Commun. Korean Math. Soc. **38** (2023), No. 3, pp. 741–754
https://doi.org/10.4134/CKMS.c220080
pISSN: 1225-1763 / eISSN: 2234-3024

EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR A CLASS OF HAMILTONIAN STRONGLY DEGENERATE ELLIPTIC SYSTEM

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Abstract. In this paper, we study the existence and nonexistence of solutions for a class of Hamiltonian strongly degenerate elliptic system with subcritical growth

$$\begin{cases} -\Delta_\lambda u - \mu v = |v|^{p-1}v & \text{in } \Omega, \\ -\Delta_\lambda v - \mu u = |u|^{q-1}u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q > 1$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$. Here Δ_λ is the strongly degenerate elliptic operator. The existence of at least a nontrivial solution is obtained by variational methods while the nonexistence of positive solutions are proven by a contradiction argument.

1. Introduction

In the past years the study on the existence and nonexistence of solutions of the following Lane-Emden elliptic system

$$\begin{cases} -\Delta u = |v|^{p-1}v & \text{in } \Omega, \\ -\Delta v = |u|^{q-1}u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p, q > 1$ and Ω is a bounded subset of $\mathbb{R}^N$, $N \geq 3$ has been considered by many authors. If $p, q > 0$ and satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > 1,$$

the so-called subquadratic case. In this case, the existence results have been established in [5, 7, 11]. In the superquadratic but subcritical case, i.e., $p, q > 0$

Received March 24, 2022; Revised February 22, 2023; Accepted March 16, 2023.
2020 *Mathematics Subject Classification.* Primary 58B34, 58J42, 81T75.
Key words and phrases. Hamiltonian elliptic system, variational methods, strongly degenerate, existence and nonexistence.

and satisfy

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

we can refer to [10, 12, 13, 15]. On the contrary, in the critical case

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N},$$

nonexistence results have been proved in [20, 26].

For more general nonlinearities, the system (1) has also received great interest, and related results can be seen in [6, 8, 24] and the references therein.

In this paper, we study the existence and nonexistence of solutions for a class of Hamiltonian strongly degenerate elliptic system has form

$$\begin{cases} -\Delta_\lambda u - \mu v = |v|^{p-1}v & \text{in } \Omega, \\ -\Delta_\lambda v - \mu u = |u|^{q-1}u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p, q > 1$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, and μ is a positive constant. Here

$$\Delta_\lambda u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\lambda_j^2(x)\frac{\partial u}{\partial x_j}\right), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

is the strongly degenerate elliptic operator which was first introduced by Franchi and Lanconelli [14], and reconsidered in [16] by Kogoj and Lanconelli under the additional assumption that the operator is homogeneous of degree two with respect to a group dilations in $\mathbb{R}^N$. This Δ_λ-Laplace operator contains many degenerate elliptic operators such as the Grushin type operator

$$G_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha > 0,\ (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

and the strongly degenerate operator $P_{\alpha,\beta}$ in [25] of the form

$$P_{\alpha,\beta} = \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z,$$

with $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ $(N_i \geq 1, i = 1, 2, 3)$, $\alpha, \beta > 0$ are two positive constants. The operator Δ_λ belongs to the class of degenerate elliptic operators which has received considerable attention over the years. For some elementary properties, typical examples and recent results of the operator Δ_λ, we refer to the papers [1–4, 9, 19, 21–23] and a recent survey paper [17].

In this note, we are interested in the subcritical growth, i.e., $p, q > 1$ and satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{Q-2}{Q}, \tag{3}$$

where $Q > 4$ (the number Q is defined in Subsection 2.1 below).

The main results of this paper are the following theorems.

Theorem 1.1. *Assume* (3) *holds. Then, there exists* $\mu_0 > 0$ *such that for all* $\mu \in (0, \mu_0)$*, the problem* (2) *admits at least two weak solutions, where at least one is nontrivial.*

Theorem 1.2. *Assume* (3) *holds. Then the problem* (2) *has no positive solutions for any* $\mu > \mu_1$*, where* μ_1 *is the first eigenvalue of* $-\Delta_\lambda$ *on* Ω *with homogeneous Dirichlet boundary condition.*

Remark 1.3. The results presented here seems to be new in the case strongly degenerate elliptic operator Δ_λ, and our results complement the existing results in [1, 4, 9, 21–23].

The plan of the paper is as follows. In Section 2, we present some preliminary results which will be used later on. In Section 3 we prove the main results, we prove the existence of a nontrivial solution in Theorem 1.1 in Subsection 3.1, while in Subsection 3.2 we prove the nonexistence in Theorem 1.2.

2. Preliminary results

2.1. Δ_λ-Laplace operator

We recall the functional setting in [16]. We consider the operator of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2 \partial_{x_i}),$$

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \dots, N$. Here the functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of class $C^1(\mathbb{R}^N \setminus \prod)$ for $i = 1, \dots, N$, where $\prod = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\}$. As in [16] we assume that λ_i satisfy the following properties:

(1) $\lambda_1(x) \equiv 1$, $\lambda_i(x) = \lambda_i\ (x_1, \dots, x_{i-1})$, $i = 2, \dots, N$;

(2) For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, \dots, N$, where
$$x^* = (|x_1|, \dots, |x_N|) \text{ if } x = (x_1, \dots, x_N);$$

(3) There exists a constant $\rho \geq 0$ such that
$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_i(x)\ \forall k \in \{1, \dots, i-1\},\ i = 2, \dots, N,$$
and for every $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i \geq 0\ \forall i = 1, \dots, N\}$;

(4) There exists a group of dilations $\{\delta_t\}_{t>0}$
$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N,\ \delta_t(x) = \delta_t(x_1, \dots, x_N) = (t^{\epsilon_1} x_1, \dots, t^{\epsilon_N} x_N),$$
where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_N$, such that λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$, i.e.,
$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1} \lambda(x)\ \forall x \in \mathbb{R}^N,\ t > 0,\ i = 1, \dots, N.$$
This implies that the operator Δ_λ is δ_t-homogeneous of degree two, i.e.,
$$\Delta_\lambda(u(\delta_t(x))) = t^2 (\Delta_\lambda u)(\delta_t(x))\ \ \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e.,

$$Q := \epsilon_1 + \cdots + \epsilon_N.$$

2.2. Function spaces and setting functional

First, we recall some function spaces which will be used to study problem (2).

For $p \geq 1$, we denote by $\overset{\circ}{W}{}^{1,p}_\lambda(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\overset{\circ}{W}{}^{1,p}_\lambda} = \left(\int_\Omega |\nabla_\lambda u|^p dx\right)^{\frac{1}{p}},$$

where $\nabla_\lambda u = (\lambda_1 \partial_{x_1} u, \ldots, \lambda_N \partial_{x_N} u)$. We define $W^{2,p}_\lambda(\Omega)$ as the space of all functions u such that

$$u \in L^p(\Omega),\ \lambda_i(x)\frac{\partial u}{\partial x_i} \in L^p(\Omega),\ \lambda_i(x)\frac{\partial}{\partial x_i}\left(\lambda_j(x)\frac{\partial u}{\partial x_j}\right) \in L^p(\Omega),\ i,j=1,\ldots,N,$$

with the norm

$$\|u\|_{W^{2,p}_\lambda} = \left(\int_\Omega \left[|u|^p + |\nabla_\lambda u|^p + \sum_{i,j=1}^N \left|\lambda_i(x)\frac{\partial}{\partial x_i}(\lambda_j(x)\frac{\partial u}{\partial x_j})\right|^p\right] dx\right)^{\frac{1}{p}}.$$

We see that $W^{2,p}_\lambda(\Omega)$ and $\overset{\circ}{W}{}^{1,p}_\lambda(\Omega)$ are Banach spaces. When $p = 2$, the spaces $W^{2,2}_\lambda(\Omega)$ and $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$ are Hilbert spaces with the following inner products

$$\begin{aligned}(u,v)_{W^{2,2}_\lambda} &= (u,v)_{L^2} + \sum_{i=1}^N (\lambda_i \frac{\partial u}{\partial x_i}, \lambda_i \frac{\partial v}{\partial x_i})_{L^2} \\ &\quad + \sum_{i,j=1}^N \left(\lambda_i \frac{\partial}{\partial x_i}(\lambda_j \frac{\partial u}{\partial x_j}), \lambda_i \frac{\partial}{\partial x_i}(\lambda_j \frac{\partial v}{\partial x_j})\right)_{L^2},\end{aligned}$$

and

$$(u,v)_{\overset{\circ}{W}{}^{1,2}_\lambda} = \sum_{i=1}^N \left(\lambda_i \frac{\partial u}{\partial x_i}, \lambda_i \frac{\partial v}{\partial x_i}\right)_{L^2},$$

respectively. The following useful embedding was established in [16].

Lemma 2.1 ([16, Proposition 3.2])**.** *The embedding*

$$\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \hookrightarrow L^{2^*_\lambda}(\Omega), \quad \textit{where}\ \ 2^*_\lambda = \frac{2Q}{Q-2},$$

is continuous. Moreover, the embedding

$$\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \hookrightarrow L^\gamma(\Omega)$$

is compact for every $\gamma \in [1, 2^*_\lambda)$.

We also have the following lemma.

Lemma 2.2 ([4, Lemma 2.2]). *The embedding* $W^{2,2}_\lambda(\Omega) \cap \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \hookrightarrow L^\gamma(\Omega)$ *is continuous with* $1 \leq \gamma \leq \frac{2Q}{Q-4}$ *and* $Q > 4$.

We consider the operator

$$-\Delta_\lambda : W^{2,2}_\lambda(\Omega) \cap \overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \to L^2(\Omega),$$

and set $A = -\Delta_\lambda$, then by Lemma 2.1, A is a linear, positive, self-adjoint operator with compact inverse. Consequently, there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\varphi_j \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega),\ j = 1, 2, \ldots$ of the operator A with eigenvalues

$$0 < \mu_1 \leq \mu_2 < \cdots \quad \text{and} \quad \mu_j \to +\infty \text{ as } j \to +\infty.$$

We denote, $E^s = D(A^s)$, $s > 0$, with the inner products

$$(u, v)_{E^s} = \int_\Omega A^s u\, A^s v\, dx, \quad u, v \in E^s,$$

where

$$D(A^s) = \left\{ \varphi = \sum_{j=1}^\infty a_j \varphi_j,\ a_j \in \mathbb{R} \,|\, \sum_{j=1}^\infty \mu_j^s a_j^2 < +\infty \right\} \text{ and } A^s \varphi = \sum_{j=1}^\infty a_j \mu_j^{\frac{s}{2}} \varphi_j.$$

We notice that, as a consequence of Lemma 2.2 and the interpolation theorem, we have the following important embeddings which will be frequently used later.

Lemma 2.3 ([4, Lemma 2.3]). *The embeddings*

$$E^s \hookrightarrow L^\nu(\Omega) \quad \textit{and} \quad E^t \hookrightarrow L^\delta(\Omega)$$

are continuous if $\frac{1}{\nu} \geq \frac{1}{2} - \frac{s}{Q}$, $\frac{1}{\delta} \geq \frac{1}{2} - \frac{t}{Q}$, *and they are compact if these inequalities are strict.*

For $s > t > 0$ such that $s + t = 2$, we consider $E = E^s \times E^t$, the Hilbert space with the inner product

$$(z, \eta)_E = (u, \varphi)_{E^s} + (v, \psi)_{E^t} \quad \text{for } z = (u, v),\ \eta = (\varphi, \psi) \in E.$$

For simplicity, we denote the norm in E by $\| \cdot \|$.

Using arguments as in [12] we can obtain the orthogonal decomposition $E = E^+ \oplus E^-$, where

$$E^+ = \{(u, A^{s-t}u) : u \in E^s\} \quad \text{and} \quad E^- = \{(u, -A^{s-t}u) : u \in E^s\}. \tag{4}$$

For any $z \in E = E^+ \oplus E^-$, we have $z = z^+ + z^-$ with $z^+ \in E^+,\ z^- \in E^-$ and if we denote by $P^- : E \to E^-$ and $P^+ : E \to E^+$ the orthogonal projections, then by (4) we obtain

$$P^-(z) = z^- = \frac{1}{2}\big(u - A^{t-s}v, v - A^{s-t}u\big)$$

and

$$P^+(z) = z^+ = \frac{1}{2}\big(u + A^{t-s}v, v + A^{s-t}u\big),$$

where $z = (u, v) \in E$. Now, we define the functional $\Phi : E = E^s \times E^t \to \mathbb{R}$ associated to the problem (2) by

$$\Phi(u,v) = \int_\Omega A^s u\, A^t v\, dx - \frac{\mu}{2}\int_\Omega (|u|^2 + |v|^2)dx - \int_\Omega H(x,u,v)\, dx, \tag{5}$$

where

$$H(x,u,v) = \frac{1}{p+1}|v|^{p+1} + \frac{1}{q+1}|u|^{q+1}.$$

Here, we have selected $s > t > 0$ such that

$$s + t = 2, \quad q + 1 < \frac{2Q}{Q-2s} \quad \text{and} \quad p + 1 < \frac{2Q}{Q-2t}.$$

After some computations, we have

$$\int_\Omega A^s u\, A^t v\, dx = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2), \quad z = (u,v) \in E^s \times E^t.$$

Thus, we can write Φ in the form

$$\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \frac{\mu}{2}\int_\Omega (|u|^2 + |v|^2)dx - \int_\Omega H(x,u,v)\, dx.$$

We can see that Φ is well-defined on E and $\Phi \in C^1(E, \mathbb{R})$ with

$$\begin{aligned}\Phi'(u,v)(\varphi,\psi) = &\int_\Omega (A^s u\, A^t\psi + A^t v\, A^s\varphi)dx - \mu\int_\Omega (u\varphi + v\psi)dx \\ &- \int_\Omega [\varphi H_u(x,u,v) + \psi H_v(x,u,v)]\, dx.\end{aligned} \tag{6}$$

One can also see that a critical point of Φ is a weak solution of the problem (2) in the following sense.

Definition 1. We say that $z = (u,v) \in E = E^s \times E^t$ is a weak solution of (2) if for all $(\varphi,\psi) \in E^s \times E^t$ we have

$$\begin{aligned}&\int_\Omega A^s u\, A^t\psi\, dx - \mu\int_\Omega v\psi dx = \int_\Omega \psi H_v(x,u,v)\, dx, \quad \forall \psi \in E^t, \\ &\int_\Omega A^t v\, A^s\varphi\, dx - \mu\int_\Omega u\varphi dx = \int_\Omega \varphi H_u(x,u,v)\, dx, \quad \forall \varphi \in E^s.\end{aligned} \tag{7}$$

2.3. Critical point theory

Let E be a Hilbert space with the inner product $(\cdot,\cdot)_E$, and $\Phi : E \to \mathbb{R}$ be a functional. Assume that $E = E^+ \oplus E^-$, where E^+, E^- are both infinite dimensional subspaces of E. We assume further that, there exist sequences of finite dimensional subspaces $E_n^+ \subset E^+$, $E_n^- \subset E^-$ such that

$$E_1^\pm \subset E_2^\pm \subset \cdots \quad \text{and} \quad \overline{\bigcup_{n=1}^\infty E_n^\pm} = E^\pm.$$

Denote

$$E_n = E_n^+ \oplus E_n^- \quad \text{and} \quad \Phi_n = \Phi|_{E_n}.$$

We have

$$E_1 \subset E_2 \subset \cdots \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} E_n} = E.$$

Definition 2. Let E be a Hilbert space and let $\Phi \in C^1(E, \mathbb{R})$. We say that Φ satisfies the $(PS)^*$ condition with respect to the scale of subspaces $(E_n)_n$ if every sequence $\{z_n\}_n$ such that

$$z_n \in E_n, \quad |\Phi_n(z_n)| \leq C$$

and

$$|\langle \Phi_n'(z_n), \eta \rangle| \leq \epsilon_n \|\eta\|_E, \quad \forall \eta \in E_n, \ \epsilon_n \to 0,$$

contains a subsequence which converges to a critical point of Φ.

We shall use the following abstract critical point result in [18, Theorem 2].

Lemma 2.4. *Let $\Phi \in C^1(E, \mathbb{R})$ such that*

(i) *Φ has a local linking at the origin, i.e., for some $r > 0$*

$$\Phi(z) \geq 0 \text{ for } z \in E^+, \text{ and } \Phi(z) \leq 0 \text{ for } z \in E^- \text{ with } \|z\|_E \leq r;$$

(ii) *Φ maps bounded sets into bounded sets*;

(iii) *$\Phi(z) \to -\infty$ as $\|z\| \to \infty$, $z \in E_n^+ \oplus E^-$ for every $n \in \mathbb{N}$*;

(iv) *Φ satisfies the $(PS)^*$ condition with respect to the scale of subspaces $(E_n)_n$.*

Then Φ has at least two critical points.

3. Proof of the main results

3.1. Existence of nontrivial solutions

In this subsection, we prove the existence of at least two weak solutions to the system (2). We first check the condition (i) of Lemma 2.4 via the following lemma.

Lemma 3.1. *Assume* (3) *holds. Then there exists a $\mu_* > 0$ such that for all $\mu \in (0, \mu_*)$, the functional Φ has a local linking at the origin.*

Proof. For $z = (u, v) \in E^+$, by the Poincaré inequality for the operator A^s we have

$$\|A^s u\|_{L^2} \geq \mu_1^{\frac{s}{2}} \|u\|_{L^2} \quad \forall u \in E^s.$$

Thus, by Lemma 2.3 we find that

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\|z\|^2 - \frac{\mu}{2} \int_\Omega (|u|^2 + |v|^2) dx \\ &\quad - \frac{1}{p+1} \int_\Omega |v|^{p+1} dx - \frac{1}{q+1} \int_\Omega |u|^{q+1} dx \end{aligned}$$

$$\geq \frac{1}{2}\|z\|^2 - \frac{\mu}{2}\left(\frac{1}{\mu_1^s}\|u\|_{E^s}^2 + \frac{1}{\mu_1^t}\|v\|_{E^t}^2\right) - C\left(\|u\|_{E^s}^{q+1} + \|v\|_{E^t}^{p+1}\right)$$

$$\geq \left(\frac{1}{2} - \frac{\mu}{2\min\{\mu_1^s, \mu_1^t\}}\right)\|z\|^2 - C\|z\|^\tau, \tag{8}$$

where $\tau > 2$. Thus, let $\mu_* = \min\{\mu_1^s, \mu_1^t\}$, then there exists a constant $r > 0$ such that

$$\Phi(z) \geq 0 \quad \forall \mu \in (0, \mu_*),\ z \in E^+,\ \|z\| \leq r.$$

Next, for $z = (u, v) \in E^-$, we have

$$\Phi(z) = -\frac{1}{2}\|z\|^2 - \frac{\mu}{2}\int_\Omega (|u|^2 + |v|^2)dx$$
$$- \frac{1}{p+1}\int_\Omega |v|^{p+1}dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx$$
$$\leq -\frac{1}{2}\|z\|^2. \tag{9}$$

Thus, $\Phi(z) \leq 0$ for $z \in E^-$ and $\|z\| \leq r$. □

Next, we check the condition (ii) of Lemma 2.4.

Lemma 3.2. Φ *maps bounded sets into bounded sets.*

Proof. Let $M \subset E = E^s \times E^t$ be a bounded set. Then for all $z = (u, v) \in M$, there exists a positive constant $C > 0$ such that

$$\|u\|_{E^s} \leq C \quad \text{and} \quad \|v\|_{E^t} \leq C. \tag{10}$$

Thus, for all $z = (u, v) \in E$, from Lemma 2.3 and the Hölder inequality, we have

$$|\Phi(z)| \leq \int_\Omega |A^s u A^t v| dx + \frac{\mu}{2}\int_\Omega (|u|^2 + |v|^2)dx$$
$$+ \frac{1}{p+1}\int_\Omega |v|^{p+1}dx + \frac{1}{q+1}\int_\Omega |u|^{q+1}dx$$
$$\leq \|A^s u\|_{L^2}\|A^t v\|_{L^2} + \frac{\mu}{2}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2\right)$$
$$+ \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{1}{q+1}\|u\|_{L^{q+1}}^{q+1}$$
$$\leq \|u\|_{E^s}\|v\|_{E^t} + C\big(\|u\|_{E^s}^2 + \|v\|_{E^t}^2 + \|u\|_{E^s}^{q+1} + \|v\|_{E^t}^{p+1}\big). \tag{11}$$

Combining (10) and (11), we infer that

$$|\Phi(z)| \leq C \quad \forall z \in M.$$

Thus we obtain the conclusion of the lemma. □

We now check the condition (iii) of Lemma 2.4.

Lemma 3.3. *Let $n \in \mathbb{N}$ be fixed and $z_k \in E_n^+ \oplus E^-$ with E_n^+ is an n-dimensional subspace of E^+. Then*

$$\Phi(z_k) \to -\infty \quad \text{whenever} \quad \|z\| \to \infty.$$

Proof. By (4), for each $k \in \mathbb{N}$, $z_k \in E_n^+ \oplus E^-$ can be written as

$$z_k = (u_k, A^{s-t}u_k) + (v_k, -A^{s-t}v_k) \qquad \text{for } u_k \in E_n^s \text{ and } v_k \in E^s.$$

Hence, we find that

$$\begin{aligned}\Phi(z_k) &= \int_\Omega |A^s u_k|^2 dx - \int_\Omega |A^s v_k|^2 dx \\ &\quad - \frac{\mu}{2}\int_\Omega \left(|u_k + v_k|^2 + |A^{s-t}(u_k - v_k)|^2\right) dx \\ &\quad - \frac{1}{q+1}\int_\Omega |u_k + v_k|^{q+1} dx - \frac{1}{q+1}\int_\Omega |A^{s-t}(u_k - v_k)|^{p+1} dx \\ &= \|u_k\|_{E^s}^2 - \|v_k\|_{E^s}^2 - \frac{\mu}{2}\int_\Omega \left(|u_k + v_k|^2 + |A^{s-t}(u_k - v_k)|^2\right) dx \\ &\quad - \frac{1}{q+1}\int_\Omega |u_k + v_k|^{q+1} dx - \frac{1}{q+1}\int_\Omega |A^{s-t}(u_k - v_k)|^{p+1} dx. \end{aligned} \tag{12}$$

Note that

$$\begin{aligned}\|z_k\|_E^2 &= \|u_k + v_k\|_{E^s}^2 + \|A^{s-t}(u_k - v_k)\|_{E^t}^2 \\ &= \|u_k + v_k\|_{E^s}^2 + \|A^t A^{s-t}(u_k - v_k)\|_{L^2}^2 \\ &= \|u_k + v_k\|_{E^s}^2 + \|A^s(u_k - v_k)\|_{L^2}^2 \\ &= \|u_k + v_k\|_{E^s}^2 + \|u_k - v_k\|_{E^s}^2 \\ &= \|u_k\|_{E^s}^2 + 2(u_k, v_k)_{E^s} + \|v_k\|_{E^s}^2 + \|u_k\|_{E^s}^2 - 2(u_k, v_k)_{E^s} + \|v_k\|_{E^s}^2 \\ &= 2(\|u_k\|_{E^s}^2 + \|v_k\|_{E^s}^2) \to \infty. \end{aligned} \tag{13}$$

Case 1: If $\|u_k\|_{E^s} \le C$, then by (13) we obtain $\|v_k\| \to \infty$. Therefore, from (12) we easy obtain

$$\Phi(z_k) \to -\infty.$$

Case 2: If $\|u_k\|_{E^s} \to \infty$, then we estimate (for some $C, C_1, C_2 > 0$)

$$\begin{aligned}\int_\Omega |u_k + v_k|^{q+1} dx &\ge C\left(\int_\Omega |u_k + v_k|^2 dx\right)^{\frac{q+1}{2}} \\ &\ge C\|u_k + v_k\|_{L^2}^{q+1},\end{aligned}$$

and since $s > t$, we get

$$\begin{aligned}\int_\Omega |A^{s-t}(u_k - v_k)|^{p+1} dx &\ge C_1\left(\int_\Omega |A^{s-t}(u_k - v_k)|^2 dx\right)^{\frac{p+1}{2}} \\ &\ge C_2\|u_k - v_k\|_{L^2}^{p+1},\end{aligned}$$

where we used the Poincaré inequality $\|A^s u\|_{L^2(\Omega)} \geq \mu_1^{\frac{s}{2}} \|u\|_{L^2(\Omega)}$ for all $u \in E^s$. Thus, for some $\tau > 2$, we get

$$\begin{aligned}\Phi(z_k) &\leq \|u_k\|_{E^s}^2 - C\left(\|u_k+v_k\|_{L^2}^{q+1} + \|u_k-v_k\|_{L^2}^{p+1}\right)\\ &\leq \|u_k\|_{E^s}^2 - C\left(\|u_k+v_k\|_{L^2} + \|u_k-v_k\|_{L^2}\right)^{\tau}\\ &\leq \|u_k\|_{E^s}^2 - C\|u_k\|_{L^2}^{\tau}, \end{aligned} \tag{14}$$

where we used the inequality $\frac{1}{2}(h(x)+h(y)) \geq h(\frac{1}{2}(x+y))$ for the convex function $h(x) = x^{\tau}$, $\tau > 2$. And by the fact that E_n^s is a finite dimensional subspace, then two norms $\|\cdot\|_{E^s}$ and $\|\cdot\|_{L^2}$ are equivalent on E_n^s. Therefore, we can conclude from (14) that $\Phi(z_k) \to -\infty$ as $k \to \infty$. □

Finally, we check the condition (iv) of Lemma 2.4.

Lemma 3.4. *There exists $\bar{\mu} > 0$ such that for all $\mu \in (0, \bar{\mu})$ the functional Φ satisfies the $(PS)^*$-condition.*

Proof. Let (z_n) be a $(PS)^*$-sequence of Φ with respect to E_n, i.e., $z_n \in E_n$ and

$$|\Phi|_{E_n}(z_n)| \leq C \quad \text{and} \quad |(\Phi'|_{E_n}(z_n), w)| \leq \epsilon_n \|w\| \quad \forall w \in E_n, \tag{15}$$

where $\epsilon_n \to 0$ as $n \to \infty$. This implies that

$$\begin{aligned}\Phi(z_n) = &\int_\Omega A^s u_n A^t v_n dx - \frac{\mu}{2}\int_\Omega (|u_n|^2 + |v_n|^2) dx\\ &- \frac{1}{q+1}\int_\Omega |u_n|^{q+1} dx - \frac{1}{p+1}\int_\Omega |v_n|^{p+1} dx \to C,\end{aligned}$$

and

$$\begin{aligned}\Phi'(z_n)(\varphi,\psi) = &\int_\Omega (A^s u_n A^t \psi + A^t v_n A^s \varphi) dx - \mu\int_\Omega (u_n\varphi + v_n\psi) dx\\ &- \int_\Omega (|u_n|^{q-1}u_n\varphi + |v_n|^{p-1}v_n\psi) dx \leq \epsilon_n \|(\varphi,\psi)\|_E,\end{aligned} \tag{16}$$

where $w = (\varphi, \psi) \in E$.

(i) We first show $\{z_n\}$ is bounded in E. Indeed, it is easy see that

$$\begin{aligned}\Phi(z_n) - \frac{1}{2}(\Phi'(z_n), z_n) &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_\Omega |u_n|^{q+1} dx\\ &\quad + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |v_n|^{p+1} dx\\ &\leq C + \epsilon_n \|z_n\|.\end{aligned}$$

Moreover, since $p, q > 1$ we have

$$\int_\Omega |u_n|^{q+1} dx \leq C + \epsilon_n(\|u_n\|_{E^s} + \|v_n\|_{E^t}) \tag{17}$$

and

$$\int_\Omega |v_n|^{p+1}dx \le C + \epsilon_n(\|u_n\|_{E^s} + \|v_n\|_{E^t}). \tag{18}$$

Note that $A^{s-t}u_n \in E^t$, indeed, since $u_n \in E^s$ implies that $A^t(A^{s-t}u_n) = A^s u_n \in L^2$, thus $A^{s-t}u_n \in E^t$. Then we can choose $w = (0, A^{s-t}u_n)$ in (16), we obtain

$$\int_\Omega |A^s u_n|^2 dx \le \mu \int_\Omega |v_n A^{s-t}u_n| dx + \int_\Omega |v_n|^p |A^{s-t}u_n| dx + \epsilon_n \|A^{s-t}u_n\|_{E^t},$$

and hence

$$\|u_n\|^2_{E^s} \le \mu \|v_n\|_{L^2} \|A^{s-t}u_n\|_{L^2} + \left(\int_\Omega |v_n|^{p+1}dx\right)^{\frac{p}{p+1}} \left(\int_\Omega |A^{s-t}u_n|^{p+1}dx\right)^{\frac{1}{p+1}} + \epsilon_n\|u_n\|_{E^s}.$$

By Lemma 2.3 and (17), we obtain

$$\|u_n\|^2_{E^s} \le \frac{\mu}{\mu_1^2}\|v_n\|_{E^t}\|u_n\|_{E^s} + \big(C + \epsilon_n(\|u_n\|_{E^s} + \|v_n\|_{E^t})\big)^{\frac{p}{p+1}} \|u_n\|_{E^s} + \epsilon_n\|u_n\|_{E^s},$$

and thus

$$\|u_n\|_{E^s} \le \frac{\mu}{\mu_1^2}\|v_n\|_{E^t} + \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big)^{\frac{p}{p+1}} + C. \tag{19}$$

Analogously, we also get that

$$\|v_n\|_{E^t} \le \frac{\mu}{\mu_1^2}\|u_n\|_{E^s} + \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big)^{\frac{q}{q+1}} + C. \tag{20}$$

Combining (19) and (20) we obtain

$$\left(1 - \frac{\mu}{\mu_1^2}\right)(\|u_n\|_{E^s} + \|v_n\|_{E^t}) \le \epsilon_n\big(\|u_n\|_{E^s} + \|v_n\|_{E^t}\big)^\theta + C, \tag{21}$$

where $\theta \le 1$. We now choose $\bar\mu = \mu_1^2$, then it follows from (21) that $\|u_n\|_{E^s} + \|v_n\|_{E^t}$ is bounded for all $\mu \in (0, \bar\mu)$.

(ii) We now prove $\{z_n\}$ converges strongly in E. By the boundedness of $\{z_n\}$, without loss of generality, we may assume that, there is a subsequence of $\{z_n\}$ (not relabel) such that

$$z_n = (u_n, v_n) \rightharpoonup z = (u, v) \text{ weakly in } E = E^s \times E^t.$$

Note that the maps $A^s : E^s \to L^2(\Omega)$ and $A^{-t} : L^2(\Omega) \to E^t$ are continuous isomorphisms, then we obtain

$$A^s(u_n - u) \rightharpoonup 0 \text{ in } L^2(\Omega),$$
$$A^{s-t}(u_n - u) \rightharpoonup 0 \text{ in } E^t.$$

By the embeddings $E^t \hookrightarrow L^{p+1}(\Omega)$ and $E^t \hookrightarrow L^2(\Omega)$ are compact, it follows that

$$A^{s-t}(u_n - u) \to 0 \text{ strongly in } L^{p+1}(\Omega) \text{ and } L^2(\Omega).$$

Next, we choose $w = (0, A^{s-t}(u_n - u)) \in E^s \times E^t$ in (16),

$$\begin{aligned}
&\left| \int_\Omega \left(|A^s u_n|^2 - A^s u_n A^s u \right) dx \right| \\
&\le \mu \int_\Omega |v_n A^{s-t}(u_n - u)| dx + \int_\Omega |v_n|^p |A^{s-t}(u_n - u)| dx + \epsilon_n \|A^{s-t}(u_n - u)\|_{E^t} \\
&\le \mu \|v_n\|_{L^2} \|A^{s-t}(u_n - u)\|_{L^2} + \|v_n\|_{p+1}^p \|A^{s-t}(u_n - u)\|_{p+1} + \epsilon_n \|u_n - u\|_{E^s} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

This implies that

$$\int_\Omega |A^s u_n|^2 dx \to \int_\Omega |A^s u|^2 dx \text{ as } n \to \infty.$$

Similarly, we also get

$$\int_\Omega |A^t v_n|^2 dx \to \int_\Omega |A^t v|^2 dx \text{ as } n \to \infty.$$

Therefore, we can conclude that $z_n \to z$ strongly in E. The proof is complete. □

Proof of Theorem 1.1. Taking $\mu_0 = \min\{\bar{\mu}, \mu_*\}$ and using the above Lemmas 3.1-3.4 we see that all conditions of Lemma 2.4 are satisfied. Thus, the problem (2) admits at least two weak solutions, where at least one solution is nontrivial. □

3.2. Nonexistence of positive solutions

Proof of Theorem 1.2. Suppose on the contrary, (u, v) is a positive solution of (2). Denote by ϕ_1 is the first eigenfunction of $-\Delta_\lambda$ on Ω with homogeneous Dirichlet boundary condition. Multiplying equations of the system (2) by ϕ_1 and integrating by part, we have

$$\mu_1 \int_\Omega v\phi_1 dx = -\int_\Omega \Delta_\lambda v \phi_1 dx = \mu \int_\Omega u\phi_1 dx + \int_\Omega u^q \phi_1 dx, \tag{22}$$

$$\mu_1 \int_\Omega u\phi_1 dx = -\int_\Omega \Delta_\lambda u \phi_1 dx = \mu \int_\Omega v\phi_1 dx + \int_\Omega v^p \phi_1 dx. \tag{23}$$

Since $v > 0$, we deduce from the second integral identity (23) that

$$\int_\Omega v\phi_1 dx \le \frac{\mu_1}{\mu} \int_\Omega u\phi_1 dx.$$

Replacing this inequality into (22), we get

$$\mu \int_\Omega u\phi_1 dx + \int_\Omega u^q \phi_1 dx \le \frac{\mu_1^2}{\mu} \int_\Omega u\phi_1 dx,$$

this is equivalent to

$$\left(\frac{\mu_1^2-\mu^2}{\mu^2}\right)\int_\Omega u\phi_1 dx \geq 0,$$

which implies that $\mu_1 \geq \mu$. This contradicts with $\mu > \mu_1$. Thus, (2) has no nontrivial positive solution for any $\mu > \mu_1$. □

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