma[\sigma(\mu_1, \mu_2) + \sigma(\mu_0, \mu_1)] \\
& + \quad \delta[\sigma(\mu_1, \mu_1) + \sigma(\mu_0, \mu_2)].
\end{aligned}
$$

So,

$$\sigma(\mu_2, \mu_1) \quad \leq \quad \frac{\beta + \gamma + \delta}{1 - \alpha - \gamma - \delta} \sigma(\mu_1, \mu_0).$$

Likewise $((\imath_0, \mu_0), (\imath_1, \mu_1) \in E(G))$, so

$$\sigma(\imath_2, \imath_1) \quad \leq \quad \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \gamma - \delta)} \sigma(\imath_1, \imath_0).$$

For $n = 1, 2, ...,$ we consider

$$\mu_{n+1} = S(\mu_n, \imath_n) \quad and \quad \imath_{n+1} = S(\imath_n, \mu_n).$$

Considering that S has the mixed G-monotone property on P, we have

$$((\mu_n, \imath_n), (\mu_{n+1}, \imath_{n+1})) \in E(G) \ and \ \ ((\imath_{n+1}, \mu_{n+1}), (\imath_n, \mu_n)) \in E(G).$$

Thereafter,

$$\sigma(\mu_{n+1}, \mu_n) \quad \leq \quad \frac{\beta + \gamma + \delta}{1 - \alpha - \gamma - \delta} \sigma(\mu_n, \mu_{n-1})$$

and

$$\sigma(\imath_{n+1}, \imath_n) \quad \leq \quad \frac{\beta + \gamma + \delta}{1 - \alpha - \gamma - \delta} \sigma(\imath_n, \imath_{n-1}).$$

Hence, for $n \in N$

$$\sigma(\mu_{n+1}, \mu_n) \quad \leq \quad (\frac{\beta + \gamma + \delta}{1 - \alpha - \gamma - \delta})^n \ \sigma(\mu_n, \mu_{n-1})$$

and

$$\sigma(\imath_{n+1}, \imath_n) \quad \leq \quad (\frac{\beta + \gamma + \delta}{(1 - \alpha - \gamma - \delta)})^n \ \sigma(\imath_n, \imath_{n-1}).$$

For $n \in N$ and $q \in N$, we have

$$
\begin{aligned}
\sigma(\mu_n, \mu_{n+q}) \quad & \leq \quad s\sigma(\mu_n, \mu_{n+1}) + s^2\sigma(\mu_{n+1}, \mu_{n+2}) + \cdots + s^n\sigma(\mu_n + q - 1, \mu_{n+q}) \\
& = \quad \frac{1}{s^{n-1}} \sum_{r=n}^{n+q-1} s^r \sigma(\mu_r, \mu_{r+1}) \\
& \leq \quad \frac{1}{s^{n-1}} \sum_{r=n}^{n+q-1} s^r (\frac{\beta + \gamma + \delta}{(1 - \alpha - \gamma - \delta)})^r \sigma(\mu_0, \mu_1).
\end{aligned}
$$

By assumptions, we get

$$\lim_{n \to +\infty} \sigma(\mu_n, \mu_{n+q}) = 0.$$

By repeating process above, we get

$$\sigma(\imath_n, \imath_{n+q}) \quad \leq \quad \frac{1}{s^{n-1}} \sum_{r=n}^{n+q-1} s^r \left(\frac{\beta+\gamma+\delta}{(1-\alpha-\gamma-\delta)}\right)^r \sigma(\imath_0, \imath_1),$$

afterwards

$$\lim_{n\to+\infty} \sigma(\imath_n, \imath_{n+q}) = 0.$$

This gives $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\imath_n\}_{n=1}^{+\infty}$ are Cauchy. Therefore, by completeness of P there exists $\mu^*, \imath^* \in P$ such that

$$\lim_{n\to+\infty} \mu_n = \mu^* \ \textit{and} \ \lim_{n\to+\infty} \imath_n = \imath^*.$$

So, by the continuity of S, we have

$$\mu^* = \lim_{n\to+\infty} \mu_n = \lim_{n\to+\infty} S(\mu_{n-1}, \imath_{n-1}) = S(\lim_{n\to+\infty} \mu_{n-1}, \lim_{n\to+\infty} \imath_{n-1}) = S(\mu^*, \imath^*),$$

$$\imath^* = \lim_{n\to+\infty} \imath_n = \lim_{n\to+\infty} S(\imath_{n-1}, \mu_{n-1}) = S(\lim_{n\to+\infty} \imath_{n-1}, \lim_{n\to+\infty} \mu_{n-1}) = S(\imath^*, \mu^*),$$

i.e., S admits $(\mu^*, \imath^*)$ as a coupled fixed point. □

By introducing following conditions, the continuity of S in Theorem 2.1, can be removed. Consider $(P, \sigma, G)$ admits property $(\Omega)$; i.e.,

- for any $\{\mu_n\}_{n=1}^{+\infty}$ in P such that $(\mu_n, \mu_{n+1}) \in E(G)$ and $\lim_{n\to+\infty} \mu_n = \mu$ then $(\mu_n, \mu) \in E(G)$,
- for any $\{\mu_n\}_{n=1}^{+\infty}$ in P such that $(\mu_{n+1}, \mu_n) \in E(G)$ and $\lim_{n\to+\infty} \mu_n = \mu$ then $(\mu, \mu_n) \in E(G)$.

**Theorem 2.2.** *consider $(P, \sigma, G)$ with the condition $(\Omega)$, let there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ with*

$$\sum_{i=0}^{+\infty} s^i \left(\frac{\beta+\gamma+\delta}{1-\alpha-\gamma-\delta}\right)^i < +\infty$$

*such that*

$$\begin{aligned} \sigma(S(\mu,\imath), S(l,v)) \leq \alpha & \frac{\sigma(\mu, S(\mu,\imath))\sigma(l, S(l,v)}{\sigma(l,v)} \\ & + \beta\sigma(\mu,l) + \gamma[\sigma(\mu, S(\mu,\imath)) + \sigma(l, S(l,v))] \\ & + \delta[\sigma(\mu, S(l,v)) + \sigma(l, S(\mu,\imath))], \end{aligned} \tag{2}$$

*satisfies for all $(\mu,\imath),(l,v) \in P \times P$ with $((\mu,\imath),(l,v)) \in E(G)$. If there exists $\mu_0, \imath_0 \in P$ such that $((\mu_0,\imath_0),(S(\mu_0,\imath_0), S(\imath_0,\mu_0))) \in E(G)$, then $S$ admits a coupled fixed point $(\mu^*, \imath^*) \in P \times P$.*

*Proof.* Due to the proof of Theorem 2.1, we prove that only $\mu^* = S(\mu^*, \imath^*)$ and $\imath^* = S(\imath^*, \mu^*)$. Consequently,

$$\lim_{n\to+\infty} \mu_{n+1} = \lim_{n\to+\infty} S(\mu_n, \imath_n) = \mu^*, \ \lim_{n\to+\infty} \imath_{n+1} = \lim_{n\to+\infty} S(\imath_n, \mu_n) = \imath^*$$

and $(\mu_n, \mu_{n+1}) \in E(G)$ and $(\imath_n, \imath_{n+1}) \in E(G)$, the condition $(\Omega)$ implies that

$$(\mu_n, \mu^*) \in E(G) \quad and \quad (\imath^*, \imath_n) \in E(G).$$

So, we have

$$\begin{aligned}\sigma(S(\mu_n, \imath_n), S(\mu^*, \imath^*)) \leq \alpha & \frac{\sigma(\mu_n, S(\mu_n, \imath_n))\sigma(\mu^*, S(\mu^*, \imath^*)}{\sigma(\mu_n, \mu^*)} \\ & + \beta\sigma(\mu_n, \mu^*) + \gamma[\sigma(\mu_n, S(\mu_n, \imath_n)) + \sigma(\mu^*, S(\mu^*, \imath^*))] \\ & + \delta[\sigma(\mu_n, S(\mu^*, \imath^*)) + \sigma(\mu^*, S(\mu_n, \imath_n)).\end{aligned}$$

Likewise, we have

$$\begin{aligned}\sigma(S(\imath_n, \mu_n), S(\imath^*, \mu^*)) \leq \alpha & \frac{\sigma(\imath_n, S(\imath_n, \mu_n))\sigma(\imath^*, S(\imath^*, \mu^*)}{\sigma(\imath_n, \imath^*)} \\ & + \beta\sigma(\imath_n, \imath^*) + \gamma[\sigma(\imath_n, S(\imath_n, \mu_n)) + \sigma(\imath^*, S(\imath^*, \mu^*))] \\ & + \delta[\sigma(\imath_n, S(\imath^*, \mu^*)) + \sigma(\imath^*, S(\imath_n, \mu_n))].\end{aligned}$$

Taking $n \to +\infty$, we get

$$\lim_{n\to+\infty} \sigma(S(\mu_n, \imath_n), S(\mu^*, \imath^*)) = 0, \ and \ \lim_{n\to+\infty} \sigma(S(\imath_n, \mu_n), S(\imath^*, \mu^*)) = 0.$$

Hence,

$$\lim_{n\to+\infty} \mu_{n+1} = S(\mu^*, \imath^*)), \ and \ \lim_{n\to+\infty} \imath_{n+1} = S(\imath^*, \mu^*).$$

So, $\mu^* = S(\mu^*, \imath^*))$ and $\imath^* = S(\imath^*, \mu^*)$ i.e., S admits $(\imath^*, \mu^*)$ as a coupled fixed point. □

## 3. Application

Let us take the following system of impulse-effect differential equations:

$$\vartheta'(\eta) = g(\eta, \vartheta(\eta), z(\eta)), \qquad z'(\eta) = g(\eta, z(\eta), \vartheta(\eta)), \tag{3}$$

$$\vartheta(\eta^+) - \vartheta(\eta^-) = I(\vartheta(\eta), z(\eta)), \quad z(\eta^+) - z(\eta^-) = I(z(\eta), \vartheta(\eta)), \tag{4}$$

$$\vartheta(0) = \vartheta_0, \quad z(0) = z_0, \tag{5}$$

where $0 < \eta < 1, K = [0, 1], \ g : K \times \Re \times \Re \to \Re, \ I \in C(\Re \times \Re, \Re)$
and the symbols $\vartheta(\eta^+) = \lim_{h\to0^+} \vartheta(\eta + h)$ and $\vartheta(\eta^-) = \lim_{h\to0^+} \vartheta(\eta - h)$.
To determine a solution for problem (3)-(5), assume a set of piecewise continuous functions:
$PC([0, 1], \Re) = \{z : K \to R, z \in C(K/\{\eta\}, \Re)$; such that $z(\eta^+)$ and $z(\eta^-)$ exist and satisfy $z(\eta^-) = z(\eta)\}$.
Define $\sigma$ on $PC([0, 1])$ by

$$\sigma(\vartheta, z) = (\sup_{t\in K} |\vartheta(t) - z(t)|)^2.$$

Let the subsequent conditions fulfill:
(a) $g : K \times \Re \times \Re \to \Re$ is continuous,
(b) for all $\vartheta, z, \mu, \nu \in PC([0, 1])$, with $\vartheta \leq \mu$ and $z \leq \nu$, we have

$$g(t, \vartheta(t), z(t)) \leq g(t, \mu(t), \nu(t)) \ and \ I(\vartheta(t), z(t)) \leq g(\mu(t), \nu(t)) \ for \ all \ t \in [0, 1];$$

(c) there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with

$$\sum_{i=0}^{+\infty} 2^i (\frac{\beta+\gamma+\delta}{1-\alpha-\gamma-\delta})^i < +\infty \quad \textit{such that}$$

$$\begin{aligned}
& |g(t,\vartheta(t),z(t)) \leq g(t,\mu(t),\nu(t))|^2 \\
\leq \quad & \frac{\alpha}{2}\frac{(|\vartheta(t)-g(t,\vartheta(t),z(t))|)^2.(|\mu(t)-g(t,\mu(t),\nu(t))|)^2}{(|\vartheta(t)-\mu(t)|^2)} \\
+ \quad & \frac{\gamma}{2}((|\vartheta(t)-g(t,\vartheta(t),z(t))|)^2+(|\mu(t)-g(t,\mu(t),\nu(t))|)^2) \\
+ \quad & \frac{\delta}{2}((|\vartheta(t)-g(t,\mu(t),\nu(t))|)^2+(|\mu(t)-g(t,\vartheta(t),z(t))|)^2)
\end{aligned}$$

and

$$|I(\vartheta(t),z(t))-I(\mu(t),\nu(t))|^2 \leq \frac{\beta}{2}(|\vartheta(t)-\mu(t)|^2)$$

for all $\vartheta, z, \mu, \nu \in PC([0, 1])$ with $\vartheta \leq \mu$ and $\nu \leq z$.
Now, we find unique solution of problems (3)-(5). These problems are equivalent to the system:

$$\begin{cases} \vartheta(t) = \vartheta_0 + \int_0^t g(r,\vartheta(r),z(r))dr + I(\vartheta(\eta),z(\eta)), & t \in K \\ z(t) = z_0 + \int_0^t g(r,z(r),\vartheta(r))dr + I(z(\eta),\vartheta(\eta)), & t \in K. \end{cases} \tag{6}$$

Define for $t \in K$, a mapping $S : PC(K,\Re) \times PC(K,\Re) \to PC(K,\Re)$ such that

$$S(\vartheta(t),z(t)) = \vartheta_0 + \int_0^t g(r,\vartheta(r),z(r))dr + I(\vartheta(\eta),z(\eta)),\ t \in K,$$

**Theorem 3.1.** *Consider that the assumptions(a)-(c) hold. Let there exists* $(\mu_0,\nu_0) \in PC(K,\Re) \times PC(K,\Re)$ *such that*

$$\mu_0(t) \leq \mu_0 + \int_0^t g(r,\mu_0(r),\nu_0(r))dr + I(\mu(\eta),\nu(\eta)),$$

$$\nu_0(t) \geq \nu_0 + \int_0^t g(r,\nu_0(r),\mu_0(r))dr + I(\nu(\eta),\mu(\eta)) forall \quad t \in K,$$

*consequently, the problems (3)-(5) have a solution.*

*Proof.* We show that the system (6) possesses a solution by proving that the mapping $S$ has a coupled fixed point. To achieve this, we must prove that S complies conditions of theorem 2.1 or theorem 2.2.
Assume the Graph G with $V(G) = PC([0,1],\Re) \times PC([0,1],\Re)$, *and*

$$E(G) = \{(\vartheta,z) \in PC([0,1],\Re) \times PC([0,1],\Re), \vartheta \leq z\},$$

and consider the product space $PC([0,1],\Re) \times PC([0,1],\Re)$ with

$$((\vartheta,z),(\mu,\nu)) \in E(G) \Leftrightarrow (\vartheta,\mu) \in E(G) \ and \ (\nu,z) \in E(G),$$

for any $(\vartheta, z), (\mu, \nu) \in PC([0,1], \Re) \times PC([0,1], \Re)$.
Now, for each $\vartheta, z, \vartheta_1, \vartheta_2, z_1, z_2 \in PC([0,1], \Re)$ and $(\vartheta_1, \vartheta_2) \in E(G)$, we have

$$\begin{aligned} S(\vartheta_1, z)(t) &= \vartheta(t)_0 + \int_0^t g(r, \vartheta_1(r), z(r))dr + I(\vartheta_1(\eta), z(\eta)), \\ &\leq \vartheta(t)_0 + \int_0^t g(r, \vartheta_2(r), z(r))dr + I(\vartheta_2(\eta), z(\eta)), \\ &= S(\vartheta_2, z)(t). \end{aligned}$$

Thus, $(S(\vartheta_1, z), S(\vartheta_1, z)) \in E(G)$. Also, if $(z_1, z_2) \in E(G)$ we have

$$\begin{aligned} S(\vartheta, z_2)(t) &= \vartheta(t)_0 + \int_0^t g(r, \vartheta(r), z_2(r))dr + I(\vartheta_1(\eta), z_2(\eta)), \\ &\leq \vartheta(t)_0 + \int_0^t g(r, \vartheta(r), z_1(r))dr + I(\vartheta(\eta), z_1(\eta)), \\ &= S(\vartheta, z_1)(t), \end{aligned}$$

therefore $(S(\vartheta, z_2), S(\vartheta, z_1)) \in E(G)$. Since, $S(\vartheta, z)$ admits the mixed G- monotone property and suppose $(\vartheta, z), (\mu, \nu) \in PC([0,1], \Re) \times PC([0,1], \Re)$ such that$(\vartheta, z), (\mu, \nu) \in E(G)$ then

$$\begin{aligned} & |S(\vartheta, z)(t) - S(\mu, \nu)(t)|^2 \\ = & \ |\int_0^t g(r, \vartheta(r), z(r))dr + I(\vartheta(\eta), z(\eta)) \\ - & \int_0^t g(r, \mu(r), \nu(r))dr - I(\mu(\eta), \nu(\eta))|^2 \\ \leq & \ 2\int_0^t |g(r, \vartheta(r), z(r)) - g(r, \mu(r), \nu(r))|^2 dr \\ + & \ 2|I(\vartheta(\eta), z(\eta)) - I(\mu(\eta), \nu(\eta))|^2 \\ \leq & \int_0^t \alpha \frac{(|\vartheta(r) - g(r, \vartheta(r), z(r))|)^2.(|\mu(r) - g(r, \mu(r), \nu(r))|)^2}{(|\vartheta(r) - \mu(r)|^2)} \\ + & \ \gamma((|\vartheta(r) - g(r, \vartheta(r), z(r))|)^2 + (|\mu(r) - g(r, \mu(r), \nu(r))|)^2) \\ + & \ \delta((|\vartheta(r) - g(r, \mu(r), \nu(r))|)^2 + (|\mu(t) - g(r, \vartheta(r), z(r))|)^2) \\ + & \ \beta(|\vartheta(t) - \mu(t)|^2). \end{aligned}$$

Hence,

$$\begin{aligned} \sigma(S(\vartheta, z), S(\mu, \nu)) \leq \alpha & \frac{\sigma(\vartheta, S(\vartheta, z))\sigma(\mu, S(\mu, \nu)}{\sigma(\vartheta, \mu)} \\ & + \beta\sigma(\vartheta, \mu) + \gamma[\sigma(\vartheta, S(\vartheta, z)) + \sigma(\mu, S(\mu, \nu))] \\ & + \delta[\sigma(\vartheta, S(\mu, \nu)) + \sigma(\mu, S(\vartheta, z))]. \end{aligned}$$

Now, by assumptions we can conclude that

$$((\mu_0, \nu_0), (S(\mu_0, \nu_0), S(\nu_0, \mu_0))) \in E(G).$$

Since, S is continuous map and $(P, \sigma, G)$ has the condition $(\Omega)$, which proves that all conditions of Theorem 2.1 and 2.2 are fulfill. Therefore, $S$ possesses a coupled fixed point . $\square$

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**Data availability** : Not applicable

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