

GENERALIZED (C, r) -HANKEL OPERATOR AND (R, r) -HANKEL OPERATOR ON GENERAL HILBERT SPACES

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ABSTRACT. Hankel operators and their variants have abundant applications in numerous fields. For a non-zero complex number r , the r -Hankel operators on a Hilbert space \mathcal{H} define a class of one such variant. This article introduces and explores some properties of two other variants of Hankel operators namely k^{th} -order (C, r) -Hankel operators and k^{th} -order (R, r) -Hankel operators ($k \geq 2$) which are closely related to r -Hankel operators in such a way that a k^{th} -order (C, r) -Hankel matrix is formed from r^k -Hankel matrix on deleting every consecutive $(k-1)$ columns after the first column and a k^{th} -order (R, r^k) -Hankel matrix is formed from r -Hankel matrix if after the first column, every consecutive $(k-1)$ columns are deleted. For $|r| \neq 1$, the characterizations for the boundedness of these operators are also completely investigated. Finally, an appropriate approach is also presented to extend these matrices to two-way infinite matrices.

1. Introduction

Hankel operators are defined as operators having infinite Hankel matrices, i.e., matrices with entries depending only on the sum of the coordinates, with respect to some orthonormal basis. After Kroncker's significant work [4] on characterization of Hankel matrices of finite rank as those whose entries are Taylor coefficients of rational functions, this domain has found plenty of applications [9] in classical problems of analysis, such as moment problems, orthogonal polynomials, etc. The description of bounded Hankel operators by Nehari [8] in 1957 turned out to be fundamental in initiating the progressive period of the study of Hankel operators. Ever since the introduction of this notion, many variants (see [1, 3, 5, 6, 9] and the references therein) have also been studied due to their multitudinal applications in the study of smoothness of wavelets, perturbation and control theory, rational approximation and so on.

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A recent work in this direction by Mirotin et al. [7] introduces the notion of μ -Hankel operators on Hilbert spaces as follows: Let μ be a complex number, $\alpha = (\alpha_n)_{n \geq 0}$ be a complex sequence and \mathcal{H}_1 and \mathcal{H}_2 be separable Hilbert spaces. The operator $A_{\mu, \alpha} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is called μ -Hankel operator if for some orthonormal bases $(e_k)_{k \geq 0}$ and $(e'_j)_{j \geq 0}$ of \mathcal{H}_1 and \mathcal{H}_2 , respectively, the matrix $(a_{jk})_{k, j \geq 0}$ of this operator consists of elements of the form $a_{jk} = \mu^k \alpha_{j+k}$. Motivated by this study, Bhola and Gupta [2] introduced and studied two new classes of operators (C, r) -Hankel operators and (R, r) -Hankel operators on general Hilbert spaces that are closely related to Hankel operators in the sense that these classes result in μ -Hankel operator if alternate columns of one or alternate rows of the other are deleted. In this article, we generalize the concepts of (C, r) -Hankel operators and (R, r) -Hankel operators to k^{th} -order (C, r) -Hankel operators and k^{th} -order (R, r) -Hankel operators on general Hilbert spaces for any integer $k \geq 2$. Interesting results are established for these operators in terms of operator equations. For $|r| \neq 1$, the characterizations for the boundedness of these operators are completely obtained. Finally, a future scope of study is also presented to extend these matrices to two-way infinite matrices.

The following are some preliminaries used in this article: The symbols \mathbb{C} , \mathbb{Z} and \mathbb{N}_0 denote the set of all complex numbers, integers and non-negative integers, respectively. The symbol k is restricted for any integer greater than or equal to 2. A bounded linear operator \mathcal{T} on a Hilbert space \mathcal{H} is said to be Hilbert-Schmidt operator if the Hilbert-Schmidt norm $\|\mathcal{T}\|_{HS}^2 = \sum_n \|\mathcal{T}(u_n)\|^2 < \infty$ for any orthonormal basis (u_n) of \mathcal{H} , where $\|\cdot\|$ represents the norm of \mathcal{H} . A bounded operator \mathcal{T} on \mathcal{H} is said to be isometry if $\mathcal{T}^* \mathcal{T} = I_{\mathcal{H}}$, and unitary if \mathcal{T} is bijective and $\mathcal{T}^* \mathcal{T} = \mathcal{T} \mathcal{T}^* = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator on \mathcal{H} . Throughout the article, we restrict the symbols \mathcal{H}_1 and \mathcal{H}_2 for any separable Hilbert spaces. If $\mathcal{H}_1 = \mathcal{H}_2$, then it is denoted by \mathcal{H} . We denote by $(u_i)_{i \in \mathbb{N}_0}$ and $(v_i)_{i \in \mathbb{N}_0}$, the orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 , respectively. The symbols \mathcal{U}_1 and \mathcal{U}_2 denote the unilateral right shift operators on \mathcal{H}_1 and \mathcal{H}_2 , respectively and are defined as $\mathcal{U}_1(u_i) = u_{i+1}$ and $\mathcal{U}_2(v_i) = v_{i+1}$ for all $i \in \mathbb{N}_0$.

2. The k^{th} -order (C, r) -Hankel operator and its properties

In this section, we introduce and study some properties of the k^{th} -order (C, r) -Hankel operator from a Hilbert space \mathcal{H}_1 to Hilbert space \mathcal{H}_2 , defined as under:

Definition. Let $k \geq 2$ be an integer, $r \in \mathbb{C}$ be a non-zero element and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a sequence of complex numbers. An operator \mathcal{T} from a Hilbert space \mathcal{H}_1 to Hilbert space \mathcal{H}_2 is said to be k^{th} -order (C, r) -Hankel operator if

$$\mathcal{T}(u_i) = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} v_j \text{ for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i \in \mathbb{N}_0}$ and $(v_i)_{i \in \mathbb{N}_0}$ are orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 , respectively. That is, for $i, j \in \mathbb{N}_0$, if $a_{i,j}$ is the $(i, j)^{th}$ -entry of the matrix representation of \mathcal{T} with respect to the orthonormal bases then

$$a_{i,j} = \langle \mathcal{T}(u_j), v_i \rangle = \left\langle \sum_{l=0}^{\infty} r^j \alpha_{j+kl} v_l, v_i \right\rangle = \sum_{l=0}^{\infty} r^j \alpha_{j+kl} \langle v_l, v_i \rangle = r^j \alpha_{j+ki},$$

and hence, the corresponding matrix is given as:

$$[\mathcal{T}] = \begin{bmatrix} \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & r^4\alpha_4 & \dots \\ \alpha_k & r\alpha_{k+1} & r^2\alpha_{k+2} & r^3\alpha_{k+3} & r^4\alpha_{k+4} & \dots \\ \alpha_{2k} & r\alpha_{2k+1} & r^2\alpha_{2k+2} & r^3\alpha_{2k+3} & r^4\alpha_{2k+4} & \dots \\ \alpha_{3k} & r\alpha_{3k+1} & r^2\alpha_{3k+2} & r^3\alpha_{3k+3} & r^4\alpha_{3k+4} & \dots \\ \alpha_{4k} & r\alpha_{4k+1} & r^2\alpha_{4k+2} & r^3\alpha_{4k+3} & r^4\alpha_{4k+4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

Remark 2.1. (A) A k^{th} -order (C, r) -Hankel operator becomes r^k -Hankel operator if after the first column, every consecutive $(k - 1)$ columns are deleted.

(B) For a non-zero complex number r and complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$, a k^{th} -order (C, r) -Hankel operator may not be bounded, in general. For example, take $r = 1 - 2i$, $\alpha_n = \frac{1}{\sqrt{n+1}}$ for all $n \in \mathbb{N}_0$ and $x = \sum_{n=0}^{\infty} \frac{1}{(1-2i)^n} u_n$. Then,

$$\|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2 = \sum_{n=0}^{\infty} \left| \frac{1}{(1-2i)^n} \right|^2$$

is finite, that is, $x \in \mathcal{H}$. If \mathcal{T} is a k^{th} -order (C, r) -Hankel operator on \mathcal{H} , then

$$\|\mathcal{T}(x)\|^2 = \sum_{j=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{1}{(1-2i)^n} r^n \alpha_{n+kj} \right|^2 = \sum_{j=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+kj+1}} \right|^2 \rightarrow \infty.$$

Therefore, \mathcal{T} is not bounded.

The above example shows that in general, a k^{th} -order (C, r) -Hankel operator may not be bounded. The following result presents a characterization for the boundedness of a k^{th} -order (C, r) -Hankel operator for $|r| < 1$.

Theorem 2.2. *Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a complex sequence. Then the k^{th} -order (C, r) -Hankel operator, $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{N}_0} |r|^{2j} |\alpha_{kn+j}|^2 < \infty$.*

Proof. Let $|r| < 1$. If \mathcal{T} is bounded, then there exists a positive constant C such that $\|\mathcal{T}(x)\|^2 \leq C\|x\|^2$ for every $x \in \mathcal{H}_1$. For each $j = 0, 1, \dots, k - 1$, take $x = u_j$, we get $\sum_{n \in \mathbb{N}_0} |r|^{2j} |\alpha_{kn+j}|^2 = \|\mathcal{T}(u_j)\|^2 \leq C\|u_j\|^2 = C$. Therefore, $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{N}_0} |r|^{2j} |\alpha_{kn+j}|^2$ is finite.

Conversely, suppose that $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{N}_0} |r|^{2j} |\alpha_{kn+j}|^2$ is finite. Consider

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} |\langle \mathcal{T}(u_i), v_l \rangle|^2 &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 (1 + |r^{2k}| + |r^{4k}| + \dots + |r^{2kn}|) \\ &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 (1 + |r|^{2k} + |r|^{4k} + \dots + |r|^{2kn}) \\ &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 (1 + |r|^{2k} + (|r|^{2k})^2 + \dots + (|r|^{2k})^n) \\ &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 \left(\frac{1 - |r|^{2k(n+1)}}{1 - |r|^{2k}} \right) \\ &\leq \left(\frac{1}{1 - |r|^{2k}} \right) \left(\sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 \right). \end{aligned}$$

This implies that $\sum_{i=0}^{\infty} \sum_{l=0}^{\infty} |\langle \mathcal{T}(u_i), v_l \rangle|^2 < \infty$. Therefore, \mathcal{T} is Hilbert-Schmidt operator and hence bounded. \square

A careful examination of the entries of a k^{th} -order (C, r) -Hankel matrix points out at a relation between the entries. The same is established via the operator equation given in the following result:

Theorem 2.3. *Let \mathcal{U}_1 and \mathcal{U}_2 be the unilateral right shift operators on \mathcal{H}_1 and \mathcal{H}_2 , respectively and r be a non-zero complex number. Then a bounded linear operator $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a k^{th} -order (C, r) -Hankel operator for a complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$ if and only if $\mathcal{T}\mathcal{U}_1^k = r^k \mathcal{U}_2^* \mathcal{T}$.*

Proof. Suppose $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a k^{th} -order (C, r) -Hankel operator for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. For each $i, l \in \mathbb{N}_0$,

$$\langle \mathcal{T}\mathcal{U}_1^k(u_i), v_l \rangle = \langle \mathcal{T}(u_{i+k}), v_l \rangle = r^{i+k} \alpha_{i+k+kl}$$

and

$$\begin{aligned} \langle r^k \mathcal{U}_2^* \mathcal{T}(u_i), v_l \rangle &= r^k \langle \mathcal{T}(u_i), \mathcal{U}_2(v_l) \rangle = r^k \langle \mathcal{T}(u_i), v_{l+1} \rangle \\ &= r^k r^i \alpha_{i+k+kl} = r^{i+k} \alpha_{i+k+kl}. \end{aligned}$$

Using the boundedness of \mathcal{T} , it follows that $\mathcal{T}\mathcal{U}_1^k = r^k \mathcal{U}_2^* \mathcal{T}$.

Conversely, let $\mathcal{T}\mathcal{U}_1^k = r^k \mathcal{U}_2^* \mathcal{T}$. We define a complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$ as $\alpha_n = (1/r)^j \langle \mathcal{T}(u_j), v_p \rangle$ if $n = kp + j$, $p \in \mathbb{N}_0$, $j \in \{0, 1, \dots, k-1\}$. Then, for all non-negative integers i, l such that $i = kp + j$, where $p \in \mathbb{N}$, $j \in \{0, 1, \dots, k-1\}$,

$$\begin{aligned} \langle \mathcal{T}(u_i), v_l \rangle &= \langle \mathcal{T}\mathcal{U}_1^k(u_{i-k}), v_l \rangle = \langle r^k \mathcal{U}_2^* \mathcal{T}(u_{i-k}), v_l \rangle = r^k \langle \mathcal{T}(u_{i-k}), \mathcal{U}_2(v_l) \rangle \\ &= r^k \langle \mathcal{T}(u_{i-k}), v_{l+1} \rangle = \dots = r^{2k} \langle \mathcal{T}(u_{i-2k}), v_{l+2} \rangle = \dots \\ &= r^{kp} \langle \mathcal{T}(u_j), v_{l+p} \rangle = r^j r^{kp} \alpha_{kp+kl+j} = r^i \alpha_{i+kl}. \end{aligned}$$

Hence, \mathcal{T} is a k^{th} -order (C, r) -Hankel operator for the sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. \square

3. The k^{th} -order (R, r) -Hankel operator and its properties

This section firstly introduces the k^{th} -order (R, r) -Hankel operator from a Hilbert space \mathcal{H}_1 to Hilbert space \mathcal{H}_2 and examines some of its properties. Meanwhile, the relationship between k^{th} -order (C, r) -Hankel operator and k^{th} -order (R, s) -Hankel operator corresponding to complex sequences $(\alpha_n)_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$, respectively, are investigated in terms of their adjoints.

Definition. Let $k \geq 2$ be an integer, r be a non-zero complex number and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a sequence of complex numbers. A linear operator \mathcal{T} from a Hilbert space \mathcal{H}_1 to Hilbert space \mathcal{H}_2 is said to be k^{th} -order (R, r) -Hankel operator if

$$\mathcal{T}(u_i) = \sum_{j=0}^{\infty} r^j \alpha_{ki+j} v_j \text{ for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i \in \mathbb{N}_0}$ and $(v_i)_{i \in \mathbb{N}_0}$ are orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 , respectively.

Observe that for $i, j \in \mathbb{N}_0$, the $(i, j)^{th}$ -entry of the matrix representation of \mathcal{T} with respect to the orthonormal bases is $b_{i,j} = \langle \mathcal{T}(u_j), v_i \rangle = r^j \alpha_{kj+i}$ and the corresponding matrix is given as:

$$[\mathcal{T}] = \begin{bmatrix} \alpha_0 & r\alpha_k & r^2\alpha_{2k} & r^3\alpha_{3k} & r^4\alpha_{4k} & \dots \\ \alpha_1 & r\alpha_{k+1} & r^2\alpha_{2k+1} & r^3\alpha_{3k+1} & r^4\alpha_{4k+1} & \dots \\ \alpha_2 & r\alpha_{k+2} & r^2\alpha_{2k+2} & r^3\alpha_{3k+2} & r^4\alpha_{4k+2} & \dots \\ \alpha_3 & r\alpha_{k+3} & r^2\alpha_{2k+3} & r^3\alpha_{3k+3} & r^4\alpha_{4k+3} & \dots \\ \alpha_4 & r\alpha_{k+4} & r^2\alpha_{2k+4} & r^3\alpha_{3k+4} & r^4\alpha_{4k+4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

Remark 3.1. (A) A k^{th} -order (R, r) -Hankel operator becomes r -Hankel operator if after the first row, every consecutive $(k - 1)$ rows are deleted.

(B) For a non-zero complex number r and complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$, a k^{th} -order (R, r) -Hankel operator, in general, may not be bounded. For an example, take $r = 1 - i$, $\alpha_n = \frac{1}{\sqrt{2n+1}}$ for all $n \in \mathbb{N}_0$ and $x = \sum_{n=0}^{\infty} \frac{1}{(1-i)^n} u_n$. Then, $\|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2 = \sum_{n=0}^{\infty} \left| \frac{1}{(1-i)^n} \right|^2$ is finite, that is, $x \in \mathcal{H}$. If \mathcal{T} is a k^{th} -order (R, r) -Hankel operator on \mathcal{H} , then $\|\mathcal{T}(x)\|^2 = \sum_{j=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{1}{(1-i)^n} r^n \alpha_{kn+j} \right|^2 = \sum_{j=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{1}{\sqrt{2kn+2j+1}} \right|^2 \rightarrow \infty$. Therefore, \mathcal{T} is not bounded.

The next result characterizes the boundedness of k^{th} -order (R, r) -Hankel operators for $|r| < 1$.

Theorem 3.2. Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a complex sequence. Then the k^{th} -order (R, r) -Hankel operator, $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$.

Proof. Let $|r| < 1$ and \mathcal{T} be a bounded operator. Then there exists a positive constant C such that $\|\mathcal{T}(x)\|^2 \leq C\|x\|^2$ for every $x \in \mathcal{H}_1$. Taking in particular $x = u_0$, we get $\sum_{n \in \mathbb{N}_0} |\alpha_n|^2 = \|\mathcal{T}(u_0)\|^2 \leq C\|u_0\|^2 = C$.

Conversely, suppose that $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$. Consider

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} |\langle \mathcal{T}(u_i), v_i \rangle|^2 &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |\alpha_{kn+j}|^2 (1 + |r|^2 + |r^2|^2 + \cdots + |r^n|^2) \\ &= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |\alpha_{kn+j}|^2 (1 + |r|^2 + (|r|^2)^2 + \cdots + (|r|^2)^n) \\ &= \left(\sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |\alpha_{kn+j}|^2 \right) \left(\frac{1 - |r|^{2(n+1)}}{1 - |r|^2} \right) \\ &\leq \left(\frac{1}{1 - |r|^2} \right) \left(\sum_{j=0}^{k-1} \sum_{n=0}^{\infty} |\alpha_{kn+j}|^2 \right) \\ &= \left(\frac{1}{1 - |r|^2} \right) \left(\sum_{n=0}^{\infty} |\alpha_n|^2 \right). \end{aligned}$$

Using $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$, it follows that $\sum_{i=0}^{\infty} \sum_{l=0}^{\infty} |\langle \mathcal{T}(u_i), v_i \rangle|^2 < \infty$. Therefore, the operator \mathcal{T} is Hilbert-Schmidt and hence bounded. \square

Proposition 3.3. *Let r be a non-zero complex number and $(\alpha_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$ be a sequence. Then the adjoint of a bounded k^{th} -order (C, r) -Hankel operator, $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a k^{th} -order (R, s) -Hankel operator, S from \mathcal{H}_2 to \mathcal{H}_1 corresponding to the complex sequence $(\beta_n)_{n \in \mathbb{N}_0}$, where $s = \frac{1}{r^k}$ and $\beta_n = \overline{r^n \alpha_n}$ for each $n \in \mathbb{N}_0$.*

Proof. Let $i, j \in \mathbb{N}_0$. Evaluating

$$\langle \mathcal{T}^*(v_j), u_i \rangle = \langle v_j, \mathcal{T}(u_i) \rangle = \overline{\langle \mathcal{T}(u_i), v_j \rangle} = \overline{r^i \alpha_{i+kj}} = \overline{r^i} \overline{\alpha_{i+kj}}$$

and

$$\langle S(v_j), u_i \rangle = s^j \beta_{i+kj} = \left(\frac{1}{r^k} \right)^j \overline{r^{i+kj} \alpha_{i+kj}} = \overline{r^i} \overline{\alpha_{i+kj}}.$$

Hence, $\mathcal{T}^* = S$, where $s = \frac{1}{r^k}$ and $\beta_n = \overline{r^n \alpha_n}$ for each $n \in \mathbb{N}_0$. \square

We have seen in Theorem 2.3 that there is a characterization of k^{th} -order (C, s) -Hankel operator in terms of an operator equation. Using this characterization and the fact that a k^{th} -order (R, r) -Hankel operator can be obtained from a k^{th} -order (C, s) -Hankel operator by taking its adjoint, the following result presents a characterization of a k^{th} -order (R, r) -Hankel operator in terms of an operator equation:

Theorem 3.4. *Let \mathcal{U}_1 and \mathcal{U}_2 be the right shift operators on \mathcal{H}_1 and \mathcal{H}_2 , respectively. Let r be a non-zero complex number. Then a bounded operator $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a k^{th} -order (R, r) -Hankel operator for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$ if and only if $\mathcal{T}\mathcal{U}_1 = r(\mathcal{U}_2^*)^k \mathcal{T}$.*

Proof. Suppose that \mathcal{T} is a k^{th} -order (R, r) -Hankel operator for a complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. Using Proposition 3.3, it follows that $\mathcal{T} = \mathcal{C}^*$, where $\mathcal{C} : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is a k^{th} -order (C, s) -Hankel operator corresponding to the sequence $(\beta_n)_{n \in \mathbb{N}_0}$, where $s = (\frac{1}{r})^{\frac{1}{k}}$ and $\beta_n = (\frac{1}{r^n}) \overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Now, Theorem 2.3 gives $\mathcal{C}\mathcal{U}_2^k = s^k \mathcal{U}_1^* \mathcal{C}$. Taking adjoint on both sides, it follows that $(\mathcal{U}_2^*)^k \mathcal{C}^* = \overline{s^k} \mathcal{C}^* \mathcal{U}_1$. That is, $\mathcal{T}\mathcal{U}_1 = r(\mathcal{U}_2^*)^k \mathcal{T}$.

Conversely, if an operator \mathcal{T} is such that $\mathcal{T}\mathcal{U}_1 = r(\mathcal{U}_2^*)^k \mathcal{T}$. Then, by reversing the steps above and by using Theorem 2.3 and Proposition 3.3, we conclude that \mathcal{T} is a k^{th} -order (R, r) -Hankel operator for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. □

Theorem 3.5. *Let $k \geq 2$ be an integer, r be a non-zero complex number such that $|r| > 1$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a complex sequence. Then the following hold:*

- (A) *The k^{th} -order (C, r) -Hankel operator $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if $\sum_{n=0}^{\infty} |r|^{2n} |\alpha_n|^2 < \infty$.*
- (B) *The k^{th} -order (R, r) -Hankel operator $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{N}_0} |r|^{-(2j/k) - 2kn - 2j} |\alpha_{kn+j}|^2 < \infty$.*

Proof. Let $|r| > 1$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a complex sequence.

(A) Let $s = \frac{1}{r^k}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ be a sequence, where $\beta_n = \overline{r^n \alpha_n}$ for each $n \in \mathbb{N}_0$. The k^{th} -order (C, r) -Hankel operator \mathcal{T} is bounded if and only if \mathcal{T}^* is bounded. Using Proposition 3.3, it follows that the operator \mathcal{T}^* is k^{th} -order (R, s) -Hankel operator corresponding to the complex sequence $(\beta_n)_{n \in \mathbb{N}_0}$. Since $|s| < 1$, therefore, using Theorem 3.2, it is concluded that \mathcal{T}^* is bounded if and only if $\sum_{n=0}^{\infty} |\beta_n|^2 < \infty$, that is, $\sum_{n=0}^{\infty} |r|^{2n} |\alpha_n|^2 < \infty$.

(B) Let $s = (\frac{1}{r})^{\frac{1}{k}}$ and $\beta_n = (\frac{1}{r^n}) \overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Since $|r| > 1$, so $|s| < 1$. The k^{th} -order (R, r) -Hankel operator \mathcal{T} is bounded if and only if \mathcal{T}^* is bounded. Using Proposition 3.3, it follows that the operator \mathcal{T}^* is k^{th} -order (C, s) -Hankel operator corresponding to the complex sequence $(\beta_n)_{n \in \mathbb{N}_0}$. Since $|s| < 1$, therefore, using Theorem 2.2, it gives \mathcal{T}^* is bounded if and only if $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{N}_0} |s|^{2j} |\beta_{kn+j}|^2 < \infty$. □

4. Commutativity

In this section, commutativity of bounded k^{th} -order (C, r) -Hankel operators and bounded k^{th} -order (R, s) -Hankel operators on a Hilbert space \mathcal{H} are investigated. Moreover, it is proved that there does not exist any unitary operator in the class of k^{th} -order (C, r) -Hankel operator or in the class of k^{th} -order (R, r) -Hankel operator for any non-zero $r \in \mathbb{C}$.

Theorem 4.1. *Let r and s be non-zero complex numbers and $(\alpha_n)_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ be two complex sequences. Then the following hold:*

- (A) *The bounded k^{th} -order (C, r) -Hankel operators \mathcal{T} commutes with the bounded k^{th} -order (C, s) -Hankel operator \mathcal{V} on a Hilbert space \mathcal{H} if and only if*

$$\sum_{j=0}^{\infty} s^i \beta_{i+kj} r^j \alpha_{j+kl} = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{j+kl}$$

for all $i, l \in \mathbb{N}_0$, provided the series on both sides converge.

- (B) *The bounded k^{th} -order (C, r) -Hankel operator \mathcal{T} commutes with the bounded k^{th} -order (R, s) -Hankel operator \mathcal{V} on \mathcal{H} if and only if*

$$\sum_{j=0}^{\infty} s^i \beta_{ki+j} r^j \alpha_{j+kl} = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{kj+l}$$

for all $i, l \in \mathbb{N}_0$, provided the series on both sides converge.

Proof. (A) For each $i \in \mathbb{N}_0$, consider

$$\begin{aligned} \mathcal{T}\mathcal{V}(u_i) &= \mathcal{T} \left(\sum_{j=0}^{\infty} s^i \beta_{i+kj} u_j \right) = \sum_{j=0}^{\infty} s^i \beta_{i+kj} \mathcal{T}(u_j) \\ &= \sum_{j=0}^{\infty} s^i \beta_{i+kj} \left(\sum_{l=0}^{\infty} r^j \alpha_{j+kl} u_l \right) \\ (1) \quad &= \sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} s^i \beta_{i+kj} r^j \alpha_{j+kl} \right) u_l. \end{aligned}$$

Similarly, we obtain that

$$(2) \quad \mathcal{V}\mathcal{T}(u_i) = \sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{j+kl} \right) u_l.$$

Since $(u_i)_{i \in \mathbb{N}_0}$ is an orthonormal basis for \mathcal{H} , therefore, using equations (1) and (2), it follows that \mathcal{T} and \mathcal{V} commute if and only if

$$\sum_{j=0}^{\infty} s^i \beta_{i+kj} r^j \alpha_{j+kl} = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{j+kl}$$

for all $i, l \in \mathbb{N}_0$.

- (B) For each $i \in \mathbb{N}_0$, evaluate

$$\mathcal{T}\mathcal{V}(u_i) = \mathcal{T} \left(\sum_{j=0}^{\infty} s^i \beta_{ki+j} u_j \right) = \sum_{j=0}^{\infty} s^i \beta_{ki+j} \mathcal{T}(u_j)$$

$$\begin{aligned}
 &= \sum_{j=0}^{\infty} s^i \beta_{ki+j} \left(\sum_{l=0}^{\infty} r^j \alpha_{j+kl} u_l \right) \\
 (3) \quad &= \sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} s^i \beta_{ki+j} r^j \alpha_{j+kl} \right) u_l.
 \end{aligned}$$

Similarly, it is obtained that

$$(4) \quad \mathcal{V}\mathcal{T}(u_i) = \sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{kj+l} \right) u_l.$$

Using equations (3) and (4), it follows that \mathcal{T} and \mathcal{V} commute if and only if

$$\sum_{j=0}^{\infty} s^i \beta_{ki+j} r^j \alpha_{j+kl} = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{kj+l}$$

for all $i, k \in \mathbb{N}_0$. □

The example below exhibits a pair of such commuting operators:

Example 4.2. If $r = s = \frac{1}{2}$, $\alpha_n = (\frac{1}{2})^n$ and $\beta_n = \frac{1^n}{2^{n+1}}$ for all $n \in \mathbb{N}_0$, then one can easily see that the k^{th} -order (C, r) -Hankel operator and k^{th} -order (C, s) -Hankel operator are bounded (using Theorem 2.2) and they satisfy the following expression:

$$\sum_{j=0}^{\infty} s^i \beta_{i+kj} r^j \alpha_{j+kl} = \sum_{j=0}^{\infty} r^i \alpha_{i+kj} s^j \beta_{j+kl}$$

for all $i, l \in \mathbb{N}_0$. Hence, these operators commute on \mathcal{H} .

Let $\mathcal{C}_{0,0}$ denote the set of all complex sequences whose only finitely many terms are non-zero.

Theorem 4.3. Let $r, s \in \mathbb{C} \setminus \{0\}$ and $\alpha, \beta \in \mathcal{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j \in \mathbb{N}_0}$ and $\beta = (\beta_j)_{j \in \mathbb{N}_0}$. Let n and m be the largest non-negative integers such that $\alpha_n \neq 0$ and $\beta_m \neq 0$. Then the k^{th} -order (R, r) -Hankel operator \mathcal{T} and k^{th} -order (R, s) -Hankel operator \mathcal{V} on a Hilbert space \mathcal{H} commute if and only if $n = m$, $r = s$ and there exists $\lambda \in \mathbb{C}$ such that $\beta_j = \lambda \alpha_j$ for all $j \in \mathbb{N}_0$.

Proof. Let if possible, $n \neq m$. Without loss of generality, we can assume that $n > m$. Let $n = kp + r_1$ and $m = kq + r_2$, where $p, q \in \mathbb{N}_0$ and $r_1, r_2 \in \{0, 1, \dots, k-1\}$. In this case, we will show that the operators do not commute. On the contrary, assume that \mathcal{T} and \mathcal{V} commute, that is, $\mathcal{T}\mathcal{V}(x) = \mathcal{V}\mathcal{T}(x)$ for all $x \in \mathcal{H}$. In particular, take $x = u_q$. Consider

$$\mathcal{T}\mathcal{V}(u_q) = \mathcal{T} \left(\sum_{j=0}^{\infty} s^q \beta_{kq+j} u_j \right) = \mathcal{T} \left(\sum_{j=0}^{r_2} s^q \beta_{kq+j} u_j \right)$$

$$\begin{aligned}
 &= \sum_{j=0}^{r_2} s^q \beta_{kq+j} \left(\sum_{i=0}^{\infty} r^j \alpha_{kj+i} u_i \right) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{r_2} s^q \beta_{kq+j} r^j \alpha_{kj+i} \right) u_i \\
 (5) \quad &= \sum_{i=0}^n \left(\sum_{j=0}^{r_2} s^q \beta_{kq+j} r^j \alpha_{kj+i} \right) u_i.
 \end{aligned}$$

Similarly, we can obtain that

$$(6) \quad \mathcal{VT}(u_q) = \sum_{i=0}^m \left(\sum_{j=0}^{n-kq} r^q \alpha_{kq+j} s^j \beta_{kj+i} \right) u_i.$$

Since the set $(u_j)_{j \in \mathbb{N}_0}$ is an orthonormal basis of \mathcal{H} , $\alpha_i = 0$ for every $i > n$ and $\beta_j = 0$ for every $j > m$, therefore, on comparing the coefficients of u_n, u_{n-1}, u_{n-2} successively to u_0 , we get $\alpha_n = 0$, a contradiction. Therefore, $n = m$.

Now, if $n = m$. Let $n = kp + r_1$, where $p \in \mathbb{N}_0$ and $r_1 \in \{0, 1, \dots, k - 1\}$. In particular, take $x = u_p$ in $\mathcal{TV}(x) = \mathcal{VT}(x)$, we have

$$(7) \quad \mathcal{TV}(u_p) = \mathcal{VT}(u_p).$$

Consider

$$\begin{aligned}
 \mathcal{TV}(u_p) &= \mathcal{T} \left(\sum_{j=0}^{\infty} s^p \beta_{kp+j} u_j \right) = \mathcal{T} \left(\sum_{j=0}^{r_1} s^p \beta_{kp+j} u_j \right) \\
 &= \sum_{j=0}^{r_1} s^p \beta_{kp+j} \left(\sum_{i=0}^{\infty} r^j \alpha_{kj+i} u_i \right) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{r_1} s^p \beta_{kp+j} r^j \alpha_{kj+i} \right) u_i \\
 (8) \quad &= \sum_{i=0}^n \left(\sum_{j=0}^{r_1} s^p \beta_{kp+j} r^j \alpha_{kj+i} \right) u_i.
 \end{aligned}$$

Similarly, we can obtain that

$$(9) \quad \mathcal{VT}(u_p) = \sum_{i=0}^n \left(\sum_{j=0}^{r_1} r^p \alpha_{kp+j} s^j \beta_{kj+i} \right) u_i.$$

Since the set $(u_j)_{j \in \mathbb{N}_0}$ is an orthonormal basis of \mathcal{H} and $\alpha_i = 0 = \beta_i$ for every $i > n$, therefore, on comparing the coefficients of u_n, u_{n-1}, u_{n-2} successively to u_0 , it follows that $s = r$ and $\beta_j = \lambda \alpha_j$ for all $0 \leq j \leq n$ where $\lambda = \frac{\beta_n}{\alpha_n}$. \square

As a consequence of this result and by using Proposition 3.3, we get the following result:

Corollary 4.4. *Let $r, s \in \mathbb{C} \setminus \{0\}$ and $\alpha, \beta \in \mathcal{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j \in \mathbb{N}_0}$ and $\beta = (\beta_j)_{j \in \mathbb{N}_0}$. Let n and m be the largest non-negative integers such that $\alpha_n \neq 0$ and $\beta_m \neq 0$. Then the k^{th} -order (C, r) -Hankel operator \mathcal{T} and k^{th} -order (C, s) -Hankel operator \mathcal{V} commute on Hilbert space*

\mathcal{H} if and only if $n = m$, $r^k = s^k$ and there exists $\lambda \in \mathbb{C}$ such that $s^j \beta_j = \lambda r^j \alpha_j$ for all $j \in \mathbb{N}_0$.

Proposition 4.5. *There does not exist any unitary k^{th} -order (C, r) -Hankel operator for any non-zero $r \in \mathbb{C}$.*

Proof. If possible, let there exist a unitary k^{th} -order (C, r) -Hankel operator \mathcal{T} for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. This implies that

$$(10) \quad \|\mathcal{T}(x)\|^2 = \|x\|^2 = \|\mathcal{T}^*(x)\|^2$$

for all $x \in \mathcal{H}$.

Case 1: If $|r| < 1$. For $x = u_0$ and u_k in equation (10), we get

$$\sum_{j=0}^{\infty} |\alpha_{kj}|^2 = 1 \text{ and } \sum_{j=1}^{\infty} |r|^{2k} |\alpha_{kj}|^2 = 1.$$

On solving simultaneously, we obtain that $|\alpha_0|^2 = 1 - \frac{1}{|r|^{2k}} < 0$, a contradiction.

Case 2: If $|r| > 1$. Using Proposition 3.3, it follows that \mathcal{T}^* is k^{th} -order (R, s) -Hankel operator corresponding to complex sequence $(\beta_n)_{n \in \mathbb{N}_0}$, where $s = \frac{1}{r^k}$ and $\beta_n = \overline{r^n \alpha_n}$ for each $n \in \mathbb{N}_0$. For $x = u_0$ in equation (10), we get

$$(11) \quad \sum_{j=0}^{\infty} |\beta_j|^2 = 1.$$

Now take $x = u_1$ in equation (10), we get

$$(12) \quad \sum_{j=0}^{\infty} |s|^2 |\beta_{k+j}|^2 = 1.$$

On solving equations (11) and (12), it follows that $\sum_{j=0}^{k-1} |\beta_j|^2 = 1 - \frac{1}{|s|^2} < 0$ (a contradiction), since $|r| > 1$ implies $|s| < 1$.

Case 3: If $|r| = 1$. For each $i \in \mathbb{N}_0$, take $x = u_i$ in equation (10), we get $\sum_{j=0}^{\infty} |\alpha_{kj}|^2 = 1$, $\sum_{j=0}^{\infty} |r|^2 |\alpha_{kj+1}|^2 = 1$, $\sum_{j=0}^{\infty} |r|^4 |\alpha_{kj+2}|^2 = 1, \dots$, so on. On solving these equations, we get $\alpha_i = 0$ for all $i \in \mathbb{N}_0$, a contradiction.

Hence, there does not exist any unitary k^{th} -order (C, r) -Hankel operator for any non-zero $r \in \mathbb{C}$. □

As a consequence of this result, we get the following results:

Corollary 4.6. *There does not exist any unitary k^{th} -order (R, r) -Hankel operator for any non-zero $r \in \mathbb{C}$.*

Corollary 4.7. *Let r be a non-zero complex number. Then the following hold:*

- (A) *If $|r| < 1$, then there does not exist any isometric k^{th} -order (C, r) -Hankel operator.*
- (B) *If $|r| > 1$, then there does not exist any isometric k^{th} -order (R, r) -Hankel operator.*

5. Future scope of study

In the previous sections, we have studied several properties of k^{th} -order (C, r) -Hankel operators and k^{th} -order (R, r) -Hankel operators from a Hilbert space \mathcal{H}_1 to a Hilbert space \mathcal{H}_2 for non-zero complex number r and complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. Also, we have seen that the following one-way infinite matrix represents the matrix representation of (C, r) -Hankel operator, \mathcal{T} with respect to the orthonormal bases:

$$(13) \quad [\mathcal{T}] = \begin{bmatrix} \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & r^4\alpha_4 & \dots \\ \alpha_k & r\alpha_{k+1} & r^2\alpha_{k+2} & r^3\alpha_{k+3} & r^4\alpha_{k+4} & \dots \\ \alpha_{2k} & r\alpha_{2k+1} & r^2\alpha_{2k+2} & r^3\alpha_{2k+3} & r^4\alpha_{2k+4} & \dots \\ \alpha_{3k} & r\alpha_{3k+1} & r^2\alpha_{3k+2} & r^3\alpha_{3k+3} & r^4\alpha_{3k+4} & \dots \\ \alpha_{4k} & r\alpha_{4k+1} & r^2\alpha_{4k+2} & r^3\alpha_{4k+3} & r^4\alpha_{4k+4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

This concept of operator matrix can be generalized to form two-way infinite matrix in the following manner:

$$(14) \quad \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \frac{1}{r^3}\alpha_{-3k-3} & \frac{1}{r^2}\alpha_{-3k-2} & \frac{1}{r}\alpha_{-3k-1} & \alpha_{-3k} & r\alpha_{-3k+1} & r^2\alpha_{-3k+2} & r^3\alpha_{-3k+3} & \dots \\ \dots & \frac{1}{r^3}\alpha_{-2k-3} & \frac{1}{r^2}\alpha_{-2k-2} & \frac{1}{r}\alpha_{-2k-1} & \alpha_{-2k} & r\alpha_{-2k+1} & r^2\alpha_{-2k+2} & r^3\alpha_{-2k+3} & \dots \\ \dots & \frac{1}{r^3}\alpha_{-k-3} & \frac{1}{r^2}\alpha_{-k-2} & \frac{1}{r}\alpha_{-k-1} & \alpha_{-k} & r\alpha_{-k+1} & r^2\alpha_{-k+2} & r^3\alpha_{-k+3} & \dots \\ \dots & \frac{1}{r^3}\alpha_{-3} & \frac{1}{r^2}\alpha_{-2} & \frac{1}{r}\alpha_{-1} & \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & \dots \\ \dots & \frac{1}{r^3}\alpha_{k-3} & \frac{1}{r^2}\alpha_{k-2} & \frac{1}{r}\alpha_{k-1} & \alpha_k & r\alpha_{k+1} & r^2\alpha_{k+2} & r^3\alpha_{k+3} & \dots \\ \dots & \frac{1}{r^3}\alpha_{2k-3} & \frac{1}{r^2}\alpha_{2k-2} & \frac{1}{r}\alpha_{2k-1} & \alpha_{2k} & r\alpha_{2k+1} & r^2\alpha_{2k+2} & r^3\alpha_{2k+3} & \dots \\ \dots & \frac{1}{r^3}\alpha_{3k-3} & \frac{1}{r^2}\alpha_{3k-2} & \frac{1}{r}\alpha_{3k-1} & \alpha_{3k} & r\alpha_{3k+1} & r^2\alpha_{3k+2} & r^3\alpha_{3k+3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix},$$

where $(\alpha_n)_{n \in \mathbb{Z}}$ is a two-way complex sequence.

It can be observed that if $|r| \neq 1$ then this type of matrix induces only unbounded operators from \mathcal{H}_1 to \mathcal{H}_2 . So, there is another way to generalize k^{th} -order (C, r) -Hankel matrix which leads to a bounded operator from \mathcal{H}_1 to \mathcal{H}_2 if r is chosen appropriately. Consider the following matrix:

$$(15) \quad \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & r^3\alpha_{-3k-3} & r^2\alpha_{-3k-2} & r\alpha_{-3k-1} & \alpha_{-3k} & r\alpha_{-3k+1} & r^2\alpha_{-3k+2} & r^3\alpha_{-3k+3} & \dots \\ \dots & r^3\alpha_{-2k-3} & r^2\alpha_{-2k-2} & r\alpha_{-2k-1} & \alpha_{-2k} & r\alpha_{-2k+1} & r^2\alpha_{-2k+2} & r^3\alpha_{-2k+3} & \dots \\ \dots & r^3\alpha_{-k-3} & r^2\alpha_{-k-2} & r\alpha_{-k-1} & \alpha_{-k} & r\alpha_{-k+1} & r^2\alpha_{-k+2} & r^3\alpha_{-k+3} & \dots \\ \dots & r^3\alpha_{-3} & r^2\alpha_{-2} & r\alpha_{-1} & \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & \dots \\ \dots & r^3\alpha_{k-3} & r^2\alpha_{k-2} & r\alpha_{k-1} & \alpha_k & r\alpha_{k+1} & r^2\alpha_{k+2} & r^3\alpha_{k+3} & \dots \\ \dots & r^3\alpha_{2k-3} & r^2\alpha_{2k-2} & r\alpha_{2k-1} & \alpha_{2k} & r\alpha_{2k+1} & r^2\alpha_{2k+2} & r^3\alpha_{2k+3} & \dots \\ \dots & r^3\alpha_{3k-3} & r^2\alpha_{3k-2} & r\alpha_{3k-1} & \alpha_{3k} & r\alpha_{3k+1} & r^2\alpha_{3k+2} & r^3\alpha_{3k+3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

This is one such generalization of k^{th} -order (C, r) -Hankel matrix, where α_0 represents its $(0, 0)^{th}$ entry. Therefore, the two-way k^{th} -order (C, r) -Hankel

operator, $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ can be defined as:

$$\mathcal{T}(u_i) = \sum_{j=-\infty}^{\infty} r^{|i|} \alpha_{i+kj} v_j \text{ for all } i \in \mathbb{Z},$$

where $(u_i)_{i \in \mathbb{Z}}$ and $(v_i)_{i \in \mathbb{Z}}$ are orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 , respectively.

The following result describes a condition for the boundedness of two-way k^{th} -order (C, r) -Hankel operators:

Theorem 5.1. *Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n \in \mathbb{Z}}$ be a two-way complex sequence. Then the two-way k^{th} -order (C, r) -Hankel operator, $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if*

$$\sum_{j=0}^{k-1} \sum_{n \in \mathbb{Z}} |r|^{2j} |\alpha_{kn+j}|^2 < \infty.$$

Proof. Let $|r| < 1$. If a two way k^{th} -order (C, r) -Hankel operator, \mathcal{T} is bounded, then there exists a positive constant C such that $\|\mathcal{T}(x)\|^2 \leq C\|x\|^2$ for every $x \in \mathcal{H}_1$. Take in particular $x = u_0$, we get $\sum_{n \in \mathbb{Z}} |\alpha_{kn}|^2 = \|\mathcal{T}(u_0)\|^2 \leq C\|u_0\|^2 = C$. Again, taking $x = u_1, u_2, \dots, u_{k-1}$ successively, it follows that $|r|^2 \sum_{n \in \mathbb{Z}} |\alpha_{kn+1}|^2 \leq C, |r|^4 \sum_{n \in \mathbb{Z}} |\alpha_{kn+2}|^2 \leq C, \dots, |r|^{2k-2} \sum_{n \in \mathbb{Z}} |\alpha_{kn+k-1}|^2 \leq C$. Therefore, $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{Z}} |r|^{2j} |\alpha_{kn+j}|^2 < \infty$.

Conversely, suppose that $\sum_{j=0}^{k-1} \sum_{n \in \mathbb{Z}} |r|^{2j} |\alpha_{kn+j}|^2 < \infty$. This implies that for each $j \in \{0, 1, 2, \dots, k-1\}$, $\sum_{n \in \mathbb{Z}} |r|^{2j} |\alpha_{kn+j}|^2 \leq C_j$ for some constant C_j (dependent upon j) and hence, $\sum_{n \in \mathbb{Z}} |\alpha_{kn+j}|^2 \leq \frac{C_j}{|r|^{2j}}$. Let $(C_{i,j})_{i,j \in \mathbb{Z}}$ be the matrix representation of the operator \mathcal{T} . Consider

$$\begin{aligned} & \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |C_{i,j}|^2 \\ &= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 (1 + 2|r^k|^2 + 2|r^{2k}|^2 + 2|r^{3k}|^2 + \dots) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} |\alpha_{kn+j}|^2 (\dots + |r^{3k-j}|^2 + |r^{2k-j}|^2 + |r^{k-j}|^2 + |r^j|^2 + |r^{k+j}|^2 + |r^{2k+j}|^2 + \dots) \\ &= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 + 2|r|^{2k} \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 (1 + |r^{2k}| + |r^{2k}|^2 + \dots) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} |\alpha_{kn+j}|^2 ((\dots + |r^{3k-j}|^2 + |r^{2k-j}|^2 + |r^{k-j}|^2) + (|r^j|^2 + |r^{k+j}|^2 + |r^{2k+j}|^2 + \dots)) \\ &= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 + 2|r|^{2k} \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 \left(\frac{1}{1 - |r|^{2k}} \right) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} |\alpha_{kn+j}|^2 (|r^{k-j}|^2 (\dots + |r^{2k}|^2 + |r^{2k}| + 1) + |r^j|^2 (1 + |r^{2k}| + |r^{2k}|^2 + \dots)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 \left(1 + \frac{2|r|^{2k}}{1-|r|^{2k}}\right) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} (|r^{k-j}|^2 + |r^j|^2) |\alpha_{kn+j}|^2 \\
&\quad (1 + |r^{2k}| + |r^{2k}|^2 + \dots) \\
&= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 \left(1 + \frac{2|r|^{2k}}{1-|r|^{2k}}\right) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} (|r^{k-j}|^2 + |r^j|^2) |\alpha_{kn+j}|^2 \left(\frac{1}{1-|r|^{2k}}\right) \\
&= \sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 \left(\frac{1+|r|^{2k}}{1-|r|^{2k}}\right) + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} (|r^{k-j}|^2 + |r^j|^2) |\alpha_{kn+j}|^2 \left(\frac{1}{1-|r|^{2k}}\right) \\
&\leq \left(\frac{2}{1-|r|^{2k}}\right) \left(\sum_{n=-\infty}^{\infty} |\alpha_{kn}|^2 + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} (|r^{k-j}|^2 + |r^j|^2) |\alpha_{kn+j}|^2\right) \\
&\leq \left(\frac{2}{1-|r|^{2k}}\right) \left(\sum_{j=0}^{k-1} \sum_{n=-\infty}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 + \sum_{j=1}^{k-1} \sum_{n=-\infty}^{\infty} |\alpha_{kn+j}|^2\right) \\
&\leq \left(\frac{2}{1-|r|^{2k}}\right) \left(\sum_{j=0}^{k-1} \sum_{n=-\infty}^{\infty} |r|^{2j} |\alpha_{kn+j}|^2 + \sum_{j=1}^{k-1} \frac{C_j}{|r|^{2j}}\right).
\end{aligned}$$

Hence, $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |C_{i,j}|^2 < \infty$. Therefore, the two-way operator \mathcal{T} is Hilbert-Schmidt and hence bounded. \square

Similar generalization can be performed for defining two-way k^{th} -order (R, r) -Hankel operators from \mathcal{H}_1 to \mathcal{H}_2 and this can be done as follows:

$$\mathcal{T}(u_i) = \sum_{j=-\infty}^{\infty} r^{|i|} \alpha_{ki+j} v_j \text{ for all } i \in \mathbb{Z},$$

where $(u_i)_{i \in \mathbb{Z}}$ and $(v_i)_{i \in \mathbb{Z}}$ are orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 , respectively and its matrix representation is given as:

$$(16) \quad \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & r^3 \alpha_{-3k-3} & r^2 \alpha_{-2k-3} & r \alpha_{-k-3} & \alpha_{-3} & r \alpha_{k-3} & r^2 \alpha_{2k-3} & r^3 \alpha_{3k-3} & \dots \\ \dots & r^3 \alpha_{-3k-2} & r^2 \alpha_{-2k-2} & r \alpha_{-k-2} & \alpha_{-2} & r \alpha_{k-2} & r^2 \alpha_{2k-2} & r^3 \alpha_{3k-2} & \dots \\ \dots & r^3 \alpha_{-3k-1} & r^2 \alpha_{-2k-1} & r \alpha_{-k-1} & \alpha_{-1} & r \alpha_{k-1} & r^2 \alpha_{2k-1} & r^3 \alpha_{3k-1} & \dots \\ \dots & r^3 \alpha_{-3k} & r^2 \alpha_{-2k} & r \alpha_{-k} & \alpha_0 & r \alpha_k & r^2 \alpha_{2k} & r^3 \alpha_{3k} & \dots \\ \dots & r^3 \alpha_{-3k+1} & r^2 \alpha_{-2k+1} & r \alpha_{-k+1} & \alpha_1 & r \alpha_{k+1} & r^2 \alpha_{2k+1} & r^3 \alpha_{3k+1} & \dots \\ \dots & r^3 \alpha_{-3k+2} & r^2 \alpha_{-2k+2} & r \alpha_{-k+2} & \alpha_2 & r \alpha_{k+2} & r^2 \alpha_{2k+2} & r^3 \alpha_{3k+2} & \dots \\ \dots & r^3 \alpha_{-3k+3} & r^2 \alpha_{-2k+3} & r \alpha_{-k+3} & \alpha_3 & r \alpha_{k+3} & r^2 \alpha_{2k+3} & r^3 \alpha_{3k+3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

The following result illustrates a characterization for the boundedness of two-way (R, r) -Hankel operators and its proof is on the similar lines as Theorem 5.1:

Theorem 5.2. *Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n \in \mathbb{Z}}$ be two-way complex sequence. Then the two-way k^{th} -order (R, r) -Hankel operator $\mathcal{T} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded if and only if $\sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty$.*

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