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Cyclic Structure Jacobi Semi-symmetric Real Hypersurfaces in the Complex Hyperbolic Quadric

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Abstract. In this paper, we introduce the notion of cyclic structure Jacobi semi-symmetric real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$. We give a classifiction of when real hypersurfaces in the complex hyperbolic quadric Q^{m*} having $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector fields have cyclic structure Jacobi semi-symmetric tensor.

1. Introduction

The complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, which is a complex hypersurface in indefinite complex hyperbolic space $\mathbb{C}H^m_1$ (see Reckziegel [16], Romero [17], [18], and Smyth [19]), is an example of a Hermitian symmetric space with rank 2 of noncompact type. Montiel-Romero [10] proved that the complex hyperbolic quadric Q^{m*} can be immersed in the indefinite complex hyperbolic space $\mathbb{C}H^{m+1}_1(-c)$, $c > 0$, by interchanging the Kähler metric with its opposite. If we change the Kähler metric of $\mathbb{C}P^{m+1}_{m-s}$ with its opposite, we have that Q^m_{m-s} endowed with its opposite metric $g' = -g$ is also an Einstein hypersurface of $\mathbb{C}H^{m+1}_{s+1}(-c)$. When $s = 0$, we know that $(Q^m_m, g' = -g)$ can be regarded as the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, which is immersed in the indefinite complex

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hyperbolic quadric $\mathbb{C}H_1^{m+1}(-c)$, $c > 0$ as a complex Einstein hypersurface.

The structure Jacobi operator R_ξ of real hypersurfaces M in Hermitian symmetric spaces $\bar{M}$ is defined by $R_\xi(X) = R(X,\xi)\xi$, where $R(X,Y)Z$ denotes the curvature tensor of M induced from the curvature tensor $\bar{R}(X,Y)Z$ of $\bar{M}$ for any tangent vector fields X, Y and Z to M. The structure Jacobi operator R_ξ in the complex 2-plane Grassmannians $G_2(C^{m+2})$, complex hyperbolic 2-plane Grassmannian $G_2^*(C^{m+2})$, complex quadrics Q^m, or complex hyperbolic quadric Q^{m*}, all in the class of Hermitian symmetric spaces, has been investigated by many geometers since the latter part of 20th century(see Berndt-Suh [1], Besse [2], Jeong-Suh [4], Jeong-Pak-Suh [5], Pérez [12], Pérez-Suh [14], and Szabo [28]).

Related to the structure Jacobi operator above, many geometers have considered the notions of the *parallel structure Jacobi operator* $\nabla R_\xi = 0$ or *of Codazzi type* $(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X$ on Riemannian manifolds, where ∇ denotes the induced connection on M from the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ (Suh [23] and [25], Pérez-Santos-Suh [15], Suh-Pérez-Woo [27]). These notions generalize the notions of parallel shape operator, $\nabla S = 0$, or parallel Ricci tensor, $\nabla Ric = 0$, from Suh [21] and [22] respectively. In particular, for real hypersurfaces in the complex projective space $\mathbb{C}P^m$ the notion of cyclic parallel structure Jacobi operator (also known as a generalized Killing structure Jacobi operator) was considered by Pérez-Santos [13], and in quaternionic projective space $\mathbb{H}P^m$ it was considered by Pérez [12].

A nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called *singular* if it is tangent to more than one maximal flat in Q^{m*}. Since the complex hyperbolic quadric Q^{m*} is a Hermitian symmetric space with rank 2, there are two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*}:

(a) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \{W \mid AW = W\}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal.*

(b) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic.*

When we consider a real hypersurface M in the complex hyperbolic quadric Q^{m*}, the unit normal vector field N of M in Q^{m*} can be either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [1], [22] and [24]). In the first case where M has an $\mathfrak{A}$-isotropic unit normal vector field N, Suh [24] considered the geometry of isometric Reeb flow in the complex hyperbolic quadric Q^{m*}, and classified this problem as follows:

Theorem A. *Let M be a real hypersurface of the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to a tube around a totally geodesic k-dimensional complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k^*} or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

In the second case of an $\mathfrak{A}$-principal unit normal vector field N, without the assumption of constant mean curvature Berndt-Suh (see [1]) introduced the following:

Theorem B. *Let M be a connected orientable real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M is a contact hypersurface if and only if M is locally congruent to one of the following hypersurfaces:*

(i) *the tube of radius r around the Hermitian symmetric space Q^{m-1^*} which is imbedded in Q^{m*} as a totally geodesic complex hypersurface,*

(ii) *a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},*

(iii) *the tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.*

In light of Theorems A and B, we define a real hypersurface in the complex hyperbolic quadric Q^{m*} to be *cyclic structure Jacobi semi-symmetric* if

$$\mathfrak{S}_{X,Y,Z}(R(X,Y)R_\xi)(Z) = 0$$

for any vector fields X, Y and $Z \in T_zM$, $z \in M$. This is a natural generalization of the notion of being structure Jacobi semi-symmetric: $R(X,Y)R_\xi(Z) = 0$ or of being structure Jacobi symmetric: $\nabla_X R_\xi = 0$, for any tangent vector fields X, Y and Z to M in Q^{m*}. By the first Bianchi identity the cyclic structure Jacobi semi-symmetric tensor is given by

$$\mathfrak{S}_{X,Y,Z}R(X,Y)R_\xi(Z) = 0.$$

The notion of a cyclic structure Jacobi semi-symmetric tensor on a Riemanian manifold and its physical meaning is due to Chaubey-Suh-De [3]. Moreover, for real hypersurfaces in the complex quadric Q^m, Suh [26] has given a classification of cyclic structure Jacobi semi-symmetric tensors. Now in this paper, in the complex hyperbolic quadric Q^{m*} we can assert the following

Main Theorem 1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-principal unit singular normal vector field. There does not exist a Hopf real hypersurface in Q^{m*} satisfying the cyclic structure Jacobi semi-symmetric operator.*

Now at each point $z \in M$ let us consider the maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM for $z \in M$. In the case that the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$ of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C} = \text{Span}\{\xi\}$ becomes $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and

AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Using Theorems A and B, and our Main Theorem 1, we give the following classification of Hopf real hypersurfaces in the complex hyperbolic quadric Q^{m*} with semi-symmetric structure Jacobi operator:

Main Theorem 2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 4$, with $\mathfrak{A}$-isotropic singular unit normal vector field. If it satisfies the cyclic structure Jacobi semi-symmetric, then M is locally congruent to a tube of radius r over a k-dimensional complex hyperbolic space $\mathbb{C}H^k$ which can be immersed as a totally geodesic in Q^{2k^*}, $m = 2k$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

Let us consider for $m = 3$ in Main Theorem 2. Then we can assert the following:

Corollary 2.1. *The structure Jacobi operator R_ξ for a Hopf real hypersurface M^5 in Q^{3^*} with $\mathfrak{A}$-isotropic unit normal vector field always satisfies the cyclic semi-symmetric property. That is,*

$$\mathfrak{S}_{X,Y,Z} R(X,Y)R_\xi(Z) = 0$$

for any tangent vector fields X, Y and $Z \in T_zM$, $z \in M$.

This paper is composed as follows: In Section 2 we give some basic material about the complex hyperbolic quadric Q^{m*}, including its Riemannian curvature tensor and a description of its singular vectors for $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field. Apart from the complex structure J there is another distinguished geometric structure on Q^{m*}, namely a parallel rank two vector bundle $\mathfrak{A}$ which covers an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*}. A maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*} is determined by one of these real structures A.

In Section 3, we study the geometry of this subbundle $\mathcal{Q}$ for real hypersurfaces in Q^{m*} and the equation of Codazzi from the curvature tensor of the complex hyperbolic quadric Q^{m*} and some important formulas from the complex conjugation A of M in Q^{m*}.

In Section 4, in order to prove our Main Theorem 1 for an $\mathfrak{A}$-principal normal vector field, the first step is to derive the structure Jacobi tensor from the equation of Gauss for real hypersurfaces M in Q^{m*}, and next by using the assumption of cyclic structure Jacobi semi-symmetric for an $\mathfrak{A}$-principal normal vector field we will get some useful formulas like Lemma 4.1 and Proposition 4.2. As a final proof of Main Theorem 1, we will prove that a contact real hypersurface in Q^{m*}, which is a tube over an m-dimensional real hyperbolic space $\mathbb{R}H^m$ in Q^{m*}, over a complex hyperbolic quadric Q^{m-1^*} in Q^{m*}, or a horosphere in Q^{m*} does not admit the cyclic structure Jacobi semi-symmetric with distributions $\xi \oplus T_\lambda = JV(A)$ and $T_\mu \oplus N = V(A)$, where $\lambda = 0$ and $\mu = \frac{2}{\alpha}$, and $T_zQ^{m*} = V(A) \oplus JV(A)$, $z \in Q^{m*}$.

In Section 5, we prove our Main Theorem 2. The first part of this proof is to give some crucial equations from the cyclic structure Jacobi semi-symmetric tensor for an $\mathfrak{A}$-isotropic unit normal vector field. Then in the middle part of the proof we will devote ourselves to the study of important formulas which can be derived from the cyclic structure Jacobi semi-symmetric tensor. Moreover, in the proof of our Main Theorem 2 we will use an important Lemma 5.1 which assures that $SA\xi = 0$ and $SAN = 0$ on the distribution $\mathcal{Q}^{\perp} = \text{Span}\,\{A\xi, AN\}$ for the complex conjugation A of T_zQ^{m*}, $z \in Q^{m*}$. Finally, the proof of our Main Theorem 2 can be divided into two cases according to the vanishing Reeb function $\alpha = 0$ or non-vanishing $\alpha \neq 0$.

2. The Complex Hyperbolic Quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric Q^{m*} different from the complex quadric Q^m. The results introduced in this section are from Berndt-Suh [1], Klein-Suh [6], and Suh [24].

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, i.e. the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m.

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface in the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [19, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, see [6] and [16]. Another related result by Smyth, [20, Theorem 1], which states that any complex hypersurface $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$. As Q^{1^*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}/SO_2$, and Q^{2^*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m+2,\mathbb{R}) \big| X^t \cdot s = -s \cdot X\}$$

(see [7, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} ,$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of dimension 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \,\Big|\, X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\} .$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} | sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \,\Big|\, X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} | sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \,\Big|\, X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X,Y) = \tfrac{1}{2}\,\mathrm{Tr}(Y^t \cdot X) = \mathrm{Tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m} .$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K .$$

As j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication by the unit complex number i with the application of the linear map

J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As for the complex quadric (again compare [1], [16] with [6] and [24]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays a fundamental role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} .$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant, and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathcal{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0 .$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the "circle of real structures"

$$\mathfrak{A}_{p_0} := \{\lambda\, A_0 | \lambda \in S^1\} .$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant, and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ resp. $JV(A)$ are the $(+1)$-eigenspace resp. the (-1)-eigenspace of A. For every unit vector $Z \in T_pQ^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [16, Proposition 3]. Here t is uniquely determined by Z. The vector Z is singular, i.e. contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z.

As for the complex quadric, the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X,Y)Z$ corresponds to $-[[X,Y],Z]$ for $X, Y, Z \in \mathfrak{m}$, see [8, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}
\bar{R}(X,Y)Z = &- g(Y,Z)X + g(X,Z)Y \\
&- g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ \\
&- g(AY,Z)AX + g(AX,Z)AY \\
&- g(JAY,Z)JAX + g(JAX,Z)JAY
\end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [16, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

As Nomizu [11, Theorem 15.3] has shown, there exists one and only one torsion-free covariant derivative $\bar{\nabla}$ on Q^{m*} so that the symmetric involutions $s_p : Q^{m*} \to Q^{m*}$ at $p \in Q^{m*}$ are all affine. $\bar{\nabla}$ is the *canonical covariant derivative* of Q^{m*}. With respect to $\bar{\nabla}$, the action of any member of G on Q^{m*} is also affine. Moreover, $\bar{\nabla}$ is the Levi-Civita connection corresponding to the Riemannian metric g, and therefore g is parallel with respect to $\bar{\nabla}$. Moreover, it is well-known that Q^{m*} becomes a Kähler manifold in this way, i.e. the complex structure J is also parallel. Finally, because the S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$ is $\mathrm{Ad}(G)$-invariant, it is also parallel with respect to the same covariant derivative $\bar{\nabla}$ induced by $\bar{\nabla}$ on $\mathrm{End}(TQ^{m*})$. Because the tangent line of the fiber of $\mathfrak{A}$ through some $A_p \in \mathfrak{A}$ is spanned by JA_p, this means precisely that for any section A of $\mathfrak{A}$ there exists a real-valued 1-form $q : TQ^{m*} \to \mathfrak{R}$ so that

$$\bar{\nabla}_v A = q(v) \cdot JA_p \quad \text{holds for } p \in Q^{m*},\ v \in T_pQ^{m*}. \tag{2.2}$$

3. Some General Equations

Let M be a real hypersurface in Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM.

The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define the maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1. (see [24]) *For each $z \in M$ in the complex hyperbolic quadric Q^{m*} we assert the following*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then we have

$$S\xi = \alpha\xi$$

with the differentiable Reeb function $\alpha = g(S\xi, \xi)$ on M, where S denotes the shape operator of M in $\mathbb{Q}^{m*}$. When we consider JX by the Kähler structure J on Q^{m*} for any vector field X on M in Q^{m*}, we may write

$$JX = \phi X + \eta(X)N$$

for a unit normal vector field N to M. Then we now consider the equation of Codazzi

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, Z)\\
&\quad = -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) - g(X, AN)g(AY, Z)\\
&\qquad + g(Y, AN)g(AX, Z) - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z).
\end{aligned}$$

Putting $Z = \xi$ we get

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&\quad = 2g(\phi X, Y) - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi)\\
&\qquad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi).
\end{aligned} \tag{3.1}$$

On the other hand, differentiating $S\xi = \alpha\xi$ implies

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&\quad = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X)\\
&\quad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y).
\end{aligned} \tag{3.2}$$

By comparing the previous two equations and putting $X = \xi$, we have the following

$$Y\alpha = (\xi\alpha)\eta(Y) + 2g(\xi, AN)g(Y, A\xi) - 2g(Y, AN)g(\xi, A\xi).$$

Inserting this formula into (3.2) gives

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&\quad = 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y)\\
&\qquad - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X)\\
&\qquad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

From this, together with (3.2), it follows that

$$\begin{aligned}
0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y)\\
& - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi)\\
& + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi)\\
& - 2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y)\\
& + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [16]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned}
\xi &= \sin(t)Z_2 - \cos(t)JZ_1,\\
AN &= \cos(t)Z_1 - \sin(t)JZ_2,\\
A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
\end{aligned}$$

By virtue of these formulas and using the orthonormality of the vectors Z_1, Z_2 in $V(A)$ we can assert that

$$g(A\xi, N) = g(AN, \xi) = 0.$$

That is, the vector field $A\xi$ is tangent to M in Q^{m*} (see also Lemma 3.2 in Suh [24]).

From the anti-commuting property between the Kähler structure and the complex conjugation A in $\mathfrak{A}$ it follows that $JA\xi = -AJ\xi = -AN$. From this, together with the above formula, and using $g(A\xi, N) = 0$ into the previous equation, we have the following

Lemma 3.2. *Let M be a Hopf hypersurface in Q^{m*} with (local) unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned}
0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) - 2g(X, AN)g(Y, A\xi)\\
& + 2g(Y, AN)g(X, A\xi) - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\}
\end{aligned}$$

holds for all vector fields X and Y on M.

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^{m*} induced from the curvature tensor $\bar{R}$ of Q^{m*} can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows: for any tangent vector fields X, Y and Z on M in Q^{m*}

$$\begin{aligned}(3.3)\quad R(X,Y)Z = &- g(Y,Z)X + g(X,Z)Y - g(JY,Z)(JX)^T + g(JX,Z)(JY)^T \\ &+ 2g(JX,Y)(JZ)^T - g(AY,Z)(AX)^T + g(AX,Z)(AY)^T \\ &- g(JAY,Z)(JAX)^T + g(JAX,Z)(JAY)^T \\ &+ g(SY,Z)SX - g(SX,Z)SY,\end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^{m*}.

Let $\{e_1, e_2, \cdots, e_{2m-1}, e_{2m} := N\}$ be a basis of the tangent vector space T_zQ^{m*} of Q^{m*} at $z \in Q^{m*}$. By definition, the Ricci operator of M in Q^{m*} is given by $\mathrm{Ric}(X) = \Sigma_{i=1}^{2m-1} R(X, e_i)e_i$. So from (3.3) it follows that

$$\begin{aligned}(3.4)\quad \mathrm{Ric}(X) = &-(2m-1)X + 3\eta(X)\xi + g(AN,N)(AX)^T - g(AX,N)(AN)^T \\ &+ g(JAN,N)(JAX)^T - g(JAX,N)(JAN)^T \\ &+ (\mathrm{Tr}\, S)SX - S^2X,\end{aligned}$$

where we have used some basic formulas induced from contracting of the curvature tensor in (3.3)(see Suh [24] and [27]).

In this paper, we consider the notion of *structure Jacobi semi-symmetric*

$$(R(X,Y)R_\xi)(Z) = 0,$$

which is weaker than the notion of parallel structure Jacobi, $\nabla_X R_\xi = 0$, where the curvature tensor $R(X,Y)Z$ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for any tangent vector fields X, Y and Z to M in the complex hyperbolic quadric Q^{m*}. Then by the Ricci formula and the first Bianchi identity on Riemannan manifolds the cyclic structure Jacobi semi-symmetric $\mathfrak{S}_{X,Y,Z}(R(X,Y)R_\xi)(Z) = 0$ implies

$$\begin{aligned}(3.5)\quad \mathfrak{S}_{X,Y,Z}R(X,Y)(R_\xi(Z)) &= R(X,Y)R_\xi(Z) + R(Y,Z)R_\xi(X) + R(Z,X)R_\xi(Y) \\ &= 0\end{aligned}$$

for any tangent vector fields X, Y and Z to M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the vector fields X, Y and Z. Hereafter, unless otherwise stated the equation is said to be *cyclic structure Jacobi semi-symmetric* or otherwise *cyclic semi-symmetric structure Jacobi operator* of M in Q^{m*}.

On the other hand, for a real structure A of Q^{m*} we decompose AX into its tangential and normal components given by

$$AX = BX + g(AX, N)N.$$

From this and the anti-commuting property between the complex structure J and the real structure A, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N. \tag{3.6}$$

In addition, from the expression of the vector fields $A\xi$ and N we obtain that $g(A\xi, N) = 0$, which means that the unit vector field $A\xi$ is tangent to M. Thus, by using the Gauss formula, $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$, we get

$$\begin{aligned}
\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\
&= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X\xi) - g(SX, A\xi)N \\
&= q(X)JA\xi + A(\nabla_X\xi + g(SX, \xi)N) - g(SX, A\xi)N \\
&= q(X)JA\xi + A\phi SX + g(SX, \xi)AN - g(SX, A\xi)N \\
&= q(X)\phi A\xi + q(X)g(A\xi, \xi)N + B\phi SX + g(\phi SX, AN)N \\
&\quad - g(SX, \xi)\phi A\xi - g(SX, \xi)g(A\xi, \xi)N - g(SX, A\xi)N \\
&= q(X)\phi A\xi + q(X)g(A\xi, \xi)N + B\phi SX \\
&\quad - g(A\xi, SX)N + g(A\xi, \xi)g(SX, \xi)N \\
&\quad - g(SX, \xi)\phi A\xi - g(SX, \xi)g(A\xi, \xi)N - g(SX, A\xi)N,
\end{aligned}$$

where we have used the formulas $(\bar{\nabla}_X A)Y = q(X)JAY$ and (3.6). From this, *by comparing the tangential and normal parts of both sides*, we can assert the following:

Lemma 3.3. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we obtain*

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi \tag{3.7}$$

and

$$q(X)g(A\xi, \xi) = 2g(SX, A\xi) \tag{3.8}$$

for any tangent vector field X on M.

4. Proof of Main Theorem 1 with $\mathfrak{A}$-principal Unit Normal Vector Field

Now in this section we consider only an $\mathfrak{A}$-principal unit normal vector field N for a real hypersurface M in the complex hyperbolic quadric Q^{m*} with cyclic structure Jacobi semi-symmetric operator. Then from (3.3) and using $AN = N$, $A\xi = -\xi$ and $AX = BX$ for an $\mathfrak{A}$-principal unit normal vector field, we have

$$R_\xi(Z) = -Z + 2\eta(Z)\xi + AZ + \alpha SZ - \alpha^2\eta(Z)\xi \tag{4.1}$$

for any tangent vector field Z on M in Q^{m*}. Now, let us use the assumption of cyclic semi-symmetric structure Jacobi operator, that is, $\mathfrak{S}_{X,Y,Z}(R(X,Y)R_\xi)Z = 0$. This is equivalent to

$$\mathfrak{S}_{X,Y,Z}R(X,Y)R_\xi(Z) = 0 \tag{4.2}$$

for any vector fields X, Y and $Z \in T_zM$, $z \in M$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the vector fields X, Y and Z. Then in order to find some geometric structures of the cyclic structure Jacobi semi-symmetric operator (4.2), we will consider some fundamental formulas as follows:

If we substitute the Reeb vector field ξ into (4.2) and using (4.1), then it follows that

$$\begin{aligned}
&\mathfrak{S}_{X,Y,Z}R(X,Y)R_\xi(Z)\\
&= -\mathfrak{S}_{X,Y,Z}R(X,Y)Z + (2-\alpha^2)\mathfrak{S}_{X,Y,Z}\eta(Z)R(X,Y)\xi\\
&\quad + \mathfrak{S}_{X,Y,Z}R(X,Y)AZ + \alpha\mathfrak{S}_{X,Y,Z}R(X,Y)SZ = 0,
\end{aligned} \tag{4.3}$$

where $S\xi = \alpha\xi$ by the assumption of M being Hopf and the Reeb function α is constant by Lemma 3.4. Here the term $R(X,Y)\xi$ in the right side of (4.3) is given by

$$\begin{aligned}
R(X,Y)\xi = &-\eta(Y)X + \eta(X)Y + \eta(Y)AX - \eta(X)AY\\
&+ \alpha\eta(Y)SX - \alpha\eta(X)SY.
\end{aligned} \tag{4.4}$$

Now let us check the equation (4.3). In order to do this, let us substitute (4.4) into (4.3) and use (4.4) in the obtained equation. Then it follows that

$$\begin{aligned}
&\eta(Z)R(X,Y)\xi + \eta(X)R(Y,Z)\xi + \eta(Y)R(Z,X)\xi\\
&\quad = -\eta(Z)\Big\{\eta(Y)X - \eta(X)Y + \eta(AY)(AX)^T - \eta(AX)(AY)^T\\
&\qquad - g(AY,N)(JAX)^T + g(AX,N)(JAY)^T - \alpha\{\eta(Y)SX - \eta(X)SY\}\Big\}\\
&\qquad - \eta(X)\Big\{\eta(Z)Y - \eta(Y)Z + \eta(AZ)(AY)^T - \eta(AY)(AZ)^T\\
&\quad - g(AZ,N)(JAY)^T + g(AY,N)(JAZ)^T - \alpha\{\eta(Z)SY - \eta(Y)SZ\}\Big\}\\
&\quad - \eta(Y)\Big\{\eta(X)Z - \eta(Z)X + \eta(AX)(AZ)^T - \eta(AZ)(AX)^T\\
&\quad - g(AX,N)(JAZ)^T + g(AZ,N)(JAX)^T - \alpha\{\eta(X)SZ - \eta(Z)SX\}\Big\}\\
&= 0.
\end{aligned} \tag{4.5}$$

Now we want to give the following lemma which will be useful in the proof of our Main Theorem 1. For an $\mathfrak{A}$-principal unit normal vector field we assert the following

Lemma 4.1. (see [9]) *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-principal singular normal vector field N. Then we obtain:*

(i) $AX = BX$

(ii) $A\phi X = -\phi AX$

(iii) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX, \xi)$

(iv) $ASX = SX - 2g(SX, \xi)\xi$ *and* $SAX = SX - 2\eta(X)S\xi$

for any $X \in T_zM$, $z \in M$, where $BX = (AX)^T$ denotes the tangential component of the vector field AX of M in Q^{m*}.

With Lemma 4.1 Lee-Suh [9] gave characterizations of Hopf hypersurfaces in Q^{m*} in terms of the singularity of the normal vector field are being investigated. Among them, as a new characterization of $\mathfrak{A}$-principal singular normal, they proved the following.

Proposition 4.2. (see [9]) *Let M be a Hopf real hypersurface in the complex quadric Q^{m*}, $m \geq 3$. Then M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is a contact real hypersurface with constant mean curvature and non-vanishing Reeb function in Q^{m*}.*

Now by virtue of (3.3) the 3rd term of (4.3) can be given by

$$
\begin{aligned}
(4.6)\qquad & \mathfrak{S}_{X,Y,Z} R(X,Y)AZ \\
&= -g(Y,AZ)X + g(X,AZ)Y - g(Z,AX)Y + g(Y,AX)Z \\
&\quad - g(X,AY)Z + g(Z,AY)X - g(JY,AZ)(JX)^T \\
&\quad + g(JX,AZ)(JY)^T - g(JZ,AX)(JY)^T + g(JY,AX)(JZ)^T \\
&\quad - g(JX,AY)(JZ)^T + g(JZ,AY)(JX)^T \\
&\quad - g(JX,Z)(JAY)^T + g(JY,Z)(JAX)^T + 2g(JX,Y)(JAZ)^T \\
&\quad - g(JY,X)(JAZ)^T + g(JZ,X)(JAY)^T + 2g(JY,Z)(JAX)^T \\
&\quad - g(JZ,Y)(JAX)^T + g(JX,Y)(JAZ)^T + 2g(JZ,X)(JAY)^T \\
&\quad - g(Y,Z)AX + g(X,Z)AY - g(Z,X)AY + g(Y,X)AZ \\
&\quad - g(X,Y)AZ + g(Z,Y)AX \\
&\quad + g(SY,AZ)SX - g(SX,AZ)SY + g(SZ,AX)SY \\
&\quad - g(SY,AX)SZ + g(SX,AY)SZ - g(SZ,AY)SX \\
&= 4\mathfrak{S}_{X,Y,Z}\, g(JX,Y)(JAZ)^T + \mathfrak{S}_{X,Y,Z}\, g((AS-SA)Y,Z)SX \\
&= 4\mathfrak{S}_{X,Y,Z}\, g(JX,Y)(JAZ)^T
\end{aligned}
$$

where in the first equality we have used $AJ = -JA$ and $A^2 = I$ in the following formula:

$$g(AY, AZ)AX = g(Y, Z)AX, \quad \text{and} \quad g(JAY, AZ)(JAX)^T = -g(JY, Z)(JAX)^T$$

and in the last equality we have used Lemma 4.1 (iv) $AS = SA$ for a Hopf hypersurface in Q^{m*}.

On the other hand, in order to calculate the 4th term $\alpha\mathfrak{S}_{X,Y,Z}R(X,Y)SZ$ let us check the following

$$\begin{aligned} R(X,Y)SZ = &- g(Y,SZ)X + g(X,SZ)Y - g(JY,SZ)(JX)^T + g(JX,SZ)(JY)^T \\ &+ 2g(JX,Y)(JSZ)^T - g(AY,SZ)(AX)^T + g(AX,SZ)(AY)^T \\ &- g(JAY,SZ)(JAX)^T + g(JAX,SZ)(JAY)^T \\ &+ g(SY,SZ)SX - g(SX,SZ)SY. \end{aligned}$$

Then it follows that

$$\begin{aligned} &\mathfrak{S}_{X,Y,Z}R(X,Y)SZ \\ &= -g(JAY,SZ)(JAX)^T - g(JAZ,SX)(JAY)^T \\ &\quad - g(JAX,SY)(JAZ)^T + g(JAX,SZ)(JAY)^T \\ &\quad + g(JAY,SX)(JAZ)^T + g(JAZ,SY)(JAX)^T \\ &\quad - g(JY,SZ)(JX)^T + g(JX,SZ)(JY)^T + 2g(JX,Y)(JSZ)^T \\ &\quad - g(JZ,SX)(JY)^T + g(JY,SX)(JZ)^T + 2g(JY,Z)(JSX)^T \\ &\quad - g(JX,SY)(JZ)^T + g(JZ,SY)(JX)^T + 2g(JZ,X)(JSY)^T. \end{aligned} \tag{4.7}$$

From (4.3), (4.6) and (4.7) it follows that

$$\begin{aligned} &\mathfrak{S}_{X,Y,Z}R(X,Y)R_\xi(Z) \\ &= 4\{g(\phi X,Y)\phi AZ + g(\phi Y,Z)\phi AX + g(\phi Z,X)\phi AY\} \\ &\quad - \alpha\{g(S\phi Z,AY)\phi AX + g(S\phi X,AZ)\phi AY + g(S\phi Y,AX)\phi AZ \\ &\quad - g(S\phi Y,AZ)\phi AX - g(S\phi Z,AX)\phi AY - g(S\phi X,AY)\phi AZ\} \\ &\quad - 2\{g(\phi Y,Z)\phi X + g(\phi Z,X)\phi Y + g(\phi X,Y)\phi Z\} \\ &\quad + 2\alpha\{g(\phi X,Y)\phi SZ + g(\phi Z,X)\phi SY + g(\phi Y,Z)\phi SX\} \\ &= 0. \end{aligned} \tag{4.8}$$

In the calculation of (4.8), by virtue of Proposition 4.2 we have used the following formula for a contact hypersurface M in Q^{m*}

$$\{g(JY,SZ) - g(JZ,SY)\}(JX)^T = \{g((S\phi + \phi S)Y,Z)\}(JX)^T = kg(\phi Y,Z)(JX)^T,$$

where $k = \frac{2}{\alpha}$ and

$$\begin{aligned}\{g(JAY,SZ) - g(JAZ,SY)\}\phi AX &= \{-g(JY,SAZ) + g(JZ,ASY)\}\phi AX\\ &= \{-g(\phi Y,SAZ) + g(\phi Z,ASY)\}\phi AX\\ &= \{-g(S\phi Y,AZ) + g(S\phi Z,AY\}\phi AX.\end{aligned}$$

It is well known that the class of contact hypersurfaces in Proposition 4.2 has 3-distinct constant principal curvatures such that α, $\lambda = 0$, $\mu = \frac{2}{\alpha}$ with multiplicities 1, $m-1$ and $m-1$ respectively (see Berndt-Suh [1]).

Now among the class of contact hypersurfaces, let us consider what kind of restrictions on the geometric structure for Hopf hypersurface in Q^{m*} permits a cyclic structure Jacobi semi-symmetric operator. Since we have assumed $m \geq 3$, we can consider $\dim T_\lambda \geq 2$, $\lambda = 0$. So let us check whether horosphere or what kind of a tube of radius r over $\mathbb{R}H^m$ or Q^{m-1^*} satisfy the cyclic structure semi-symmetric Jacobi operator.

This means that the cyclic structure Jacobi semi-symmetric holds for contact hypersurface in Proposition 4.2 with distributions $\xi\oplus T_\lambda = JV(A)$ and $N\oplus T_\mu = V(A)$, where $\lambda = 0$ and $\mu = \frac{2}{\alpha}$, and $T_zQ^{m*} = V(A)\oplus JV(A)$. By Lemma 4.1. (iv), we know that $T_\lambda \subset JV(A)$ and $T_\mu \subset V(A)$ for $\lambda = 0$ and $\mu = \frac{2}{\alpha}$.

Now let us consider that $X, Y \in T_\lambda \cap JV(A)$ and $Z \in T_\mu \cap V(A)$.

Then $SX = SY = 0$ and $SZ = \mu Z$, where $\mu = \frac{2}{\alpha}$. Moreover, $S\phi Z = 0$, $S\phi X = \mu\phi X$ and $S\phi Y = \mu\phi Y$. From the intersection of real structure with eigen space T_λ and T_μ in this case we know that $AX = -X$ and $AY = -Y$ and $AZ = Z$. Accordingly from (4.8) let us assume that the cyclic structure Jacobi semi-symmetric holds. Then it follows that

$$\begin{aligned}(4.9)\qquad &\mathfrak{S}_{X,Y,Z} R(X,Y)R_\xi(Z)\\ &= 4\{g(\phi X,Y)\phi Z - g(\phi Y,Z)\phi X - g(\phi Z,X)\phi Y\}\\ &\quad - 2\{g(\phi Y,Z)\phi X + g(\phi Z,X)\phi Y + g(\phi X,Y)\phi Z\}\\ &\quad + 2\alpha g(\phi X,Y)\mu\phi Z\\ &\quad - \alpha\Big[- \big\{g(S\phi Z,AY) - g(S\phi Y,AZ)\big\}\phi X\\ &\quad - \big\{g(S\phi X,AZ) - g(S\phi Z,AX)\big\}\phi Y\\ &\quad + \big\{g(S\phi Y,AX) - g(S\phi X,AY)\big\}\phi Z\Big]\\ &= 4\{g(\phi X,Y)\phi Z - g(\phi Y,Z)\phi X - g(\phi Z,X)\phi Y\}\\ &\quad - 2\{g(\phi Y,Z)\phi X + g(\phi Z,X)\phi Y + g(\phi X,Y)\phi Z\}\\ &\quad + 4g(\phi X,Y)\phi Z - 2\big\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(\phi X,Y)\phi Z\big\}\\ &\qquad = 2g(\phi X,Y)\phi Z - 8g(\phi Y,Z)\phi X - 8g(\phi Z,X)\phi Y\\ &\qquad = 0.\end{aligned}$$

Here if we put $Z = \phi Y$ and using $g(\phi Y, X) = 0$, $\phi Y \in T_\mu$, and $g(X, Y) = 0$ for $X, Y \in T_\lambda$, then $\phi X = 0$ for any $X \in T_\lambda$. This gives a contradiction, because $\dim T_\lambda = \dim T_\mu = m - 1 \geq 2$ for $m \geq 3$. So this kind of case can not be appeared.

Summing up all of the facts for real hypersurfaces M in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field and cyclic semi-symmetric structure Jacobi operator mentioned above, we can give a complete proof of our Main Theorem 1 in the introduction.

5. Proof of Main Theorem 2 with $\mathfrak{A}$-isotropic Unit Normal Vector Field

In Section 4, we gave a property of the cyclic structure Jacobi semi-symmetric real hypersurfaces in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field. Motivated by the result of Section 4, in this section we give a complete proof of our Main Theorem 2 for real hypersurfaces with *cyclic structure Jacobi semi-symmetric real hypersurfaces* when M has an $\mathfrak{A}$-isotropic unit normal vector field. Then by putting $Z = \xi$ in the assumption of cyclic structure Jacobi semi-symmetric it is given by

$$\mathfrak{S}_{X,Y,Z} R(X, Y) R_\xi(Z) = 0 \tag{5.1}$$

for any tangent vector fields X, Y and Z on M in Q^{m*}. Since we assumed that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in Section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of the $\mathfrak{A}$-isotropic unit normal vector field, the equation (3.3) gives $N = \frac{1}{\sqrt{2}} Z_1 + \frac{1}{\sqrt{2}} J Z_2$. This implies that

$$g(\xi, A\xi) = 0,\ g(\xi, AN) = 0,\ g(AN, N) = 0,\ g(A\xi, N) = 0,$$

and

$$g(JAN, \xi) = -g(AN, N) = 0.$$

Then the vector fields AN and $A\xi$ become tangent vector fields to M in Q^{m*}.

Since AN is a tangent vector field for an $\mathfrak{A}$-isotropic normal vector field, we know that

$$\nabla_Y(AN) = \{(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N\}^T = \{q(Y)JAN - ASY\}^T,$$

and

$$\nabla_Y(A\xi) = -q(Y)AN + B\phi SY + g(SY, \xi)AN,$$

where we have used (3.4) and (3.6), and $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^{m*}.

Now we assert an important lemma which gives a key role in the proof of our Main Theorem 2 as follows:

Lemma 5.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field N. Then we have the following:*

$$SA\xi = 0 \quad \text{and} \quad SAN = -S\phi A\xi = 0.$$

Proof. Let us denote by $\mathcal{Q}^{\perp} = \text{Span}\{A\xi, AN\}$, where $\mathcal{Q}$ is the maximal $\mathfrak{A}$-invariant subspace in the complex subbundle of $\mathcal{C}$. By differentiating $g(AN, N) = 0$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned}0 &= g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N)\\ &= g(q(X)JAN - ASX, N) - g(AN, SX)\\ &= -2g(ASX, N).\end{aligned}$$

Then $SAN = 0$. From (3.4), we obtain $AN = -\phi A\xi$. So, it implies that $S\phi A\xi = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have:

$$\begin{aligned}0 &= g(\bar{\nabla}_X(A\xi), N) + g(A\xi, \bar{\nabla}_X N)\\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X)\\ &= -2g(SA\xi, X)\end{aligned}$$

for any $X \in T_zM$, $z \in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

Putting $Y = Z = \xi$ in (3.3) and using $g(A\xi, \xi) = 0$, then the structure Jacobi operator for $\mathfrak{A}$-isotropic unit normal becomes

$$\begin{aligned}(5.2)\qquad R_\xi(X) &= R(X, \xi)\xi\\ &= -X + \eta(X)\xi + g(AX, \xi)A\xi + g(AX, N)AN + \alpha SX - \alpha^2\eta(X)\xi\\ &= -X + (1 - \alpha^2)\eta(X)\xi + \eta(AX)A\xi + g(AX, N)AN + \alpha SX.\end{aligned}$$

Then the assumption of the cyclic semi-symmetric structure Jacobi operator and the 1st Bianchi identity gives the following

$$\begin{aligned}(5.3)\qquad &\mathfrak{S}_{X,Y,Z}R(X, Y)R_\xi(Z)\\ &= \mathfrak{S}_{X,Y,Z}(1 - \alpha^2)\eta(Z)R(X, Y)\xi + \mathfrak{S}_{X,Y,Z}\eta(AZ)R(X, Y)A\xi\\ &\quad + \mathfrak{S}_{X,Y,Z}g(AZ, N)R(X, Y)AN + \alpha\mathfrak{S}_{X,Y,Z}R(X, Y)SZ.\end{aligned}$$

Now let us calculate the terms in the right side of (5.3) one by one as follows:

In order to do this, first by putting $Z = \xi$ in (3.3) and using M being Hopf, we have

$$\begin{aligned} R(X,Y)\xi = & -\eta(Y)X + \eta(X)Y - \eta(AY)(AX)^T + \eta(AX)(AY)^T \\ & + g(AY,N)(JAX)^T - g(AX,N)(JAY)^T \\ & + \alpha\Big\{\eta(Y)SX - \eta(X)SY\Big\}. \end{aligned}$$

From this, the 1st term in the right side of (5.3) becomes the following

$$\begin{aligned} & \mathfrak{S}_{X,Y,Z}(1-\alpha^2)\eta(Z)R(X,Y)\xi \\ & = (1-\alpha^2)\Big[\big\{-\eta(Z)\eta(AY) + \eta(Y)\eta(AZ)\big\}(AX)^T \\ & \quad + \big\{-\eta(X)\eta(AZ) + \eta(Z)\eta(AX)\big\}(AY)^T \\ & \quad + \big\{-\eta(Y)\eta(AX) + \eta(X)\eta(AY)\big\}(AZ)^T \\ & \quad + \big\{-\eta(Y)g(AZ,N) + \eta(Z)g(AY,N)\big\}(JAX)^T \\ & \quad + \big\{-\eta(Z)g(AX,N) + \eta(X)g(AZ,N)\big\}(JAY)^T \\ & \quad + \big\{-\eta(X)g(AY,N) + \eta(Y)g(AX,N)\big\}(JAZ)^T\Big]. \end{aligned} \tag{5.4}$$

On the other hand, by putting $Z = A\xi$ in (3.3) and using Lemma 5.1 and $g(JAY, A\xi) = 0$, the following term is given by

$$\begin{aligned} & \eta(AZ)R(X,Y)A\xi \\ & = \eta(AZ)\big\{-g(Y,A\xi)X + g(X,A\xi)Y - g(JY,A\xi)(JX)^T \\ & \quad + g(JX,A\xi)(JY)^T + 2g(JX,Y)(JA\xi)^T - g(AY,A\xi)(AX)^T + g(AX,A\xi)(AY)^T\big\} \\ & = -\eta(AZ)\big\{\eta(AY)X - \eta(AX)Y + g(Y,AN)(JX)^T \\ & \quad - g(X,AN)(JY)^T + 2g(JX,Y)(AN)^T + \eta(Y)(AX)^T - \eta(X)(AY)^T\big\}. \end{aligned}$$

From this, the 2nd term in the right side of (5.3) can be expressed in detail as follows:

$$\begin{aligned} & \mathfrak{S}_{X,Y,Z}\eta(AZ)R(X,Y)A\xi \\ & = -\eta(AZ)\big\{\eta(AY)X - \eta(AX)Y + g(Y,AN)(JX)^T \\ & \quad - g(X,AN)(JY)^T + 2g(JX,Y)(AN)^T + \eta(Y)(AX)^T - \eta(X)(AY)^T\big\} \\ & \quad - \eta(AX)\big\{\eta(AZ)Y - \eta(AY)Z + g(Z,AN)(JY)^T - g(Y,AN)(JZ)^T \\ & \quad + 2g(JY,Z)(AN)^T + \eta(Z)(AY)^T - \eta(Y)(AZ)^T\big\} \\ & \quad - \eta(AY)\big\{\eta(AX)Z - \eta(AZ)X + g(X,AN)(JZ)^T - g(Z,AN)(JX)^T \\ & \quad + 2g(JZ,X)(AN)^T + \eta(X)(AZ)^T - \eta(Z)(AY)^T\big\}. \end{aligned} \tag{5.5}$$

On the other hand, by putting $Z = AN$ in (3.3) and using $JAN = A\xi$ and Lemma 5.1, we know the following

$$R(X,Y)AN = -g(Y,AN)X + g(X,AN)Y + \eta(AY)(JX)^T - \eta(AX)(JY)^T + 2g(JX,Y)A\xi + \eta(Y)(JAX)^T - \eta(X)(JAY)^T.$$

From this, the 3rd term in (5.3) is given by

$$\begin{aligned}(5.6)\qquad &\mathfrak{S}_{X,Y,Z}g(AZ,N)R(X,Y)AN\\ &= g(AZ,N)\{-g(Y,AN)X + g(X,AN)Y + \eta(AY)(JX)^T\\ &\quad - \eta(AX)(JY)^T + 2g(JX,Y)A\xi + \eta(Y)(JAX)^T - \eta(X)(JAY)^T\}\\ &\quad + g(AX,N)\{-g(Z,AN)Y + g(Y,AN)Z + \eta(AZ)(JY)^T\\ &\quad - \eta(AY)(JZ)^T + 2g(JY,Z)A\xi + \eta(Z)(JAY)^T - \eta(Y)(JAZ)^T\}\\ &\quad + g(AY,N)\{-g(X,AN)Z + g(Z,AN)X + \eta(AX)(JZ)^T\\ &\quad - \eta(AZ)(JX)^T + 2g(JZ,X)A\xi + \eta(X)(JAZ)^T - \eta(Z)(JAX)^T\}.\end{aligned}$$

Now by the definition of the curvature tensor $R(X,Y)Z$ of M in Q^{m*}, the final term of the right side in (5.3) is given by

$$\begin{aligned}(5.7)\quad &\alpha\mathfrak{S}_{X,Y,Z}R(X,Y)SZ\\ &= \alpha\mathfrak{S}_{X,Y,Z}\{g(Y,SZ)X - g(X,SZ)Y - g(JY,SZ)(JX)^T\\ &\quad + g(JX,SZ)(JY)^T + 2g(JX,Y)(JSZ)^T - g(AY,SZ)(AX)^T\\ &\quad + g(AX,SZ)(AY)^T - g(JAY,SZ)(JAX)^T + g(JAX,SZ)(JAY)^T\\ &\quad + g(SY,SZ)SX - g(SX,SZ)SY\}\\ &= \alpha\mathfrak{S}_{X,Y,Z}\{-g(JY,SZ)(JX)^T + g(JX,SZ)(JY)^T + 2g(JX,Y)(JSZ)^T\\ &\quad - g(AY,SZ)(AX)^T + g(AX,SZ)(AY)^T - g(JAY,SZ)(JAX)^T\\ &\quad + g(JAX,SZ)(JAY)^T\}.\end{aligned}$$

Then by virtue of (5.4), (5.5), (5.6) and (5.7), the cyclic structure Jacobi semi-symmetric operator in (5.3) gives the following

$$\begin{aligned}0 &= (1-\alpha^2)\mathfrak{S}_{X,Y,Z}\eta(Z)R(X,Y)\xi + \mathfrak{S}_{X,Y,Z}\eta(AZ)R(X,Y)A\xi\\ &\quad + \mathfrak{S}_{X,Y,Z}g(AZ,N)R(X,Y)AN + \alpha\mathfrak{S}_{X,Y,Z}R(X,Y)SZ.\end{aligned}$$

This equation is equivalent to the following

$$\begin{aligned}(5.8)\qquad 0 &= \alpha^2\big[\{\eta(Z)\eta(AY) - \eta(Y)\eta(AZ)\}(AX)^T\\ &\quad + \{\eta(X)\eta(AZ) - \eta(Z)\eta(AX)\}(AY)^T\\ &\quad + \{\eta(Y)\eta(AX) - \eta(X)\eta(AY)\}(AZ)^T\\ &\quad + \{\eta(Y)g(AZ,N) - \eta(Z)g(AY,N)\}(JAX)^T\end{aligned}$$

$$
\begin{aligned}
&+ \{\eta(Z)g(AX,N) - \eta(X)g(AZ,N)\}(JAY)^T \\
&+ \{\eta(X)g(AY,N) - \eta(Y)g(AX,N)\}(JAZ)^T\big] \\
&- 2\eta(AZ)\{g(Y,AN)(JX)^T - g(X,AN)(JY)^T + g(JX,Y)(AN)^T\} \\
&- 2\eta(AX)\{g(Z,AN)(JY)^T - g(Y,AN)(JZ)^T + g(JY,Z)(AN)^T\} \\
&- 2\eta(AY)\{g(X,AN)(JZ)^T - g(Z,AN)(JX)^T + g(JZ,X)(AN)^T\} \\
&+ 2\{g(AZ,N)g(JX,Y) + g(AX,N)g(JY,Z) + g(AY,N)g(JZ,X)\}A\xi \\
&- \alpha\big\{g(JY,SZ)(JX)^T - g(JX,SZ)(JY)^T - 2g(JX,Y)(JSZ)^T \\
&+ g(JZ,SX)(JY)^T - g(JY,SX)(JZ)^T - 2g(JY,Z)(JSX)^T \\
&+ g(JX,SY)(JZ)^T - g(JZ,SY)(JX)^T - 2g(JZ,X)(JSY)^T \\
&+ g(AY,SZ)(AX)^T - g(AX,SZ)(AY)^T + g(AZ,SX)(AY)^T \\
&- g(AY,SX)(AZ)^T + g(AX,SY)(AZ)^T - g(AZ,SY)(AX)^T \\
&+ g(JAY,SZ)(JAX)^T - g(JAX,SZ)(JAY)^T \\
&+ g(JAZ,SX)(JAY)^T - g(JAY,SX)(JAZ)^T \\
&+ g(JAX,SY)(JAZ)^T - g(JAZ,SY)(JAX)^T\big\}.
\end{aligned}
$$

Now let us consider an open subset $\mathcal{U} = \{p \in M | \alpha(p) \neq 0\}$ of M in Q^{m*}. Then on this open subset $\mathcal{U}$ of M, let us take $X, Y, Z \in \mathcal{Q}$, where the distribution $\mathcal{Q}$ is the orthogonal complement of the distribution $\mathcal{Q}^\perp = \mathrm{Span}\{A\xi, AN\}$ in $\mathcal{C}$, and put $Z = \phi X$ in (5.8). By Lemma 3.2 we can use the formulas $SX = \lambda X$ and $S\phi X = \mu\phi X$, where $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$. Then (5.8) becomes

$$
\begin{aligned}
(5.9)\qquad &\alpha\big\{\mu g(Y,X)\phi X - \mu\phi Y - 2\mu g(\phi X,Y)\phi^2 X + g(\phi^2 X,SX)\phi Y \\
&- g(\phi Y,SX)\phi^2 X - 2g(\phi Y,\phi X)\phi SX \\
&+ g(\phi X,SY)\phi^2 X - g(\phi^2 X,SY)\phi X - 2g(\phi^2 X,X)\phi SY \\
&+ g(AY,S\phi X)(AX)^T - g(AX,S\phi X)(AY)^T + g(AZ,SX)(AY)^T \\
&- g(AY,SX)(AZ)^T + g(AX,SY)(A\phi X)^T - g(A\phi X,SY)(AX)^T \\
&+ g(JAY,S\phi X)\phi AX - g(JAX,S\phi X)\phi AY + g(JA\phi X,SX)\phi AY \\
&- g(JAY,SX)\phi AZ + g(JAX,SY)\phi A\phi X - g(JA\phi X,SY)\phi AX\big\} \\
&= 0.
\end{aligned}
$$

From this, we consider $X, Y \in T_\lambda \cap V(A)$ and $Z = \phi X \in T_\mu \cap JV(A)$. This implies $AX = X$, $AY = Y$ and $AZ = A\phi X = -\phi X$. Then for an open subset $\mathcal{U} = \{p \in M | \alpha(p) \neq 0\}$ in M it follows that

$$
\begin{aligned}
(5.10)\qquad &\mu g(Y,X)\phi X - \mu\phi Y + 2\mu g(\phi X,Y)X - \lambda\phi Y \\
&+ \lambda g(\phi Y,X)X - 2\lambda g(\phi Y,\phi X)\phi X \\
&- \lambda g(\phi X,Y)X + \lambda g(X,Y)\phi X + 2\lambda\phi Y \\
&+ \mu g(Y,\phi X)X - \mu g(X,\phi X)Y - \lambda g(\phi X,X)Y
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda g(Y,X)\phi X-\lambda g(X,Y)\phi X+\lambda g(\phi X,Y)X\\
&+\mu g(\phi Y,\phi X)\phi X-\mu g(\phi X,\phi X)\phi Y+\lambda g(X,X)\phi Y\\
&-\lambda g(\phi Y,X)X+\lambda g(\phi X,Y)\phi^2X-\lambda g(X,Y)\phi X=0.
\end{aligned}
$$

For $m \geq 4$, $\dim T_\lambda = m-1 \geq 3$ and $\dim T_\mu = m-1 \geq 3$. This enables us to take $X, Y \in T_\lambda$ such that $g(X,Y)=0$ for $SX=\lambda X$ and $SY=\lambda Y$. So in (5.10) let us take $g(X,Y)=0$ for $X,Y \in T_\lambda$. Then $g(\phi X, Y)=0$, because $Y \in T_\lambda$ and $\phi X \in T_\mu$ by Lemma 3.2. From this and (5.10) it follows that

$$-\mu\phi Y-\lambda\phi Y+2\lambda\phi Y-\mu\phi Y+\lambda\phi Y=2(\lambda-\mu)\phi Y=0. \tag{5.11}$$

This implies $\lambda=\mu$, where $\mu=\frac{\alpha\lambda-2}{2\lambda-\alpha}$. If $2\lambda=\alpha$, then by using Lemma 3.2, we get $\alpha\lambda=2$. This gives $\alpha=2$ and $\lambda=1$ with multiplicities 1 and $2(m-1)$ respectively. Then this gives that the shape operator S commutes with the structure tesor ϕ, that is, $S\phi=\phi S$. Consequently, by the result due to Suh [24], M is locally congruent to a tube of radius r over a totally geodesic k-dimensional complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k^*}, $m=2k$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.

Now we consider an open subset $\mathcal{V}=Int(M-\mathcal{U})$ in M, where "Int" denotes the set of interior points of the set $M-\mathcal{U}$. Then on this open subset $\mathcal{V}$ the Reeb function α is identically vanishing. Then let us put $X=AN$ in (5.8) with the Reeb function $\alpha=0$. Then it follows that

$$
\begin{aligned}
&\eta(AZ)\{g(Y,AN)A\xi-(JY)^T+g(A\xi,Y)(AN)^T\}\\
&\quad+\eta(AY)\{(JZ)^T-g(Z,AN)A\xi-g(Z,A\xi)(AN)^T\}\\
&\quad-\{g(AZ,N)g(A\xi,Y)+g(JYZ)-g(Y,AN)g(Z,A\xi)\}A\xi\\
&=2\eta(AZ)g(Y,AN)A\xi-\eta(AZ)(JY)^T-2\eta(AY)g(Z,AN)A\xi\\
&\quad-g(JY,Z)A\xi+\eta(AY)(JZ)^{\perp}\\
&=0
\end{aligned} \tag{5.12}
$$

for any vector fields X, Y and Z on M in Q^{m*}.

From this, we take the inner product of (5.12) with the vector field $A\xi$ and use $g(JY, A\xi)=g(Y,AN)$. Then it follows that

$$\eta(AZ)g(Y,AN)-\eta(AY)g(Z,AN)-g(JY,Z)=0. \tag{5.13}$$

Now in this equation let us take $X,Y,Z \in \mathcal{Q}$, where the distribution $\mathcal{Q}$ is the orthogonal complement of the distribution $\mathcal{Q}^\perp=\mathrm{Span}\{A\xi, AN\}$ in $\mathcal{C}$. So $\eta(AZ)=0$ and $\eta(AY)=0$ for any vector fields Y,Z in $\mathcal{Q}$. Then (5.13) gives $g(\phi Y,Z)=0$. But if we take $Z=\phi Y$, we get a contradiction for $m\geq 4$. Accordingly, the open subset $\mathcal{V}=Int(M-\mathcal{U})$ must be empty. This means that $M=\mathcal{U}\cup\partial\mathcal{U}$, where $\partial\mathcal{U}$ denotes the set of boundary points of the set $\mathcal{U}$ in M and is discrete and measure zero by the induced Riemannian metric g on M. For any point $p\in\partial\mathcal{U}$ there exists a sequence

$p_n \in \mathfrak{U}$ such that $lim_{n\to\infty}\alpha(p_n) = \alpha(p) = 0$. But on the subset $\mathfrak{U}$ we know that $\alpha(p_n) = 2\coth 2r \neq 0$ is constant and $\coth 2r \geq 2$ including the case $r \to \infty$. This gives a contradiction. Accordingly, the set $\partial\mathfrak{U}$ also should be empty.

Now summing up all of the facts mentioned above, we give a complete proof of our Main Theorem 2 in the introduction.

As in the Corollary 2.1 in the introduction, let us give the proof as follows: For $m = 3$ it can be checked that a real hypersurface M^5 in the complex hyperbolic quadric Q^{6^*} with $\mathfrak{A}$-isotropic unit normal vector field N always satisfies the semi-symmetric cyclic structure Jacobi operator, that is, $\mathfrak{S}_{X,Y,Z} R(X,Y)R_\xi(Z) = 0$.

In fact, for $X \in T_\lambda \cap V(A)$ and $Y = \phi X \in T_\mu \cap JV(A)$ we have $SX = \lambda X$, $SY = \mu Y$, and $AX = X$ and $AY = A\phi X = -\phi X$. So putting $Z = Y = \phi X$ in (5.8) implies the following

$$
\begin{aligned}
(5.14)\quad & \mathfrak{S}_{X,Y,Z} R(X,\phi X)R_\xi(\phi X) \\
&= g(\phi X, S\phi X)X + 2\mu g(\phi X,\phi X)X + g(\phi^2 X, SX)\phi^2 X - g(\phi^2 X, SX)\phi^2 X \\
&\quad + g(\phi X, S\phi X)\phi^2 X - 2g(\phi^2 X, X)\phi S\phi X + g(A\phi X, S\phi X)AX \\
&\quad - g(A\phi X, S\phi X)AX - g(\phi AX, S\phi X)\phi A\phi X + g(\phi A\phi X, SX)\phi A\phi X \\
&\quad - g(\phi A\phi X, SX)\phi A\phi X + g(\phi AX, S\phi X)\phi A\phi X \\
&= \mu X + 2\mu X + \lambda X - \lambda X - \mu X - 2\mu X - \mu X + \mu X - \mu X \\
&\quad + \lambda X - \lambda X + \mu X \\
&= 0.
\end{aligned}
$$

That is, the cyclic structure Jacobi semi-symmetric holds on M^5 in Q^{6^*}. Consequently, we give a complete proof of Corollary 2.1 in the introduction.

Data Availability. Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Competing Interests Declaration. We had full access to all of the data in this study and take complete responsibility for the integrity of the data and the accuracy of the data analysis.

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