J. Appl. Math. & Informatics Vol. **41***(2023), No. 4, pp. 707 - 722*
https://doi.org/10.14317/jami.2023.707

EPIMORPHISMS, DOMINIONS FOR GAMMA SEMIGROUPS AND PARTIALLY ORDERED GAMMA SEMIGROUPS

PHOOL MIYAN*, SELESHI DEMIE AND GEZEHEGN TEREFE

Abstract. The purpose of this paper is to obtain the commutativity of a gamma dominion for a commutative gamma semigroup by using Isbell zigzag theorem for gamma semigroup and we prove some gamma semigroup identities are preserved under epimorphism. Moreover, we extend epimorphism, dominion and Isbell zigzag theorem for partially ordered semigroup to partially ordered gamma semigroup.

AMS Mathematics Subject Classification : 06F05, 18A20, 20M10.
Key words and phrases : Gamma semigroup, partially ordered gamma semigroup, epimorphism, dominion, zigzag.

1. Introduction

The investigations on Γ semigroups were done by several mathematicians. For example one may see Saha [10, 11], Sen and Saha [12], Rasouli and Shabani [9] and Kwon and Lee [8]. Sen [13], defined a Γ semigroup S as follows: for any non-empty sets S and Γ. if there exists a mapping from $S \times \Gamma \times S$ to S which maps (a,α,b) to $a\ \alpha\ b$ satisfying the condition $(a\alpha b)\gamma c = a\alpha(b\gamma c)$ for all $a, b, c \in S$ and $\alpha\ b,\gamma \in \Gamma$, this concept of a Γ semigroup is generalization of a semigroup. Also, Sumanto et al, [16] defined Γ semigroup by considering Γ as a collection of associative binary operations on S

The concept of partially ordered Γ semigroup was defined by Kwon and Lee [8] as, a Γ semigroup S is said to be a partially ordered Γ semigroup if S is partially ordered set such that $a \le b$ implies $a\gamma c \le b\gamma c$ and $c\gamma a \le c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$ and also it has been studied by Kehayopulu [6, 7]. Rasouli and Shabani [9] defined a Γ semigroup homomorphism. Also, gamma semigroup homomorphism defined by Ashraf et al. [2] as: pair of mappings $f_1 : S_1 \longrightarrow S_2$ and $f_2 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2$ is said to be a homomorphism from (S_1, Γ_1) to (S_2, Γ_2) if $(s_1\ \gamma\ s_2)\ f_1 = ((s_1)\ f_1)((\gamma)\ f_2)((s_2) f_1)$ for all s_1, $s_2 \in S_1$ and $\gamma \in \Gamma_1$.

Received March 3, 2022. Revised June 3, 2022. Accepted May 4, 2023. *Corresponding Author.

Dominion and zig zag theorem for semigroups were first studied in 1965 by Isbell [5] with connection to epimorphisms and he proved that dominion of commutative semigroup is commutative. For subsemigroup U of a semigroup S, we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta,\gamma : S \longrightarrow T$, such that $u\beta = u\gamma$ for each $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the dominion of U in S and we denote it by $dom(U, S)$.

Dominion for Γ semigroups is defined in 2021 by Ashraf et al. [2]. Following his definition, we say that U dominates an element d of S if for every $\Gamma^{'}$ semigroup T and for all homomorphisms $\beta,\alpha : S \longrightarrow T$ and $\beta^{'},\alpha^{'} : \Gamma \longrightarrow \Gamma^{'}$ with $u\alpha = u\beta$ and $\gamma\ \alpha^{'} = \gamma\ \beta^{'}$ for all $u \in U \Rightarrow d\alpha = d\beta$. The set of all elements of S dominated by U is called Γ dominion of U in S and we denote it by Γ $dom(U, S)$. It can easily be verified that Γ $dom(U, S)$ is Γ subsemigroup of S containing U. The Isbell zig zag theorem for gamma semigroups was proved by Ashraf et al. [2]. In his study Isbell zig zag theorem was generalized for the class of Γ semigroups, which is used as an important tool throughout in this paper to prove dominion of commutative Γ semigroup to be commutative and also to prove some Γ semigroup identities to be preserved under epimorphisms.

Following Ashraf [1], for a Γ semigroup S, each element of S will be called variable and let $u(x_1, ..., x_n) = g_1\ (x_i\ , \gamma_j)$ and $v(y_1, y_2, ..., y_m) = g_2\ (y_k, \lambda_l)$ be two words, where $x_i\ , y_k \in S$, $\gamma_j\ , \lambda_l \in \Gamma$ and $1 \leq i \leq n, 1 \leq$ j $\leq n-1, 1 \leq k \leq m, 1 \leq l \leq m-1$. Then the pair of words $(u(x_1, ..., x_n)), v(y_1, y_2, ..., y_m))$ is called an identity and usually written as $u = v$. By permutation identity in the variables
$x_1, ..., x_n$ $(n \geq 2)$ we mean an identity $x_1\gamma_1x_2\gamma_2 ...\ x_{n-1}\gamma_{n-1}x_n = x_{i1}\gamma_{j2} \ . \ . \ . \ x_{in-1}\gamma_{jn-1})x_{in}$, where i, j are permutation on the sets $1, 2, ..., n$ and $1, 2, ..., n-1$ respectively. A Γ semigroup is called permutative if it satisfies permutation identity. The identities

$x_1\gamma_1\ x_2 = x_2\ \gamma_1x_1$	Commutativity (C)
$x_1\gamma_1x_2\gamma_2x_3 = x_1\gamma_1x_3\gamma_2x_2$	Left Normality (LN)
$x_1\gamma_1x_2\gamma_2x_3 = x_2\gamma_1x_1\gamma_2x_3$	Right Normality (RN)
$x_1\gamma_1x_2\gamma_2x_3\gamma_3x_4 = x_1\gamma_1x_3\gamma_2x_2\gamma_3x_4$	Normality(N)

are all permutation identities.

Note. Let C be the class of Γ semigroups and let $D = \{S \in C : a\gamma_1 b = c\gamma_2 d \Rightarrow \gamma_1 = \gamma_2$ for all $a, b, c, d \in S$ and for all $\gamma_1,\gamma_2 \in \Gamma\ \}$. Throughout rest of the paper, we prove the results for the class D of Γ semigroups with further mention. The semigroup theoretic notations and conventions of Clifford and Preston [3] and Howie [4] will be used with explicit mention.

We define epimorphism for gamma semigroup as follows:

Definition 1.1. A morphisms $\alpha : S \longrightarrow T$ and $\alpha^{'} : \Gamma \longrightarrow \Gamma^{'}$ in the category of all Γ semigroups is called epimorphism, if for all $\Gamma^{''}$ semigroup N and for all pairs of morphisms $\beta, \theta : T \longrightarrow N$ and $\beta^{'}, \theta^{'} : \Gamma^{'} \longrightarrow \Gamma^{''}$, with $\beta\alpha = \theta\alpha$ and $\beta^{'} \alpha^{'} (\gamma) = \theta^{'} \alpha^{'} (\gamma) \Rightarrow \beta = \theta$, for all $\gamma \in \Gamma$.

The following Isbell zig zag Theorem 1.2 is proved in [2] for the same class D of Γ semigroups which is of basic importance to investigations.

Theorem 1.2 (Ashraf et al.[2]). *Let U be a Γ subsemigroup of a Γ semigroup S and let $d \in S$. Then $d \in \Gamma$ $Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$\begin{aligned} d &= a_o\gamma t_1 \\ &= y_1\gamma a_1\gamma t_1 \\ &= y_1\gamma a_2\gamma t_2 \\ &= y_2\gamma\ a_3\gamma t_2 \\ &\vdots \\ &= y_m\gamma a_{2m-1}\gamma t_m \\ &= y_m\gamma a_{2m} \end{aligned} \qquad (1)$$

Where $m \geq 1$,$a_i \in U$ $(i = 0,1,\cdots,2m)$, y_i,$t_i \in S$ $(i = 1,2,\cdots,m)$, and $a_o = y_1\gamma a_1$, $a_{2m-1}\gamma t_m = a_{2m}$ $a_{2i-1}\gamma t_i = a_{2i}\gamma t_{ti+1}$, $y_i\gamma\ a_{2i} = y_{i+1}\gamma a_{2i+1}$ $(1 \leq i \leq m$ - 1) Such a series of factorization is called a zig zag in S over U with value d, length m and spine a_o,a_1,$\cdots$,a_{2m}.

2. Epimorphically Preserved Gamma Semigroup Identities

In this section first we generalize Isbell's result [5] from commutative semigroups to commutative Γ semigroups. Further, we show that some permutation identities for Γ semigroup are preserved under epimorphism.

Definition 2.1 (Ashraf [1]). An identity μ is said to be preserved under epimorphism in conjunction with an identity τ if whenever S satisfies τ and μ, and $\varphi : S \longrightarrow T$ is an epimorphism in the category of all semigroups, then T also satisfies τ and μ ; or equivalently, whenever U satisfies τ and μ and $dom(U,S)$ $= S$, then S also satisfies τ and μ.

Lemma 2.2. *let U be commutative gamma subsemigroup of commutative gamma semigroup S. For any $u \in U$ and $d \in \Gamma$ $dom(U,S)\backslash U$, then $u\gamma^{'}d = d\gamma^{'}u$.*

Proof. For any $u \in U$ and $d \in \Gamma$ $dom(U,S)\backslash U$. Then, by Theorem 1.2, d has zig zag equation in S over U of length m.Now,

$u\gamma^{'}d = u\gamma a_o\gamma t_1$ using zig zag equation (1)
$= a_o\gamma^{'}u\gamma t_1$ since U is commutative
$= y_1\gamma a_1\gamma^{'}u\gamma t_1$ using zig zag equation (1)

$= y_1\gamma u\gamma' a_1\gamma t_1$ since U is commutative
$= y_1\gamma u\gamma' a_2\gamma t_2$ using zig zag equation (1)
$= y_1\gamma a_2\gamma' u\gamma t_2$ since U is commutative
$= y_2\gamma a_3\gamma' u\gamma t_2$ using zig zag equation (1)
$\vdots$
$= y_m\gamma a_{2m-1}\gamma' u\gamma t_m$
$= y_m\gamma u\gamma' a_{2m-1}\gamma t_m$ since U is commutative
$= y_m\gamma u\ \gamma' a_{2m}$ using zig zag equation (1)
$= y_m\gamma a_{2m}\ \gamma' u$ since U is commutative
$= d\gamma' u$ using zig zag equation (1) □

Theorem 2.3. *Let U be the commutative Γ subsemigroup of commutative Γ semigroup S, then Γ $dom(U,S)$ is also commutative.*

Proof. We used the result which is proved by Lemma 2.2 above:
Now to complete the proof of the theorem, let for all d_1, $d_2 \in \Gamma$ $dom(U,S)$. Then
$d_1\gamma' d_2 = d_1\gamma' a_o\gamma t_1$ by zig zag equation (1)
$= a_o\gamma'\ d_1\gamma t_1$ by Lemma 2.2
$= y_1\gamma a_1\gamma' d_1\gamma t_1$ by zig zag equation (1)
$= y_1\gamma d_1\gamma' a_1\gamma t_1$ by Lemma 2.2
$= y_1\gamma d_1\gamma' a_2\gamma t_2$ by zig zag equation (1)
$= y_1\gamma a_2\gamma' d_1\gamma t_2$ by Lemma 2.2
$= y_2\gamma a_3\gamma' d_1\gamma t_2$ by zig zag equation (1)
$\vdots$
$= y_m\gamma a_{2m-1}\gamma' d_1\gamma t_m$
$= y_m\gamma\ d_1\ \gamma' a_{2m-1}\gamma t_m$ by Lemma 2.2
$= y_m\gamma d_1\gamma' a_{2m}$ by zig zag equation (1)
$= y_m\gamma a_{2m}\gamma' d_1$ by Lemma 2.2
$= d_2\gamma' d_1$ by zig zag equation (1)
Hence theorem follows. □

Corollary 2.4. *Let U be a Γ subsemigroup of a Γ semigroup S and φ: $U \longrightarrow S$ is epimorphism. For x_1,$x_2$$\in$$S$ and $\gamma$$\in$$\Gamma$. Then commutative identity ($x_1\gamma x_2 = x_2\gamma x_1$) is preserved under epimorphism in the category of Γ semigroups.*

Theorem 2.5. *Let U be a Γ subsemigroup of a Γ semigroup S and φ : $U \longrightarrow S$ is epimorphism. For x_1,x_2,$x_3 \in S$ and $\gamma \in \Gamma$, then following identities are preserved under epimorphism.*

(i) $x_1\gamma x_2\gamma x_3 = x_1\gamma x_3\gamma x_2$
(ii) $x_1\gamma x_2\gamma x_3 = x_2\gamma x_1\gamma x_3$

Proof. Let U satisfies $x_1\gamma x_2\gamma x_3 = x_1\gamma x_3\gamma x_2$ and Γ $dom(U,S) = S$.
Case 1. Let $x_1 \in S\setminus U$, x_2, $x_3 \in U$. Then, by Theorem 1.2, x_1 has zig zag

equation in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3 = y_m\gamma a_{2m}\gamma x_2\gamma x_3$ using zig zag equation (1)
$= y_m\gamma a_{2m}\gamma x_3\gamma x_2$ since$U$ satisfies identity(i)
$= x_1\gamma x_3\gamma x_2$ using zig zag equation (1)

Case 2. Let x_1, $x_2 \in S\setminus U$. Then, by Theorem 1.2, x_2 has zig zag equation of type (1) in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3 = x_1\gamma y_m\gamma a_{2m}\gamma x_3$ using zig zag equation (1)
$= x_1\gamma y_m\gamma x_3\gamma a_{2m}$ by case 1
$= x_1\gamma y_m\gamma x_3\gamma a_{2m-1}\gamma t_m$ using zig zag equation (1)
$= x_1\gamma y_m\gamma a_{2m-1}\gamma x_3\gamma t_m$ by case 1
$= x_1\gamma y_{m-1}\gamma a_{2m-2}\gamma x_3\gamma t_m$ using zig zag equation (1)
$= x_1\gamma y_{m-1}\gamma x_3\gamma a_{2m-2}\gamma t_m$ by case 1
$= x_1\gamma y_{m-1}\gamma\ x_3\gamma a_{2m-3}\gamma t_{m-1}$ using zig zag equation (1)
$\vdots$
$= x_1\gamma y_1\gamma x_3\gamma a_1\gamma t_1$ using zig zag equation (1)
$= x_1\gamma y_1\gamma a_1\gamma x_3\gamma t_1$ by case 1
$= x_1\gamma a_o\gamma x_3\gamma t_1$ using zig zag equation (1)
$= x_1\gamma x_3\gamma a_o\ \gamma t_1$ by case 1
$= x_1\gamma x_3\gamma x_2$ using zig zag equation (1)

Case 3. Let x_1, x_2, $x_3 \in S\setminus U$. Then, by Theorem 1.2, x_3 has zig zag equation in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3 = x_1\gamma x_2\gamma a_o\gamma t_1$ by zig zag equation (1)
$= x_1\gamma a_o\gamma x_2\gamma t_1$ by case 2
$= x_1\gamma y_1\gamma a_1\gamma x_2\gamma t_1$ by zig zag equation (1)
$= x_1\gamma y_1\gamma x_2\gamma a_1\gamma t_1$ by case 2
$= x_1\gamma y_1\gamma x_2\gamma a_2\gamma t_2$ by zig zag equation (1)
$= x_1\gamma y_1\gamma a_2\gamma x_2\gamma t_2$ by case 2
$= x_1\gamma y_2\gamma a_3\gamma x_2\gamma t_2$ by zig zag equation (1)
$= x_1\gamma y_2\gamma x_2\gamma a_3\gamma t_2$ by case 2
$\vdots$
$= x_1\gamma y_m\gamma x_2\gamma a_{2m-1}\gamma t_m$
$= x_1\gamma y_m\gamma x_2\gamma a_{2m}$ by zig zag equation (1)
$= x_1\gamma y_m\gamma a_{2m}\gamma x_2$ by case 2
$= x_1\gamma x_3\gamma x_2$ by zig zag equation (1)

Therefore, $x_1\gamma x_2\gamma x_3 = x_1\gamma x_3\gamma x_2$ is preserved under epimorphism.
Dually, we can prove $x_1\gamma x_2\gamma x_3 = x_2\gamma x_1\gamma x_3$ is preserved under epimorphism. □

Lemma 2.6. *Let U be a Γ subsemigroup of a Γ semigroup S and Γ $dom(U,S)$ $= S$. Then for any $d \in S\setminus U$ and any positive integer k, there exist a_1,a_2,$\cdots$, $a_k \in U$,γ_1,γ_2,$\cdots$,$\gamma_k \in \Gamma$ and $d_1 \in S\setminus U$ such that $d= a_1\gamma_1 a_2\gamma_2 \cdots a_k\gamma_k d_k$.*

Proof. Since, $d \in S\setminus U$ and Γ $dom(U,S) = S$, by zig zag equation, there exist $a_1 \in U$,$d_1 \in S\setminus U$ such that $d = a_1\gamma_1 d_1$.

Applying again zig zag equation d_1, we get $d = a_1\gamma_1 a_2\gamma_2 d_k$ for some $a_1 \in U$ and $d_k \in S\setminus U$. Continuing this process gives us the required result. □

Theorem 2.7. *Let U be a Γ subsemigroup of a Γ semigroup S and φ: $U \longrightarrow S$ is epimorphism. for x_1,x_2,x_3,$x_4 \in S$ and $\gamma \in \Gamma$. Then normality identity $x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma x_3\gamma x_2\gamma x_4$ is preserved under epimorphism.*

Proof. Before proving the theorem, first we prove some propositions. □

Proposition 2.8. *Let U satisfies identity $x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma x_3\gamma x_2\gamma x_4$, and Γ dom$(U,S) = S$. Then the identity holds for x_1,$x_2 \in S$ and x_3,$x_4 \in U$.*

Proof. **Case 1.** Let $x_1 \in S \setminus U$, x_2,x_3,$x_4 \in U$. Then, by Theorem 1.2, x_1 has zig zag equation in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3\gamma x_4 = y_m\gamma a_{2m}\gamma x_2\gamma x_3\gamma x_4$ using zig zag equation (1)
$= y_m\gamma a_{2m}\gamma x_3\gamma x_2\gamma x_4$ since U satisfies normality identity
$= x_1\gamma x_3\gamma x_2\gamma x_4$ using zig zag equation (1)

Case 2. Let x_1,$x_2 \in S\setminus U$, x_3,$x_4 \in U$. Then, by Theorem 1.2, x_2 has zig zag equation of type (1) in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma y_m\gamma a_{2m}\gamma x_3\gamma x_4$ by zig zag equation (1)
$= x_1\gamma y_m\gamma x_3\gamma a_{2m}\gamma x_4$ by case 1
$= x_1\gamma y_m\gamma x_3\gamma a_{2m-1}\gamma t_m\gamma x_4$ using zig zag equation (1)
$= x_1\gamma y_m\gamma x_3\gamma a_{2m-1}\gamma u'_m\gamma t'_m\gamma x_4$ by Lemma 2.6, $t_m = u'_m\gamma t'_m$, $u'_m \in U$, $t'_m \in S\setminus U$
$= x_1\gamma y_m\gamma a_{2m-1}\gamma x_3\gamma u'_m\gamma t'_m\gamma x_4$ by case 1
$= x_1\gamma y_{m-1}\gamma a_{2m-2}\gamma x_3\gamma u'_m\gamma t'_m\gamma x_4$ using zig zag equation (1)
$= x_1\gamma y_{m-1}\gamma x_3\gamma a_{2m-2}\gamma u'_m\gamma t'_m\gamma x_4$ by case 1
$= x_1\gamma y_{m-1}\gamma x_3\gamma a_{2m-2}\gamma t_m\gamma x_4$ by Lemma 2.6, $t_m = u'_m\gamma t'_m$
$= x_1\gamma y_{m-1}\gamma x_3\gamma a_{2m-3}\gamma t_{m-1}\gamma x_4$ using zig zag equation (1)
$\vdots$
$= x_1\gamma y_2\gamma x_3\gamma a_3\gamma t_2\gamma x_4$
$= x_1\gamma y_2\gamma x_3\gamma a_3\gamma u'_2\gamma t'_2\gamma x_4$ by Lemma 2.6, $t_2 = u'_2\gamma t'_2$, $u'_2 \in U$, $t'_2 \in S\setminus U$
$= x_1\gamma y_2\gamma a_3\gamma x_3\gamma u'_2\gamma t'_2\gamma x_4$ by case 1
$= x_1\gamma y_1\gamma a_2\gamma x_3\gamma u'_2\gamma t'_2\gamma x_4$ using zig zag equation (1)
$= x_1\gamma y_1\gamma x_3\gamma a_2\gamma u'_2\gamma t'_2\gamma x_4$ by case 1.
$= x_1\gamma y_1\gamma x_3\gamma a_2 t_2\gamma x_4$ by Lemma 2.6, $t_2 = u'_2\gamma t'_2$
$= x_1\gamma y_1\gamma x_3\gamma a_1 t_1\gamma x_4$ by zig zag equation (1)
$= x_1\gamma y_1\gamma x_3\gamma a_1\gamma u'_1\gamma t'_1\gamma x_4$ by Lemma 2.6, $t_1 = u'_1\gamma t'_1$, $u'_1 \in U$, $t'_1 \in S\setminus U$

$= x_1\gamma y_1\gamma a_1\gamma x_3\gamma u_1^{'}\gamma t_1^{'}\gamma x_4$ by case 1
$= x_1\gamma a^{'}\gamma x_3\gamma u_1^{'}\gamma t_1^{'}\gamma x_4$ by zig zag equation (1)
$= x_1\gamma x_3\gamma a^{'}\gamma u_1^{'}\gamma t_1^{'}\gamma x_4$ by case 1
$= x_1\gamma x_3\gamma a^{'}\gamma t_1\gamma x_4$ by Lemma 2.6, $t_1 = u_1^{'}\gamma t_1^{'}$, $u_1^{'} \in U$,
$= x_1\gamma x_3\gamma x_2\gamma x_4$ by zig zag equation (1) □

Dually, we can prove following proposition.

Proposition 2.9. *Let U satisfies normality identity $x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma x_3\gamma x_2\gamma x_4$ and Γ dom$(U,S) = S$, then normality identity holds for all $x_1,x_2 \in U$ and $x_3,x_4 \in S$.*

Proof. **Case 1.** Take $x_1,x_2,x_3 \in S\setminus U$, $x_4 \in U$. Then, by Theorem 1.2, x_3 has zig zag equation in S over U of length m. Now,

$x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma x_2\gamma a^{'}\gamma t_1\gamma x_4$ by zig zag equation (1)
$= x_1\gamma x_2\gamma a^{'}\gamma b_1^{'}\gamma t_1^{'}\gamma x_4$ by Lemma 2.6, $t_1 = b_1^{'}\gamma t_1^{'}$, $b_1^{'} \in U$, $t_1^{'} \in S\setminus U$.
$= x_1\gamma a^{'}\gamma x_2\gamma b_1^{'}\gamma t_1^{'}\gamma x_4$ by Proposition 2.8
$= x_1\gamma y_1\gamma a_1\gamma x_2\gamma t_1\gamma x_4$ by zig zag (1),(Lemma 2.6)
$= x_1\gamma y_1^{'}\gamma c_1^{'}\gamma a_1\gamma x_2\gamma t_1\gamma x_4$ by Lemma 2.6, $y_1 = y_1^{'}\gamma c_1^{'}$, $c_1^{'} \in U$, $y_1^{'} \in S\setminus U$
$= x_1\gamma y_1^{'}\gamma c_1^{'}\gamma x_2\gamma a_1\gamma t_1\gamma x_4$ by Proposition 2.8
$= x_1\gamma y_1\gamma x_2\gamma a_2\gamma t_2\gamma x_4$ by Lemma 2.6 and zig zag (1)
$= x_1\gamma y_1\gamma x_2\gamma a_2\gamma b_2^{'}\gamma t_2^{'}\gamma x_4$ by Lemma 2.6, $t_2 = b_2^{'}\gamma t_2^{'}$, $b_2^{'} \in U$, $t_2^{'} \in S\setminus U$.
$= x_1\gamma y_1\gamma a_2\gamma x_2\gamma b_2^{'}\gamma t_2^{'}\gamma x_4$ by Proposition 2.8
$= x_1\gamma y_2\gamma a_3\gamma x_2\gamma t_2\gamma x_4$ by Lemma 2.6 and zig zag (1)
$\vdots$
$= x_1\gamma y_m\gamma a_{2m-1}\gamma x_2\gamma t_m\gamma x_4$
$= x_1\gamma y_m^{'}\gamma c_m^{'}\gamma a_{2m-1}\gamma x_2\gamma t_m\gamma x_4$ by Lemma 2.6, $y_m = y_m^{'}\gamma c_m^{'}$, $c_m^{'} \in U$, $y_m^{'} \in S\setminus U$
$= x_1\gamma y_m^{'}\gamma c_m^{'}\gamma x_2\gamma a_{2m-1}\gamma t_m\gamma x_4$ by Proposition 2.8
$= x_1\gamma y_m\gamma x_2\gamma a_{2m}\gamma x_4$ by Lemma 2.8 and zig zag (1)
$= x_1\gamma y_m\gamma a_{2m}\gamma x_2\gamma x_4$ by Proposition 2.8
$= x_1\gamma x_3\gamma x_2\gamma x_4$ by zig zag equation (1)

Case 2. Take $x_1,x_2,x_3,x_4 \in S\setminus U$. Then, by Theorem 1.2, x_4 has zig zag equation in S over U of length m.

$x_1\gamma x_2\gamma x_3\gamma x_4 = x_1\gamma x_2\gamma x_3\gamma a^{'}\gamma t_1$ by zig zag equation (1)
$= x_1\gamma x_3\gamma x_2\gamma a^{'}\gamma t_1$ by previous case
$= x_1\gamma x_3\gamma x_2\gamma x_4$ by zig zag equation (1) □

Hence by proof of Proposition 2.8 and 2.9 we got Theorem 2.7 is proved.

3. Epimorphisms and Dominions for Partially Ordered Gamma Semigroups

In this section, we defined homomorphism between two partially ordered gamma semigroups. Also, we used it to extend some results of Sohail and Tart [15] for partially ordered semigroup to partially ordered gamma semigroups. Finally we proved that subpartially ordered gamma semigroup is epimorphically embedded if partially ordered gamma semigroup identities are preserved under epimorphisms.

Note. The investigations mentioned below are performed by considering gamma as the set of associative binary operations on partially ordered gamma semigroups.

Definition 3.1. A *po* Γ semigroup homomorphism between two *po* Γ semigroups $(S,\, \Gamma,\, \leq_s)$ and *po* $\Gamma^{'}$ semigroup $(T,\, \Gamma^{'},\, \leq_T)$ in the category of *po* Γ semigroups is defined as:
a morphism α: $(S,\, \Gamma,\, \leq_s) \longrightarrow (T,\, \Gamma^{'},\, \leq_T)$ is said to be *po* Γ semigroup homomorphism, then α satisfies
$\alpha(a^{'}\gamma b) = \alpha(a^{'})\gamma^{'}\alpha(b)$. For all $a^{'},\, b \in (S,\, \Gamma,\, \leq_s)$, $\gamma \in \Gamma$ and $\gamma^{'} \in \Gamma^{'}$.

Definition 3.2. A morphism α : $(S,\, \Gamma,\, \leq_s) \longrightarrow (T,\, \Gamma^{'},\, \leq_T)$ such that $\alpha(a^{'}\gamma b) = \alpha(a^{'})\gamma^{'}\alpha(b)$ For all $a^{'}, b \in (S,\, \Gamma,\, \leq_s)$, $\gamma \in \Gamma$ and $\gamma^{'} \in \Gamma^{'}$ in the category of *po* Γ semigroup is called epimorphism. If for all *po* $\Gamma^{''}$ semigroup $(N,\, \Gamma^{''},\, \leq_N)$ and for every pair of *po* Γ semigroup homomorphism: $\beta,\, \theta$: $(T,\, \Gamma^{'},\, \leq_T) \longrightarrow (N,\, \Gamma^{''},\, \leq_N)$ such that $\beta(a\gamma^{'}b) = \beta(a)\gamma^{''}\beta(b)$ and $\theta(a\gamma^{'}b) = \theta(a)\gamma^{''}\theta(b)$ With $\beta\alpha = \theta\alpha \Rightarrow \beta = \theta$. For all $a, b \in (T,\, \Gamma^{'},\, \leq_T)$, $\gamma^{'} \in \Gamma^{'}$ and $\gamma^{''} \in \Gamma^{''}$.

Definition 3.3. Let $(U,\, \Gamma,\, \leq_u)$ be a sub*po* Γ semigroup of *po* Γ semigroup $(S,\, \Gamma,\, \leq_S)$. We say that $(U,\, \Gamma,\, \leq_u)$ dominates an element $d \in (S,\, \Gamma,\, \leq_s)$. If for every *po* $\Gamma^{'}$semigroup $(T,\, \Gamma^{'},\, \leq_T)$ and for every *po* Γ semigroup homomorphism $\beta,\, \alpha$: $(S,\, \Gamma,\, \leq_s) \longrightarrow (T,\, \Gamma^{'},\, \leq_T)$ such that $\beta(a^{'}\gamma b) = \beta(a^{'})\gamma^{'}\beta(b)$ and $\alpha(a^{'}\gamma b) = \alpha(a^{'})\gamma^{'}\alpha(b)$ With $u\alpha = u\beta \Rightarrow d\alpha = d\beta$. For all $u \in (U,\, \Gamma,\, \leq_u)$, $\gamma \in \Gamma$, $\gamma^{'} \in \Gamma^{'}$ and $a^{'},\, b \in (S,\, \Gamma,\, \leq_s)$.
The set of all elements of $(S,\, \Gamma,\, \leq_S)$ dominated by $(U,\, \Gamma,\, \leq_u)$ is called the *Po* Γ dominion of $(U,\, \Gamma,\, \leq_u)$ in $(S,\, \Gamma,\, \leq_S)$ and we denote it by *Po* Γ $dom_S(U)$.
It may be easily seen that *Po* Γ $dom_S(U)$ is a subsemigroup of $(S,\, \Gamma,\, \leq_S)$ containing $(U,\, \Gamma,\, \leq_u)$.

Lemma 3.4. *Take a subpo gamma semigroup* $(U,\, \Gamma,\, \leq_u)$ *of a po gamma semigroup* $(S,\, \Gamma,\, \leq_S)$*. Then an element* d *of* $(S,\, \Gamma,\, \leq_S)$ *is in po* Γ $dom(U, S)$ *if and*

only if $d \otimes 1 = 1 \otimes d$ *in* $s^1 \otimes_{(u^1)} s^1$.
where $(U^1, \Gamma, \leq_{(U^1)})$ *and* $(S^1, \Gamma, \leq_{(S^1)})$ *are the po gamma monoids obtained from* $(U, \Gamma, \leq_u)$ *and* $(S, \Gamma, \leq_S)$, *respectively, by adjoining an incomparable external identity whether or not they already have one.*

Theorem 3.5. *Let* $(U, \Gamma, \leq_u)$ *be a subpo* Γ *semigroup of po* Γ *semigroup* $(S, \Gamma, \leq_S)$ *and* $d \in (S, \Gamma, \leq_S)$. *Then* $d \in$ *po* Γ $dom_S(U)$ *if and only if* $d \in (U, \Gamma, \leq_u)$ *or there exists a series of factorizations of* d *as follows:*

$$\begin{array}{ll}
d \leq s_1\gamma u_1 & u_1 \leq v_1\gamma t_1 \\
s_1\gamma v_1 \leq s_2\gamma u_2 & u_2\gamma t_1 \leq v_2\gamma t_2 \\
\vdots & \vdots \\
s_{n-1}\gamma v_{n-1} \leq u_n & u_n\gamma t_{n-1} \leq d \\
 & d \leq v_n\gamma t_n \\
v_n \leq s_{n+1}\gamma u_{n+1} & \\
s_{n+1}\gamma v_{n+1} \leq s_{n+2}\gamma u_{n+2} & u_{n+1}\gamma t_n \leq v_{n+1}\gamma t_{n+1} \\
\vdots & \vdots \\
s_{n+m}\gamma v_{n+m} \leq d & u_{n+m}\gamma t_{n+m-1} \leq v_{n+m}
\end{array} \qquad (2)$$

with elements $u_1, \ldots u_{n+m}, v_1, \ldots, v_{n+m} \in (U, \Gamma, \leq_u)$, $s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_{n+m}, t_1, \ldots, t_{n+m-1} \in (S, \Gamma, \leq_s)$ *and* $\gamma \in \Gamma$. *Such a series of factorization is called Isbell zig-zag theorem for po* Γ *semigroup in* $(S, \Gamma, \leq_s)$ *over* $(U, \Gamma, \leq_u)$ *with value* d, *length* $n + m$.

Proof. Suppose that $d \in$ *po* Γ $dom(U, S)$. Then by above lemma $d \otimes 1 = 1 \otimes d$ in $s^1 \otimes_{(u^1)} s^1$. Following the results of Shi et al.[14] there exists a system of inequalities,

$$\begin{array}{ll}
d \leq s_1\gamma u_1 & u_1 \leq v_1\gamma t_1 \\
s_1\gamma v_1 \leq s_2\gamma u_2 & \\
\vdots & u_2\gamma t_1 \leq v_2\gamma t_2 \\
s_{n-1}\gamma v_{n-1} \leq s'_n\gamma u_n & \vdots \\
s'_n\gamma v'_n \leq 1 & u_n\gamma t_{n-1} \leq v'_n\gamma d \\
1 \leq s'_{n+1}\gamma u'_{n+1} & \\
s'_{n+1}\gamma v_n \leq s_{n+1}\gamma u_{n+1} & u'_{n+1}\gamma d \leq v_n\ \gamma t_n \\
\vdots & u_{n+1}\gamma t_n \leq v_{n+1}\gamma t_{n+1} \\
s_{n+m-1}\gamma v_{n+m-1} \leq s_{n+m}\gamma u_{n+m} & \vdots \\
s_{n+m}\gamma v_{n+m} \leq d & u_{n+m}\ \gamma t_{n+m-1} \leq v_{n+m},
\end{array}$$

where $u_1, \ldots u_{n+m}, v_1, \ldots v_{n+m}, v'_n, u'_{n+1} \in U^1$; $s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_{n+m}, t_1, \ldots, t_{n+m-1}, s'_n, s'_{n+1} \in S^1$ and $\gamma \in \Gamma$.
Since 1 is incomparable and Γ is associative binary operation on $(S,\Gamma,\leq_S)$, $s'_n\gamma v'_n \leq 1$ implies $s'_n\ \gamma\ v'_n = 1$.

then we have $s'_n = v'_n = 1$,
by adjoining 1, we have also
$s'_{n+1} = u'_{n+1} = 1$.
We can rewrite the above set of inequalities as follows:

$$\begin{array}{ll}
d \leq s_1\gamma u_1 & u_1 \leq v_1\gamma t_1 \\
s_1\gamma v_1 \leq s_2\gamma u_2 & u_2\gamma t_1 \leq v_2\gamma t_2 \\
\vdots & \vdots \\
s_{n-1}\gamma v_{n-1} \leq u_n & u_n\gamma t_{n-1} \leq d \\
 & d \leq v_n\gamma t_n \\
v_n \leq s_{n+1}\gamma u_{n+1} & \\
s_{n+1}\gamma v_{n+1} \leq s_{n+2}\gamma u_{n+2} & u_{n+1}\gamma t_n \leq v_{n+1}\gamma t_{n+1} \\
\vdots & \vdots \\
s_{n+m}\gamma v_{n+m} \leq d & u_{n+m}\gamma t_{n+m-1} \leq v_{n+m}
\end{array}$$

By following Sohail and Tart [15] proved that zig zag theorem for *po* semigroup in the ordered context and Ashraf et al.[2] proved that zig zag theorem for gamma semigroup in the unordered context. We next show that all the elements in the above set of inequalities may be assumed to lie in $(S, \Gamma, \leq_S)$.
We perform this by assuming that u_i, v_i, s_i or t_i is not in $(S, \Gamma, \leq_S)$ for some i and this will results collapsing of inequality.
To preform these we took the upper half portion only as follows:

$$\begin{array}{ll}
d \leq s_1\gamma u_1 & u_1 \leq v_1\ \gamma t_1 \\
s_1\gamma v_1 \leq s_2\gamma u_2 & u_2\gamma t_1 \leq v_2\gamma t_2 \\
\vdots & \vdots \\
s_{n-1}\gamma v_{n-1} \leq u_n & u_n\gamma t_{n-1} \leq d
\end{array} \quad (3)$$

We proceed the proof by taking all possibilities.
If $u_1 = 1$, then (from $u_1 \leq v_1\gamma t_1$) $v_1 = t_1 = 1$, and we can write

$$\begin{array}{ll}
d \leq s_2\gamma u_2 & u_2 \leq v_2\gamma t_2 \\
s_2\gamma v_2 \leq s_3\gamma u_3 & u_3\gamma t_2 \leq v_3\gamma t_3 \\
\vdots & \vdots \\
s_{n-1}\gamma v_{n-1} \leq u_n & u_n\gamma t_{n-1} \leq d
\end{array}$$

If $v_1 = 1$ then $s_1 \leq s_2\gamma u_2$, and we can calculate

$$\begin{array}{ll}
d \leq s_2\gamma(u_2\gamma u_1) & u_2\gamma u_1 \leq v_2\gamma t_2 \\
s_2\gamma v_2 \leq s_3\gamma u_3 & u_3\gamma t_2 \leq v_3\gamma t_3 \\
\vdots & \vdots \\
s_{n-1}\gamma v_{n-1} \leq u_n & u_n\ \gamma t_{n-1} \leq d
\end{array}$$

If $u_i = 1$, $2 \le i \le n-1$, then the set of inequalities
$s_{i-2}\gamma v_{i-2} \le s_{i-1}\gamma u_{i-1}$ $u_{i-1}\gamma t_{i-2} \le v_{i-1}\gamma\ t_{i-1}$
$s_{i-1}\gamma v_{i-1} \le s_i\ \gamma u_i$ $u_i\gamma t_{i-1} \le v_i\ \gamma t_i$
$s_i\gamma v_i \le s_{i+1}\gamma u_{i+1}$ $u_{i+1}\gamma t_i \le v_{i+1}\gamma t_{i+1}$
Collapses to
$s_{i-2}\gamma v_{i-2} \le s_{i-1}\ \gamma\ u_{i-1}$ $u_{i-1}\ \gamma t_{i-2} \le v_{i-1}\ \gamma v_i\gamma t_i$
$s_{i-1}\gamma v_{i-1}\gamma v_i \le s_{i+1}\gamma u_{i+1}$ $u_{i+1}\gamma t_i \le v_{i+1}\gamma\ v_i t_{i+1}$
If $v_i = 1$, $2 \le i \le n-2$, then we obtained the same results as we got for u_i
If $s_n = 1$ then $d = u_1 \in U$ and there is nothing to prove,
If $s_i = 1$, $2 \le i \le n-1$, such that $s_j \in (S,\Gamma,\le_S)$ for all $j \le i-1$, then starting from the top of in equality we can write
$d \le s_1\gamma u_1 \le s_1\ \gamma v_1\gamma t_1 \le s_2\gamma u_2\gamma t_1 \le \cdots \le u_i\gamma t_{i-1}$.
On the other hand, we also have
$d \ge u_n\gamma t_{n-1} \ge s_{n-1}\gamma v_{n-1}\ \gamma t_{n-1} \ge s_{n-1}\gamma u_{n-1}\gamma t_{n-2} \ge \cdots \ge u_i\gamma t_{i-1}$.
Thus $d = u_i\gamma t_{i-1}$ and we may shorten the inequalities (3) to
$d \le s_1\gamma u_1$ $u_1 \le v_1\gamma t_1$
$s_1\gamma v_1 \le s_2\gamma u_2$ $u_2\gamma t_1 \le v_2\gamma t_2$
$\vdots$ $\vdots$
$s_{i-1}\gamma v_{i-1} \le u_i$ $u_i\gamma t_{i-1} \le d$
Similarly,
if $t_i = 1$, $1 \le i \le n-1$, with $t_i \in (S,\ \Gamma,\ \le_S)$ for all $j \le i-1$, then we have
$d \le s_1\gamma u_1 \le s_1\gamma v_1\gamma t_1 \le s_2\gamma v_2\gamma t_1 \le \cdots \le s_i\gamma u_i\gamma t_{i-1} \le s_i\ \gamma v_i$
This gives
$d = s_i\gamma v_i = s_{i+1}\gamma u_{i+1}$ and
one can shorten (3) to
$d \le s_{i+1}\gamma u_{i+1}$ $u_{i+1} \le v_{i+1}\gamma t_{i+1}$
$s_{i+1}\gamma v_{i+1} \le s_{i+2}\gamma u_{i+2}$ $u_{i+2}\gamma t_{i+1} \le v_{i+2}\gamma t_{i+2}$
$\vdots$ $\vdots$
$s_{n-1}\gamma v_{n-1} \le u_n$ $u_n\gamma t_{n-1} \le d$.
This completes the proof of the direct part.

Conversely, suppose that $d \in (S,\ \Gamma,\ \le_S)$, and there exist the system of zigzag inequality (3)
We need to show that $d \in po\ \Gamma\ dom(U, S)$.
Suppose that $(T,\ \Gamma,\ \le_T)$ is *po* Γ' semigroup and a homomorphism
$\beta,\ \alpha:\ (S,\ \Gamma,\ \le_S) \longrightarrow (T,\ \Gamma',\ \le_T)$ such that
$\beta(a'\gamma b) = \beta(a')\gamma'\ \beta(b)$ and $\alpha(a'\gamma b) = \alpha(a')\gamma'\ \alpha(b)$ is *po* Γ semigroup homomorphism (agreeing on $(U,\ \Gamma,\ \le_U)$) With $u\alpha = u\beta$.
For all $u \in (U,\ \Gamma,\ \le_U)$,$\gamma \in \Gamma$, $\gamma' \in \Gamma'$ and a',$b \in (S,\ \Gamma,\ \le_S)$.
Now, by using zig zag Theorem 3.1, we have
$\alpha(\mathrm{d}) \le \alpha(s_1\gamma u_1)$
$\le \alpha(s_1)\gamma'\alpha(u_1)$

$\leq \alpha(s_1)\gamma^{'}\beta(v_1\gamma t_1)$
$\leq \alpha(s_1)\gamma^{'}\beta(v_1)\gamma^{'}\ \beta(t_1)$
$\leq \alpha(s_1\gamma v_1)\gamma^{'}\ \beta(t_1)$
$\vdots$
$\leq \alpha(s_{n-1}\gamma v_{n-1})\gamma^{'}\ \beta(t_{n-1})$
$\leq \beta(u_n)\gamma^{'}\beta(t_{n-1})$
$\leq \beta(u_n\gamma t_{n-1})$
$\leq \beta(d)$,
thus, $\alpha\ (d) \leq \beta\ (d)$ (4)
And
conversely, by Appling zigzag on d, we have
$\beta(d) \leq \beta(v_n\gamma t_n)$
$\leq \beta(v_n)\gamma^{'}\beta(t_n)$
$\leq \alpha(v_n)\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1}\gamma u_{n+1})\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\alpha(u_{n+1})\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(u_{n+1}\gamma t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(v_{n+1}\gamma t_{n+1})$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(v_{n+1})\gamma^{'}\beta(t_{n+1})$
$\leq \alpha(s_{n+2}\gamma u_{n+2})\gamma^{'}\beta(t_{n+1})$
$\vdots$
$\leq \alpha(s_{n+m}\gamma u_{n+m})\gamma^{'}\beta(t_{n+m-1})$
$\leq \alpha(s_{n+m})\gamma^{'}\beta(u_{n+m}\gamma t_{n+m-1})$
$\leq \alpha(s_{n+m})\gamma^{'}\beta(v_{n+m})$
$\leq \alpha(s_{n+m}\gamma v_{n+m})$
$\leq \alpha(d)$
thus, $\beta\ (d) \leq \alpha\ (d)$ (5)
From (4) and (5) we have $\alpha\ (d) = \beta\ (d) \Rightarrow d \in po\ \mathit{\Gamma}\ dom(U, S)$ by definition of dominion for *po* gamma semigroup.
Therefore, $d \in po\ \mathit{\Gamma}\ dom(U, S)$, this complete the proof of the Theorem.

□

Theorem 3.6. *Let* $(U,\ \mathit{\Gamma},\ \leq_u)$ *be subpo* $\mathit{\Gamma}$ *semigroup of po* $\mathit{\Gamma}$ *semigroup* $(S,\ \mathit{\Gamma},\ \leq_S)$. *If po* $\mathit{\Gamma}$ *semigroups identities are preserved under epimorphism, then subpo* $\mathit{\Gamma}$ *semigroup* $(U,\ \mathit{\Gamma},\ \leq_U)$ *is epimorphically embedded in* $(S,\ \mathit{\Gamma},\ \leq_S)$.
i.e. po $\mathit{\Gamma}$ $dom(U, S) = (S,\ \mathit{\Gamma},\ \leq_S)$.

Proof. Suppose that $\varphi:\ (U,\ \mathit{\Gamma},\ \leq_U) \longrightarrow (S,\ \mathit{\Gamma},\ \leq_S)$ is epimorphism and $U^{'} = V^{'}$ is identities that preserved under epimorphism.
Now take for all $u^{'} \in U^{'} = v^{'} \in V$ such that $u^{'} = v^{'}$ (6)
Since $U^{'}$and $V^{'}$ are preserved under epimorphism then $(U,\ \mathit{\Gamma},\ \leq_U)$ satisfies $U^{'} = V^{'}$ and also, $(S,\ \mathit{\Gamma},\ \leq_S)$ satisfies $U^{'} = V^{'}$.

Thus, $u^{'}, v^{'} \in (S, \Gamma, \leq_S)$
To complete the proof of the theorem, we Claim that the followings:
po Γ $dom(U,S) \subseteq (S, \Gamma, \leq_S)$
$(S, \Gamma, \leq_S) \subseteq$ *po* Γ $dom(U,S)$
To show that *po* Γ $dom(U,S) \subseteq (S,\Gamma,\leq_S)$
Ashraf et al.[2] examined that set of all elements of S dominated by U is called the dominion of U in S and is denoted by $dom(U,S)$. It may easily be seen that Γ $dom(U,S)$ is a Γ subsemigroup of S containing U.
Suppose that set of all elements of $(S, \Gamma, \leq_S)$ dominated by $(U, \Gamma, \leq_u)$ and it is denoted by *po* Γ $dom_S(U)$.
According to Ashraf et al. [2] and definition (3.3) (dominion of *po* Γ semigroup), we have
For all $x \in$ *po* Γ $dom_S(U) \Rightarrow x \in (S, \Gamma, \leq_S)$
Thus *po* Γ $dom(U,S) \subseteq (S, \Gamma, \leq_S)$. (7)
To show that $S \subseteq$ *po* Γ $dom_S(U)$:
Let $(T, \Gamma^{'}, \leq_T)$ be a *po* $\Gamma^{'}$-semigroup and β,α: $(S, \Gamma, \leq_s) \longrightarrow (T, \Gamma^{'}, \leq_T)$ be *po* Γ homomorphisms agreed on $(U, \Gamma, \leq_u)$ such that $\beta(a^{'}\gamma b) = \beta(a^{'})\gamma^{'}\beta(b)$ and $\alpha(a^{'}\gamma b) = \alpha(a^{'})\gamma^{'}\alpha(b)$ with $u\alpha = u\beta$, for all $u \in (U, \Gamma, \leq_u) \subseteq (S, \Gamma, \leq_S)$, $\gamma \in \Gamma$, $\gamma^{'} \in \Gamma^{'}$ and $a^{'},b \in (S, \Gamma, \leq_s)$.
Now, for all $u^{'}, v^{'} \in (S, \Gamma, \leq_S)$ and by zig zag Theorem 3.1
$\alpha(u^{'}) \leq \alpha(s_1\gamma u_1)$
$\leq \alpha(s_1)\gamma^{'}\alpha(u_1)$
$\leq \alpha(s_1)\gamma^{'}\beta(v_1\gamma t_1)$
$\leq \alpha(s_1)\gamma^{'}\beta(v_1)\gamma^{'}\beta(t_1)$
$\leq \alpha(s_1\gamma v_1)\gamma^{'}\beta(t_1)$
$\vdots$
$\leq \alpha(s_{n-1}\gamma v_{n-1})\gamma^{'}\beta(t_{n-1})$
$\leq \alpha(u_{n-1})\gamma^{'}\beta(t_{n-1})$
$\leq \beta(u_{n-1}\gamma t_{n-1})$
$\leq \beta(u^{'})$
$\alpha\ (u^{'}) \leq \beta(u^{'})$ (8)
And
conversely, by appling zigzag on $(u^{'})$
$\beta(u^{'}) \leq \beta(v_n\gamma t_n)$
$\leq \beta(v_n)\gamma^{'}\beta(t_n)$
$\leq \alpha(v_n)\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1}\gamma u_{n+1})\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\alpha(u_{n+1})\gamma^{'}\beta(t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(u_{n+1}\gamma t_n)$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(v_{n+1}\gamma t_{n+1})$
$\leq \alpha(s_{n+1})\gamma^{'}\beta(v_{n+1})\gamma^{'}\beta(t_{n+1})$

$\leq \alpha(s_{n+2}\gamma u_{n+2})\gamma^{'}\beta(t_{n+1})$
$\vdots$
$\leq \alpha(s_{n+m}\gamma u_{n+m})\gamma^{'}\beta(t_{n+m-1})$
$\leq \alpha(s_{n+m})\gamma^{'}\beta(u_{n+m}\gamma t_{n+m-1})$
$\leq \alpha(s_{n+m})\gamma^{'}\beta(v_{n+m})$
$\leq \alpha(s_{n+m}\gamma v_{n+m})$
$\leq \alpha(u^{'})$
Thus, $\beta(u^{'}) \leq \alpha(u^{'})$ (9)
Thus, from (8) and (9) we have $\alpha(u^{'}) = \beta(u^{'})$, then we get
$\alpha(u^{'}) = \beta(u^{'}) = \alpha(v^{'}) = \beta(v^{'}) \Rightarrow u^{'}, v^{'} \in po\ \Gamma dom(U, S)$, by (6)).
This results $(S, \Gamma, \leq_S) \subseteq po\ \Gamma dom(U, S)$ (10)
Therefore from the result in (7) and (10), we get
$po\ \Gamma dom(U, S) = (S, \Gamma, \leq_S)$. This completes the proof of theorem. □

Corollary 3.7. *Let $(U, \Gamma, \leq_U)$ be subpo Γ semigroup of po Γ semigroup$(S, \Gamma, \leq_s)$. A homomorphism $\alpha : (U, \Gamma, \leq_u) \longrightarrow (S, \Gamma, \leq_s)$ such that $\alpha(a^{'}\gamma b) = \alpha(a^{'})\gamma\ \alpha(b)$, for all $a^{'}, b \in U, \gamma \in \Gamma$ is epimorphism if and only if $dom(U, S) = S$.*

4. Conclusions

In this paper, we proved commutativity of a gamma dominion for a commutative gamma semigroup by using Isbell zigzag theorem for gamma semigroup. We also showed that some gamma semigroup identities are preserved under epimorphism. Homomorphism between two partially ordered gamma semigroups has been defined and used to extend epimorphism, dominion and zigzag theorem for partially ordered semigroups to partially ordered gamma semigroups. Finally, we proved subpartially ordered gamma semigroup is epimorphically embedded if partially ordered gamma semigroups identities are preserved under epimorphism.

Conflicts of Interest : The authors declare that there is no conflict of interest.

Data Availability : Not applicable

Acknowledgments : The authors are grateful to Haramaya University, Office of Vice President for Research Affairs, for the financial support. The authors would also like to thank the anonymous referees for their valuable comments and suggestions which helped in improving the quality of the paper.

References

1. W. Ashraf, *Epimorphically preserved semigroup identities*, J. Semigroup Theory Appl. **1** (2017).
2. W. Ashraf, S.A. Ahanger, A. Salam and N.M. Khan, *DominionS and zig-zag theorem for gamma semigroups*, J. Math. Extn. **15** (2021), 1-14.
3. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc. Providence, R.I., vol. I, 1961, vol. II, 1967.
4. J.M. Howie, *Fundamentals of semigroup theory*, London Math., Soc. Monographs New-series, 1995.
5. J.R. Isbell, *Epimorphisms and dominions*, Proc. of the conf. on Categorical Algebra, La Jolla, (1965), 232-246, Lange and Springer, Berlin, 1966.
6. N. Kehayopulu, *On ordered Gamma semigroups*, Scientiae Mathematicae Japonicae **71** (2010), 179-185.
7. N. Kehayopulu, *On prime, weakly prime ideals in po Gamma semigroups*, Lobachevskii J. Math. **30** (2009), 257–262.
8. Y.I. Kwon, and S.K. Lee, *The weakly semiprime ideals of PO Gamma Semigroups*, Korean J. Math. **5** (1997), 135–139.
9. H. Rasouli and A.R. Shabani, *On gamma acts over gamma semigroups*, European J. pure and Appl. Math. **10** (2017), 739-748.
10. N.K. Saha, *On Gamma semigroup II*, Bull. Cal. Math. Soc. **79** (1987), 331-335.
11. N.K. Saha, *On Gamma semigroup III*, Bull. Cal. Math. Soc. **80** (1988), 1-12.
12. M.K. Sen and N.K. Saha, *On Gamma semigroup I*, Bull. Cal. Math. Soc. **78** (1986), 180-186.
13. M.K. Sen, *On Gamma semigroup*, Proc. of the Int. conf. on Alg. and its appl., Decker Publication, New York, **311**, 1981.
14. X. Shi, Z. Liu, F. Wang and S. Bulman-Fleming, *Indecomposable, projective and flat S-posets*, Comm. Algebra **33** (2005), 235–251.
15. N. Sohail, and L. Tart, *Dominions, zig zags and epimorphisms for partially ordered semigroup*, Acta et Commen. Univer. Tart. de Math. **18** (2014), 81–91.
16. A. Sumanto, Solikhin, and S. Hariyanto, *Gamma Semigroup generated by a semigroup*, J. of Physics, Conference Series **1** (2020), 012042.

Phool Miyan received Ph.D. Degree in Mathematics in 2014 from Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India. Received M.Phil. Degree and M.Sc. Degree in Mathematics in 2009 and 2006 from Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India. He started his teaching career in 2014 as an Assistant Professor in the Department of Mathematics, Gagan College of Management and Technology, Aligarh, India. He jointed as an Assistant Professor in the Department of Mathematics Haramaya University, Ethiopia in 2016. Currently he is working as an Assistant Professor in the same University in Ethiopia. His Area of Research is Near Ring Theory, Ring Theory and Semigroup Theory.

Department of Mathematics, College of Natural and Computational Sciences, Haramaya University, P. O. Box 138, Dire Dawa, Ethiopia.
e-mail: phoolmiyan83@gmail.com

Seleshi Demie received Ph.D. Degree in Applied Mathematics from Andhra University, India in 2015 and M.Sc. Degree in Mathematics from Addis Ababa University, Ethiopia in 1999. He is currently working as an Assistant Professor at Haramaya University, Department of Mathematics. His Area of Research are Operations Research and Optimization, Mathematical Modeling, Algebra (Semigroup Theory and Near Ring Theory) and Mathematics Education.

Department of Mathematics, College of Natural and Computational Sciences, Haramaya University, P. O. Box 138, Dire Dawa, Ethiopia.
e-mail: seleshidemie@gmail.com

Gezehegn Terefe received B.Sc. Degree in Mathematics in 2017 from Department of Mathematics, Wachemo University, Ethiopia. Very recently in May 2022 he successfully defended his M.Sc. Thesis and Awarded his M.Sc. Degree in Mathematics from the Department of Mathematics, Haramaya University, Ethiopia. His Area of Research is Semigroup Theory.

Department of Mathematics, College of Natural and Computational Sciences, Haramaya University, P. O. Box 138, Dire Dawa, Ethiopia.
e-mail: gezehegn43@gmail.com