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Spherical Indicatrix of a New Approach to Bertrand Curves in Euclidean 3-space

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Abstract. In this paper, we investigate the spherical indicatrices of a new relationship between Bertrand pair curves in Euclidean 3-space. We obtain necessary and sufficient conditions for this type of Bertrand pair curves to be slant helix, and provide an example.

1. Introduction

In 1802, Lancret [17] defined a helix as a curve whose tangent vector makes a constant angle with a fixed straight line, called the directrix. Later in 1845, Saint Venant [21] showed that a necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion be a constant. In 1995, Scofield studied closed-form arc-length parametrizations for curves of constant precession and slant helices with a constant speed of precession [23]. In 2004, Izumiya and Takeuchi introduced the concept of slant helix in $\mathbb{E}^3$ saying that the principal normal lines make a constant angle with a fixed direction. They showed a curve to be a slant helix if and only if the principal image of the major normal indicatrix has a constant geodesic curvature [14]. In 2005, Kula and Yayli investigated spherical indicatrices of a slant helix and showed that the curve of constant precession is a slant helix in $\mathbb{E}^3$ [15]. A family of curves with constant curvature but non-constant torsion is called Salkowski and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski [22]. In 2009, Monterde studied some characterizations of these curves and he proved that the principal normal vector makes a constant angle with a fixed straight line [20]. In 2010, Kula et al. studied the relationship between slant helices and helices and they characterized slant helices

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in $\mathbb{E}^3$ in terms of differential equations [16]. In 2011, Ali and Lopez studied slant helix in Minkowski 3-space [1]. In 2012, Ali studied the position vector of a slant helix with respect to the standard frame in terms of Frenet equations in $\mathbb{E}^3$ [2]. In 2013, Camci et al. studied spherical slant helix in $\mathbb{E}^3$ [8]. In 2014, Menninger studied a generic characterization of the slant helix in $\mathbb{E}^3$ in terms of its curvature and torsion and derived an explicit arc-length parametrization of its tangent vector [19]. In 2019, Yilmaz and Has studied the position vectors of slant helices using an alternative moving frame in Minkowski 3-space [29]. In the fields of computer-aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion, the design of highways, etc. [28]. Helix and slant helix curves play an important role in curve theory, and have numerous applications in biological sciences, physics, etc. For instance, in biological sciences, curves are used in the analysis of Deoxyribonucleic Acid (DNA), and in physics, they are used in characterizing the motion of particles in a magnetic field.

In 1845, Saint Venant [21] posed a question whether the principal normal of a curve is the principal normal of another curve on the surface generated by the principal normal of the given one. Bertrand [7] gave an answer to this question in 1850 and introduced curves with a property that the principal normal vector of a curve α coincides with the principal normal vector of another curve α^* at their corresponding points. Further, these curves were characterized in $\mathbb{E}^3$ with condition $ak + b\tau = 1$, where a and b are nonzero constants and k and τ are the curvature and torsion of the curve, respectively [11]. In [18], Matsuda and Yorozu studied generalized Bertrand curves in $\mathbb{E}^4$. Llater they were studied in $\mathbb{E}^n$ by Cheng and Lin [10]. The spherical indicatrix is useful to visualize the motion of an indicatrix on a sphere with the help of the Frenet frame of the curve. In [5], Babaarslan and Yayli examined Bertrand curves of the tangent, normal, binormal, and *Darboux* indicatrices of a space curve in $\mathbb{E}^3$. In [25], Tuncer and Unal studied a new representation of spherical indicatrices of Bertrand pair curves in $\mathbb{E}^3$. On the other hand, the Bertrand pair curves were studied in pseudo-Riemannian spaces by many authors, please see [4, 6, 13, 26, 27] and the references therein.

In [9], Camci, et al. introduced a new relationship between Bertrand pair curves α and α^* in $\mathbb{E}^3$ by not taking the vector $\overrightarrow{\alpha^*\alpha}$ parallel to a normal vector of Bertrand curve α. Using this approach, the authors introduced and studied a new parametrization of Bertrand partner D-curves in Euclidean 3-space [24].

In view of this, we study the spherical indicatrices of such Bertrand pair curves in $\mathbb{E}^3$. The paper is organized as follows: In Section 2 we recall some basic notation for curves that will be used in the rest of the paper. In Section 3, we study the spherical indicatrices of homothetic Bertrand pair curves. Section 4 is devoted to the study of spherical indicatrices of non-homothetic Bertrand pair curves, and to obtain a characterization of the Bertrand pair curves that are slant helix. Also, in this section, we provide an example of such pair curves.

2. Some Basic Concepts and Related Results

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean 3-space where s denotes arc-length. The Frenet formula of the curve α is given by [11]

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where triplet (T, N, B) denotes the Frenet frame and k, τ are curvature and torsion of α, respectively.

Definition 2.1. For a given point, a set of curves such that any straight line through the point intersects all the curves in the set at the same angle, then these set of curves are called homothetic curves [12]. Otherwise, it is called non-homothetic curves.

In [9], the authors introduced a new relationship between a pair of Bertrand curves α and α^* as follows:

$$\alpha^*(s^*) = \alpha(s) + u(s)T(s) + v(s)N(s) + w(s)B(s), \tag{2.2}$$

where $u(s)$, $v(s)$ and $w(s)$ are differentiable functions. Furthermore, they obtained the following result:

Theorem 2.2. ([9]) *Let α and α^* be Bertrand pair curves with Frenet frames (T, N, B) and (T^*, N^*, B^*) satisfying (2.2). If*

(i) there exist a homothety map between Bertrand pair curves α and α^*, then

$$T = \epsilon_1 T^*,\ N = \epsilon_1 N^*,\ B = \epsilon_1 B^*, \tag{2.3}$$

$$T^* = \epsilon_1 T, N^* = \epsilon_1 N, B^* = \epsilon_1 B, \tag{2.4}$$

$$k = \epsilon_1 k^*(1 + u' - vk),\ \ \tau = \epsilon_1 \tau^*(1 + u' - vk), \tag{2.5}$$

$$k^* = \frac{\epsilon_1 k}{(1 + u' - vk)}\ ,\ \tau^* = \frac{\epsilon_1 \tau}{(1 + u' - vk)}, \tag{2.6}$$

$$\frac{ds^*}{ds} = \epsilon_1(1 + u' - vk). \tag{2.7}$$

(ii) there does not exist homothety map between Bertrand pair curves α and α^*, then

$$\begin{cases} T &= \frac{m_1m_2m_3}{\sqrt{1+h^2}}(m_2m_3hT^* - B^*),\ N = m_1m_2N^*, \\ B &= \frac{m_1m_2m_3}{\sqrt{1+h^2}}(m_2m_3T^* + hB^*), \end{cases} \tag{2.8}$$

$$T^* = \frac{m_1}{h^2+1}(hT+B),\ N^* = m_1m_2N,\ B^* = \frac{m_1m_2m_3}{\sqrt{h^2+1}}(-T+hB), \tag{2.9}$$

$$\begin{cases} k &= m_1m_2m_3(w'+v\tau)(hm_3k^* + m_2\tau^*), \\ \tau &= -m_1m_2m_3(w'+v\tau)(m_3k^* - m_2h\tau^*), \end{cases} \tag{2.10}$$

$$k^* = \frac{m_1m_2(hk-\tau)}{(w'+v\tau)(h^2+1)},\ \tau^* = \frac{m_1m_3(k+h\tau)}{(w'+v\tau)(h^2+1)}, \tag{2.11}$$

$$\frac{ds^*}{ds} = m_1(w'+v\tau)\sqrt{h^2+1}, \tag{2.12}$$

where $\epsilon_1, m_1, m_2, m_3 \in \{-1, 1\}$, $w' + v\tau \neq 0$, k, τ, s *and* k^*, τ^*, s^* *are non-zero curvatures, torsions and arc-lengths of* α *and* α^**, respectively and* h *is a constant given by* $h = \frac{1+u'-vk}{w'+v\tau}$.

Izumiya and Takeuchi [14] characterized a curve to be a slant helix if and only if the principal image of the major normal indicatrix has a constant geodesic curvature k_g, i.e.,

$$k_g = \left(\frac{\tau}{k}\right)' \frac{k^2}{(k^2+\tau^2)^{3/2}} \tag{2.13}$$

is a constant function.

Following are the examples of slant helices:

Example 2.3. ([9]) The curve α given by

$$\alpha(s) = \Big(\frac{-3}{\sqrt{2}}\cos\sqrt{2}s\sin s + 2\cos s\sin\sqrt{2}s, \frac{-3}{\sqrt{2}}\sin\sqrt{2}s\sin s - 2\cos s\cos\sqrt{2}s, \frac{1}{\sqrt{2}}\sin s\Big)$$

is a slant helix with $k(s) = \sin s$, $\tau(s) = \cos s$ and $k_g = -1$.

Example 2.4. ([3]) The curve γ given by

$$\gamma(s) = \Big(\frac{-1}{3}\sqrt{s^2+1}\ \cos(3\arctan s), \frac{-1}{3}\sqrt{s^2+1}\ \sin(3\arctan s), \frac{2\sqrt{2}}{3}\sqrt{s^2+1}\Big)$$

is a slant helix with $k(s) = \frac{2\sqrt{2}}{(s^2+1)^{3/2}}$, $\tau(s) = \frac{2\sqrt{2}s}{(s^2+1)^{3/2}}$ and $k_g = \frac{1}{2\sqrt{2}}$.

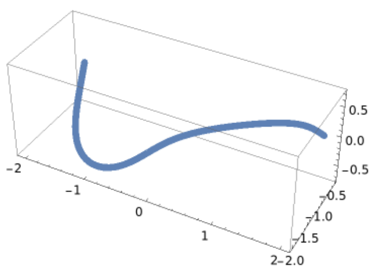

Figure 1: curve α.

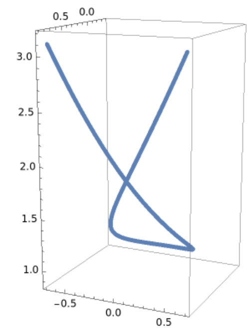

Figure 2: curve γ.

Example 2.5. ([2]) The Salkowski curve given by

$$\psi(t) = \big(\psi_1(t),\, \psi_2(t), \psi_3(t)\big),$$

where

$$\begin{cases} \psi_1(t) = \frac{n}{4m}\left[\frac{n-1}{2n+1}\cos[(2n+1)t] + \frac{n+1}{2n-1}\cos[(2n-1)t] - 2\cos[t]\right], \\ \psi_2(t) = \frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t] + \frac{n+1}{2n-1}\sin[(2n-1)t] - 2\sin[t]\right], \\ \psi_3(t) = -\frac{n}{4m^2}\cos[2nt] \end{cases}$$

is a slant helix with arc-length parameter $s = \frac{sin(nt)}{m}$, $k(s) = 1$, $\tau(s) = \frac{ms}{\sqrt{1-m^2s^2}}$ and $k_g = m$.

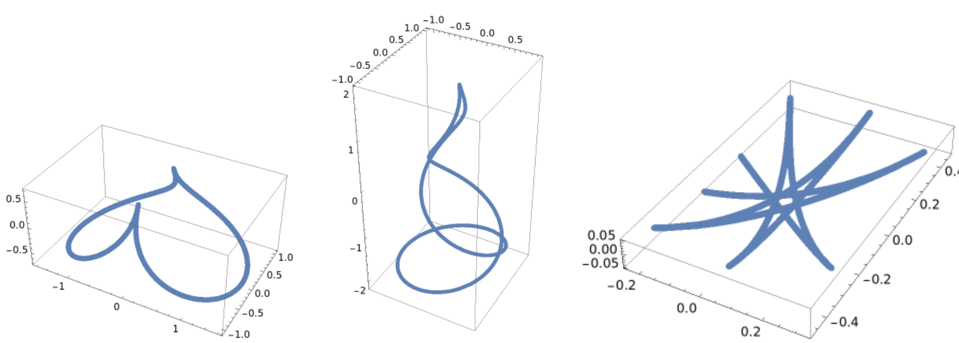

Figure 3: Slant helices with $m = \frac{1}{\sqrt{1-n^2}}$ and $n = \frac{1}{3}, \frac{1}{8}, \frac{10}{11}$.

Now onwards, we denote by Γ, Γ^*, Γ_t, Γ_b, Γ_t^*, Γ_b^* the geodesic curvature of the principal image of the principal normal indicatrix of a Bertrand pair curve α, α^* and their tangent indicatrix α_t, α_t^* and binormal indicatrix α_b, α_b^*, respectively.

Proposition 2.6. *Let α and α^* be Bertrand pair curves satisfying* (2.2)*, then*

$$\Gamma = \frac{H'}{k^*(1+H^2)^{3/2}}, \tag{14}$$

where $H = \frac{\tau^*}{k^*}$ and $H' = \frac{dH}{ds^*}$.

Proof. Suppose α and α^* are homothetic. Using (2.5) in (2.13), we get

$$\Gamma = \frac{H'}{\epsilon_1 k^*(1+u'-vk)(1+H^2)^{3/2}} \frac{ds^*}{ds}. \tag{15}$$

Using (2.7) in (15), we obtain (14).

Suppose α and α^* are not homothetic. Using (2.10) in (2.13), we get

$$\Gamma = \frac{m_1 H' k^{*2}}{(w'+v\tau)\sqrt{1+h^2}(k^{*2}+\tau^{*2})^{3/2}} \frac{ds^*}{ds}. \tag{16}$$

Using (2.12) in (16), we obtain (14). This completes the proof. □

Proposition 2.7. *Let α and α^* be Bertrand pair curves satisfying* (2.2)*. If α^* and α are homothetic, then*

$$\Gamma^* = \frac{f'}{k(1+f^2)^{3/2}}. \tag{17}$$

If α^* and α are not homothetic, then

$$\Gamma^* = \frac{m_3 f'}{k(1+f^2)^{3/2}}, \tag{18}$$

where $f = \frac{\tau}{k}$.

Proof. Using (2.13), we have

$$\Gamma^* = \left(\frac{\tau^*}{k^*}\right)' \frac{k^{*2}}{(k^{*2}+\tau^{*2})^{3/2}}. \tag{19}$$

Suppose α^* and α are homothetic. Using (2.6) in (19), we get

$$\Gamma^* = \frac{(1+u'-vk)f'}{\epsilon_1 k(1+f^2)^{3/2}} \frac{ds}{ds^*}. \tag{20}$$

Using (2.7) in (20), we obtain (17).

Suppose α^* and α are not homothetic. Using (2.11) in (19), we have

$$\Gamma^* = \frac{m_3 f' k^2 (w'+v\tau)\sqrt{1+h^2}}{m_1(k^2+\tau^2)^{3/2}} \frac{ds}{ds^*}. \tag{21}$$

Using (2.12) in (21), we obtain (18). Whereby proof is complete. □

3. Spherical Indicatrices of Homothetic Bertrand Pair Curves in Euclidean 3-space

In this section, we study the tangent, normal, and binormal indicatrices of Bertrand pair curves satisfying (2.2) in $\mathbb{E}^3$.

First of all, we study tangent indicatrices $\alpha_t = T$ and $\alpha_t^* = T^*$. We have:

Theorem 3.1. *Let α and α^* be homothetic Bertrand pair curves satisfying (2.2), then the tangent indicatrices α_t and α_t^* of Bertrand pair curves α and α^* are also homothetic. Also, we have,*

$$T_t = \epsilon_1 T_t^*,\ N_t = \epsilon_1 N_t^*,\ B_t = \epsilon_1 B_t^*, \tag{3.1}$$

$$\begin{cases} T_t & = \epsilon_1 N^*,\ N_t = \frac{\epsilon_1}{\sqrt{1+H^2}}(-T^* + HB^*), B_t = \frac{\epsilon_1}{\sqrt{1+H^2}}(B^* + HT^*), \\ k_t & = k_t^* = \sqrt{1+H^2},\ \tau_t = \tau_t^* = \frac{H'}{k^*(1+H^2)},\ \frac{ds_t}{ds^*} = \frac{ds_t^*}{ds^*} = k^*, \end{cases} \tag{3.2}$$

where $(T_t,\ N_t,\ B_t)$ and $(T_t^*,\ N_t^*,\ B_t^*)$ the Frenet frames, k_t, k_t^*, τ_t, τ_t^* and s_t, s_t^* are the curvatures, torsions and arc-lengths of α_t and α_t^*, respectively.

Proof. The Frenet formula for the tangent indicatrix α_t is given by

$$T_t{}' = k_t N_t, N_t{}' = -k_t T_t + \tau_t B_t, B_t{}' = -\tau_t N_t. \tag{3.3}$$

Since, the tangent indicatrix of α and α^* is given by $\alpha_t = T$ and $\alpha_t^* = T^*$. Then from (2.3), we have

$$\alpha_t = \epsilon_1 \alpha_t^*. \tag{3.4}$$

Differentiating (3.4) with respect to s_t and using (2.1) and (3.3), we obtain

$$T_t = \epsilon_1 T_t^* \Big(\frac{ds_t^*}{ds_t}\Big). \tag{3.5}$$

Taking the inner product of (3.5) with itself, we have

$$T_t = \epsilon_1 T_t^*,\ ds_t = ds_t^*. \tag{3.6}$$

From the first relation of (3.3) and using (3.6), we get

$$k_t = k_t^*,\ \ N_t = \epsilon_1 N_t^*. \tag{3.7}$$

From the second relation of (3.3) and using (3.6) and (3.7), we get

$$\tau_t = \tau_t^*,\ B_t = \epsilon_1 B_t^*. \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we obtain (3.1).

Now, we will find the Frenet frame of α_t. From (2.3), we have

$$\alpha_t = \epsilon_1 T^*. \tag{3.9}$$

Differentiating (3.9) with respect to s^* and using (2.1) and then taking inner product with itself, we obtain

$$T_t = \epsilon_1 N^*, \ \frac{ds_t}{ds^*} = k^*. \tag{3.10}$$

Differentiating (3.10) with respect to s_t and using (2.1) and first relation of (3.3), we obtain

$$N_t = \frac{\epsilon_1}{\sqrt{1+H^2}}(-T^* + HB^*), \ k_t = \sqrt{1+H^2}. \tag{3.11}$$

Differentiating N_t in (3.11) with respect to s_t and using (2.1) and (3.10), we obtain

$$\frac{dN_t}{ds_t} = \frac{\epsilon_1(-k^*(1+H^2)^2 N^*) + H'(B^* + HT^*)}{k^*(1+H^2)^{3/2}}. \tag{3.12}$$

Using (3.10), (3.11) and (3.12) in second relation of (3.3), we get

$$B_t = \frac{\epsilon_1}{\sqrt{1+H^2}}(B^* + HT^*), \ \tau_t = \frac{H'}{k^*(1+H^2)}. \tag{3.13}$$

Using (3.10), (3.11) and (3.13). We obtain (3.2). This completes the proof of the Theorem. □

Next, we study normal indicatrices $\alpha_n = N$ and $\alpha_n^* = N^*$. We have:

Theorem 3.2. *Let α and α^* be homothetic Bertrand pair curves satisfying* (2.2)*, then normal indicatrices α_n and α_n^* of α and α^* are also homothetic. Also we have,*

$$T_n = \epsilon_1 T_n^*, \ N_n = \epsilon_1 N_n^*, \ B_n = \epsilon_1 B_n^*, \tag{3.14}$$

$$\begin{cases} T_n &= \frac{\epsilon_1}{\sqrt{1+H^2}}(-T^* + HB^*), \\ N_n &= \frac{\epsilon_1}{\rho\sqrt{1+H^2}}(HH'T^* - k^*(1+H^2)^2 N^* + H'B^*), \\ B_n &= \frac{\epsilon_1\left((\sqrt{1+H^2}P-\rho)T^* + \sqrt{1+H^2}QN^* + (\sqrt{1+H^2R+\rho H})B^*\right)}{\sqrt{(\sqrt{1+H^2}P-\rho)^2 + (1+H^2)Q^2 + (\sqrt{1+H^2}R+\rho H)^2}}, \\ \tau_n &= \tau_n^* = \frac{\sqrt{(\sqrt{1+H^2}P-\rho)^2 + (1+H^2)Q^2 + (\sqrt{1+H^2}R+\rho H)^2}}{k^*(1+H^2)^2}, \\ k_n &= k_n^* = \frac{\rho}{k^*(1+H^2)^{3/2}}, \ \frac{ds_n}{ds^*} = \frac{ds_n^*}{ds^*} = k^*\sqrt{1+H^2}, \end{cases} \tag{3.15}$$

where

$$\begin{cases} \rho &= \sqrt{H'^2 + (1+H^2)^3 k^{*2}}, \\ P &= -\rho' H H'(1+H^2) + \rho H H''(1+H^2) + \rho^3, \\ Q &= \rho' k^*(1+H^2)^3 - k^{*'}\rho(1+H^2)^3 - 3\rho H H' k^*(1+H^2)^2, \\ R &= -H'\rho'(1+H^2) - HH'^2\rho + H''\rho(1+H^2) - k^*\tau^*\rho(1+H^2)^3, \end{cases} \tag{3.16}$$

(T_n, N_n, B_n) and (T_n^*, N_n^*, B_n^*) the Frenet frames, k_n, k_n^*, τ_n, τ_n^*, s_n, s_n^* are the curvatures, torsions and arc-lengths of α_n and α_n^*, respectively.

Proof. The Frenet formula for the normal indicatrix α_n is given by

$$T_n{}' = k_n N_n, N_n{}' = -k_n T_n + \tau_n B_n, B_n{}' = -\tau_n N_n, \tag{3.17}$$

Since, the normal indicatrix of α and α^* is given by $\alpha_n = N$ and $\alpha_n^* = N^*$. Then from (2.3), we have

$$\alpha_n = \epsilon_1 \alpha_n^*. \tag{3.18}$$

Differentiating (3.18) with respect to s_n and using (2.1) and (3.17), we obtain

$$T_n = \epsilon_1 T_n^* \Big(\frac{ds_n^*}{ds_n}\Big). \tag{3.19}$$

Taking the inner product of (3.5) with itself, we have

$$T_n = \epsilon_1 T_n^*, \ ds_n = ds_n^*. \tag{3.20}$$

From the first relation of (3.17) and using (3.20), we get

$$k_n = k_n^*, \ \ N_n = \epsilon_1 N_n^*. \tag{3.21}$$

From the second relation of (3.17) and using (3.20) and (3.21), we get

$$\tau_n = \tau_n^*, \ B_n = \epsilon_1 B_n^*. \tag{3.22}$$

Using (3.20), (3.21) and (3.22), we obtain (3.14).

Now, we will find the Frenet frame of α_n. From (2.3), we have

$$\alpha_n = \epsilon_1 N^*. \tag{3.23}$$

Differentiating (3.23) with respect to s^* and using (2.1) and then taking inner product with itself, we have

$$T_n = \frac{\epsilon_1}{\sqrt{1+H^2}}(-T^* + HB^*), \ \frac{ds_n}{ds^*} = k^*\sqrt{1+H^2}. \tag{3.24}$$

Differentiating (3.24) with respect to s_n and using (2.1) and first relation of (3.17), we get

$$N_n = \frac{\epsilon_1}{\rho\sqrt{1+H^2}}(HH'T^* - k^*(1+H^2)^2N^* + H'B^*),\ k_n = \frac{\rho}{k^*(1+H^2)^{3/2}}. \tag{3.25}$$

Differentiating N_n in (3.25) with respect to s_n and using (2.1) and (3.24), we have

$$\frac{dN_n}{ds_n} = \frac{\epsilon_1(PT^* + QN^* + RB^*)}{\rho k^*(1+H^2)^{3/2}}. \tag{3.26}$$

Using (3.24), (3.25) and (3.26) in second relation of (3.17), we get

$$\begin{cases} \tau_n &= \frac{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}{k^*(1+H^2)^2}, \\ B_n &= \frac{\epsilon_1((\sqrt{1+H^2}P-\rho)T^*+\sqrt{1+H^2}QN^*+(\sqrt{1+H^2}R+\rho H)B^*)}{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}. \end{cases} \tag{3.27}$$

From (3.24), (3.25) and (3.27), we obtain (3.15). Whereby proof is complete. □

Now, we study binormal indicatrices $\alpha_b = B$ and $\alpha_b^* = B^*$. We have:

Theorem 3.3. *Let α and α^* be homothetic Bertrand pair curves satisfying* (2.2), *then the binormal indicatrices α_b and α_b^* of Bertrand pair curves α and α^* are also homothetic. Also we have,*

$$T_b = \epsilon_1 T_b^*,\ N_b = \epsilon_1 N_b^*,\ B_b = \epsilon_1 B_b^*, \tag{3.28}$$

$$\begin{cases} T_b &= -\epsilon_1 N^*,\ N_b = \frac{\epsilon_1}{\sqrt{1+H^2}}(T^* - HB^*),\ B_b = -\frac{\epsilon_1}{\sqrt{1+H^2}}(B^* + HT^*), \\ k_b = & k_b^* = \frac{\sqrt{1+H^2}}{H},\ \tau_b = \tau_b^* = \frac{H'}{\tau^*(1+H^2)},\ \frac{ds_b}{ds^*} = \frac{ds_b^*}{ds^*} = \tau^*, \end{cases} \tag{3.29}$$

where (T_b, N_b, B_b) and (T_b^*, N_b^*, B_b^*) the Frenet frames, k_b, k_b^*, τ_b, τ_b^* and s_b, s_b^* are the curvatures, torsions and arc-length of α_b and α_b^*, respectively.

Proof. The Frenet formula for the binormal indicatrix α_b is given by

$${T_b}' = k_b N_b, {N_b}' = -k_b T_b + \tau_b B_b, {B_b}' = -\tau_b N_b, \tag{3.30}$$

Since, the binormal indicatrix of α and α^* is given by $\alpha_b = B$ and $\alpha_b^* = B^*$. Then from (2.3), we have

$$\alpha_b = \epsilon_1 \alpha_b^*. \tag{3.31}$$

Differentiating (3.31) with respect to s_b and using (2.1) and (3.30), we obtain

$$T_b = \epsilon_1 T_b^* \Big(\frac{ds_b^*}{ds_b}\Big). \tag{3.32}$$

Taking inner product of (3.32) with itself, we have

$$T_b = \epsilon_1 T_b^*,\ ds_b = ds_b^*. \tag{3.33}$$

From the first relation of (3.30) and using (3.33), we get

$$k_b = k_b^*,\ \ N_b = \epsilon_1 N_b^*. \tag{3.34}$$

From the second relation of (3.30) and using (3.33) and (3.34), we get

$$\tau_b = \tau_b^*,\ B_b = \epsilon_1 B_b^*. \tag{3.35}$$

Using (3.33), (3.34) and (3.35), we obtain (3.28).

Now, we will find the Frenet frame of α_b. From (2.3), we have

$$\alpha_b = \epsilon_1 B^*, \tag{3.36}$$

Differentiating (3.36) with respect to s^* and using (2.1) and then taking inner product with itself, we have

$$T_b = -\epsilon_1 N^*,\ \frac{ds_b}{ds^*} = \tau^*. \tag{3.37}$$

Differentiating (3.37) with respect to s_b and using (2.1) and first relation of (3.30), we obtain

$$N_b = \frac{\epsilon_1}{\sqrt{1+H^2}}(T^* - HB^*),\ k_b = \frac{\sqrt{1+H^2}}{H}. \tag{3.38}$$

Differentiating N_b in (3.38) with respect to s_b and and using (2.1) and (3.37), we obtain

$$\frac{dN_b}{ds_b} = \frac{\epsilon_1(k^*(1+H^2)^2 N^* - H'(B^* + HT^*))}{\tau^*(1+H^2)^{3/2}}. \tag{3.39}$$

Using (3.37), (3.38) and (3.39) in second relation of (3.30), we get

$$B_b = -\frac{\epsilon_1}{\sqrt{1+H^2}}(B^* + HT^*),\ \tau_b = \frac{H'}{\tau^*(1+H^2)}, \tag{3.40}$$

Using (3.37), (3.38) and (3.40). We obtain (3.29). Thus, proof is complete. □

4. Spherical Indicatrices of Non-homothetic Bertrand Pair Curves in Euclidean 3-space

In this section, we study the tangent, normal, and binormal indicatrices of a non-homothetic Bertrand pair curves satisfying (2.2) in $\mathbb{E}^3$.

Now, we study tangent indicatrices of non-homothetic Bertrand pair curves $\alpha_t = T$ and $\alpha_t^* = T^*$. We have:

Theorem 4.1. *Let* α *and* α^* *be non-homothetic Bertrand pair curves satisfying* (2.2) *with their tangent indicatrices* α_t *and* α_t^*. *Then*

$$\begin{cases} T_t &= m_1m_2m_3N^*,\ N_t = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(-T^*+HB^*), \\ B_t &= \frac{m_1m_2m_3}{\sqrt{1+H^2}}(B^*+HT^*),\ s_t = \int \frac{1}{\sqrt{h^2+1}}(m_2m_3hk^*-\tau^*)ds^*, \\ k_t &= \frac{k^*\sqrt{1+H^2}\sqrt{1+h^2}}{(m_2m_3hk^*-\tau^*)},\ \tau_t = \frac{H'\sqrt{1+h^2}}{(1+H^2)(m_2m_3hk^*-\tau^*)}, \end{cases} \tag{4.1}$$

$$\begin{cases} T_t^* &= m_1N, N_t^* = \frac{m_1}{\sqrt{1+f^2}}(-T+fB),\ B_t^* = \frac{m_1}{\sqrt{1+f^2}}(B+fT), \\ k_t^* &= \frac{\sqrt{1+h^2}\sqrt{1+f^2}}{(h-f)},\ \tau_t^* = \frac{f'\sqrt{1+h^2}}{(1+f^2)k(h-f)},\ s_t^* = \int \frac{1}{\sqrt{h^2+1}}k(h-f)ds, \end{cases} \tag{4.2}$$

Proof. From (2.8), we have

$$\alpha_t = \frac{m_1m_2m_3}{\sqrt{1+h^2}}(m_2m_3hT^* - B^*). \tag{4.3}$$

Differentiating (4.3) with respect to s^*, using (2.1) and then taking inner product with itslef, we have

$$T_t = m_1m_2m_3N^*,\ \frac{ds_t}{ds^*} = \frac{1}{\sqrt{h^2+1}}(m_2m_3hk^* - \tau^*). \tag{4.4}$$

Differentiating (4.4) with respect to s_t and using (2.1) and first relation of (3.3), we obtain

$$N_t = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(-T^*+HB^*),\ k_t = \frac{k^*\sqrt{1+H^2}\sqrt{1+h^2}}{(m_2m_3hk^*-\tau^*)}. \tag{4.5}$$

Differentiating N_t in (4.5) with respect to s_t and using (2.1) and (4.4), we obtain

$$\frac{dN_t}{ds_t} = m_1m_2m_3\frac{\big(-(1+H^2)^2k^*N^* + H'(B^*+HT^*)\big)\sqrt{h^2+1}}{(1+H^2)^{3/2}(m_2m_3hk^*-\tau^*)}. \tag{4.6}$$

Using (4.4), (4.5) and (4.6) in second relation of (3.3), we get

$$B_t = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(B^*+HT^*),\ \tau_t = \frac{H'\sqrt{1+h^2}}{(1+H^2)(m_2m_3hk^*-\tau^*)}. \tag{4.7}$$

Using (4.4), (4.5) and (4.7). We obtain (4.1).

From (2.8), we have

$$\alpha_t^* = \frac{m_1}{\sqrt{1+h^2}}(hT + B). \tag{4.8}$$

Differentiating (4.8) with respect to s and using (2.1) and then taking inner product with itself, we have

$$T_t^* = m_1 N, \ \frac{ds_t^*}{ds} = \frac{1}{\sqrt{h^2+1}}(hk - \tau). \tag{4.9}$$

Differentiating (4.9) with respect to s_t^* and using (2.1) and first relation of (3.3), we obtain

$$N_t^* = \frac{m_1}{\sqrt{1+f^2}}(-T + fB), \ k_t^* = \frac{\sqrt{1+f^2}\sqrt{1+h^2}}{(h-f)}. \tag{4.10}$$

Differentiating N_t^* in (4.10) with respect to s_t^* and using (2.1) and (4.9), we obtain

$$\frac{dN_t^*}{ds_t^*} = m_1 \frac{\big(-(1+f^2)^2 kN + f'(B + fT)\big)\sqrt{h^2+1}}{k(1+f^2)^{3/2}(hk - f)}. \tag{4.11}$$

Using (4.9), (4.10) and (4.11) in second relation of (3.3), we get

$$B_t^* = \frac{m_1}{\sqrt{1+f^2}}(B + fT), \ \tau_t^* = \frac{f'\sqrt{1+h^2}}{k(1+f^2)(h-f)}. \tag{4.12}$$

Using (4.9), (4.10)and (4.12), we obtain (4.2). Thus, the proof is complete. □

Now, we study normal indicatrices of non-homothetic Bertrand pair curves $\alpha_n = N$ and $\alpha_n^* = N^*$. We have:

Theorem 4.2. *Let α and α^* be non-homothetic Bertrand pair curves satisfying* (2.2) *with their normal indicatrices α_n and α_n^*. Then*

$$\begin{cases} T_n &= \frac{m_1 m_2}{\sqrt{1+H^2}}(-T^* + HB^*), \\ N_n &= \frac{m_1 m_2}{\rho\sqrt{1+H^2}}(HH'T^* - k^*(1+H^2)^2 N^* + H'B^*), \\ B_n &= \frac{m_1 m_2\big((\sqrt{1+H^2}P-\rho)T^* + \sqrt{1+H^2}QN^* + (\sqrt{1+H^2R+\rho H})B^*\big)}{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}, \\ \tau_n &= \frac{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}{k^*(1+H^2)^2}, \\ k_n &= \frac{\rho}{k^*(1+H^2)^{3/2}}, \ s_n = \int k^*\sqrt{1+H^2}ds^*, \end{cases} \tag{4.13}$$

$$(4.14)\quad \begin{cases} T_n^* &= \frac{m_1m_2}{\sqrt{1+f^2}}(-T+fB),\ N_n^* = \frac{m_1m_2}{\phi\sqrt{1+f^2}}(ff'T-k(1+f^2)^2N+f'B),\\ B_n^* &= \frac{m_1m_2\left((\sqrt{1+f^2}U-\phi)T+\sqrt{1+f^2}VN+(\sqrt{1+f^2}W+\phi f)B\right)}{\sqrt{(\sqrt{1+f^2}U-\phi)^2+(1+f^2)V^2+(\sqrt{1+f^2}W+\phi f)^2}},\\ \tau_n^* &= \frac{\sqrt{(\sqrt{1+f^2}U-\phi)^2+(1+f^2)V^2+(\sqrt{1+f^2}W+\phi f)^2}}{k(1+f^2)^2},\\ k_n^* &= \frac{\phi}{k(1+f^2)^{3/2}},\ s_n^* = \int k\sqrt{1+f^2}ds, \end{cases}$$

where,

$$(4.15)\quad \begin{cases} \rho &= \sqrt{H'^2+(1+H^2)^3{k^*}^2},\\ P &= -\rho' HH'(1+H^2)+\rho HH''(1+H^2)+\rho^3,\\ Q &= \rho' k^*(1+H^2)^3-k^{*'}\rho(1+H^2)^3-3\rho HH'k^*(1+H^2)^2,\\ R &= -H'\rho'(1+H^2)-HH'^2\rho+H''\rho(1+H^2)-k^*\tau^*\rho(1+H^2)^3, \end{cases}$$

$$(4.16)\quad \begin{cases} \phi &= \sqrt{f'^2+(1+f^2)^3k^2},\\ U &= -\phi' ff'(1+f^2)+\phi ff''(1+H^2)+\phi^3,\\ V &= \phi' k^*(1+f^2)^3-k'\phi(1+f^2)^3-3\phi ff'k(1+f^2)^2,\\ W &= -f'\phi'(1+f^2)-ff'^2\phi+f''\phi(1+f^2)-k\tau\phi(1+f^2)^3. \end{cases}$$

Proof. From (2.3), we have

$$(4.17)\quad \alpha_n = m_1m_2N^*.$$

Differentiating (4.17) with respect to s^* and using (2.1) and then taking inner product with itself, we have

$$(4.18)\quad T_n = \frac{m_1m_2}{\sqrt{1+H^2}}(-T^*+HB^*),\ \frac{ds_n}{ds^*} = k^*\sqrt{1+H^2}.$$

Differentiating (4.18) with respect to s_n and using (2.1) and first relation of (3.17), we get

$$(4.19)\quad N_n = \frac{m_1m_2}{\rho\sqrt{1+H^2}}(HH'T^*-k^*(1+H^2)^2N^*+H'B^*),\ k_n = \frac{\rho}{k^*(1+H^2)^{3/2}}.$$

Differentiating N_n in (4.19) with respect to s_n and using (2.1) and (3.24), we have

$$(4.20)\quad \frac{dN_n}{ds_n} = \frac{m_1m_2(PT^*+QN^*+RB^*)}{\rho k^*(1+H^2)^{3/2}}.$$

Using (4.18), (4.19) and (4.20) in second relation of (3.17), we get

$$(4.21)\qquad \begin{cases} \tau_n &= \frac{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}{k^*(1+H^2)^2}, \\ B_n &= \frac{m_1m_2((\sqrt{1+H^2}P-\rho)T^*+\sqrt{1+H^2}QN^*+(\sqrt{1+H^2}R+\rho H)B^*)}{\sqrt{(\sqrt{1+H^2}P-\rho)^2+(1+H^2)Q^2+(\sqrt{1+H^2}R+\rho H)^2}}. \end{cases}$$

From (4.18), (4.19) and (4.21). We obtain (4.13).

From (2.3), we have

$$(4.22)\qquad \alpha_n^* = \epsilon_1 N.$$

Differentiating (4.22) with respect to s and using (2.1) and then taking inner product with itself, we find

$$(4.23)\qquad T_n^* = \frac{\epsilon_1}{\sqrt{1+f^2}}(-T+fB),\ \frac{ds_n^*}{ds} = k\sqrt{1+f^2}.$$

Differentiating (4.23) with respect to s_n^* and using (2.1) and first relation of (3.17), we get

$$(4.24)\qquad N_n^* = \frac{\epsilon_1}{\phi\sqrt{1+f^2}}(ff'T^* - k(1+f^2)^2N + f'B),\ k_n^* = \frac{\phi}{k(1+f^2)^{3/2}}.$$

Differentiating N_n^* in (4.24) with respect to s_n^* and using (2.1) and (4.23), we have

$$(4.25)\qquad \frac{dN_n^*}{ds_n^*} = \frac{\epsilon_1 UT + VN + WB}{\phi k(1+f^2)^{3/2}}.$$

Using (4.23), (4.24) and (4.25) in second relation of (3.17), we find

$$(4.26)\qquad \begin{cases} \tau_n^* &= \frac{\sqrt{(\sqrt{1+f^2}U-\phi)^2+(1+f^2)V^2+(\sqrt{1+f^2}W+\phi f)^2}}{k(1+f^2)^2}, \\ B_n^* &= \frac{\epsilon_1\left((\sqrt{1+f^2}U-\phi)T+\sqrt{1+f^2}VN+(\sqrt{1+f^2}W+\phi f)B\right)}{\sqrt{(\sqrt{1+f^2}U-\phi)^2+(1+f^2)V^2+(\sqrt{1+f^2}W+\phi f)^2}}. \end{cases}$$

From (4.23), (4.24) and (4.26), we obtain (4.14). Whereby proof is complete. □

Now, we study binormal indicatrices of non-homothetic Bertrand pair curves $\alpha_b = B$ and $\alpha_b^* = B^*$. We have:

Theorem 4.3. *Let α and α^* be non-homothetic Bertrand pair curves satisfying (2.2) with their binormal indicatrices α_b and α_b^*. Then*

$$(4.27)\qquad \begin{cases} T_b &= m_1m_2m_3N^*,\ N_b = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(-T^*+HB^*), \\ B_b &= \frac{m_1m_2m_3}{\sqrt{1+H^2}}(B^*+HT^*),\ s_b = \int\frac{1}{\sqrt{h^2+1}}(m_2m_3k^*-h\tau^*)ds^*, \\ k_b &= \frac{k^*\sqrt{1+H^2}\sqrt{1+h^2}}{(m_2m_3k^*-h\tau^*)},\ \tau_b = \frac{H'\sqrt{1+h^2}}{(1+H^2)(m_2m_3k^*-h\tau^*)}, \end{cases}$$

$$(4.28)\qquad \begin{cases} T_b^* &= -m_1m_2m_3N, N_b^* = \frac{m_1m_2m_3}{\sqrt{1+f^2}}(T-fB),\\ B_b^* &= -\frac{m_1m_2m_3}{\sqrt{1+f^2}}(B+fT),\ s_b^* = \int \frac{k(1+hf)}{\sqrt{h^2+1}}ds,\\ k_b^* &= \frac{\sqrt{1+f^2}\sqrt{1+h^2}}{(1+hf)},\ \tau_b^* = \frac{f'\sqrt{1+h^2}}{k(1+hf)(1+f^2)}. \end{cases}$$

Proof. From (2.8), we have

$$(4.29)\qquad \alpha_b = \frac{m_1m_2m_3}{\sqrt{1+h^2}}(m_2m_3T^* + hB^*).$$

Differentiating (4.29) with respect to s^* and using (2.1) and then taking inner product with itself, we have

$$(4.30)\qquad T_b = m_1m_2m_3N^*,\ \frac{ds_b}{ds^*} = \frac{1}{\sqrt{h^2+1}}(m_2m_3k^* - h\tau^*).$$

Differentiating (4.30) with respect to s_b and using (2.1) and first relation of (3.30), we obtain

$$(4.31)\qquad N_b = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(-T^* + HB^*),\ k_b = \frac{k^*\sqrt{1+H^2}\sqrt{1+h^2}}{(m_2m_3k^* - h\tau^*)}.$$

Differentiating N_b in (4.31) with respect to s_b and using (2.1) and (4.30), we obtain

$$(4.32)\qquad \frac{dN_b}{ds_b} = m_1m_2m_3\frac{\sqrt{1+h^2}\big(-k^*(1+H^2)^2N^* + H'(B^*+HT^*)\big)}{(1+H^2)^{3/2}(m_2m_3k^* - h\tau^*)}.$$

Using (4.30), (4.31) and (4.32) in second relation of (3.30), we get

$$(4.33)\qquad B_b = \frac{m_1m_2m_3}{\sqrt{1+H^2}}(B^* + HT^*),\ \tau_b = \frac{H'\sqrt{1+h^2}}{(1+H^2)(m_2m_3k^* - h\tau^*)}.$$

Using (4.30), (4.31) and (4.33). We obtain (4.27).

From (2.8), we have

$$(4.34)\qquad \alpha_b^* = \frac{m_1m_2m_3}{\sqrt{1+h^2}}(-T + hB).$$

Differentiating (4.34) with respect to s and using (2.1) and taking inner product with itslef, we have

$$(4.35)\qquad T_b^* = -m_1m_2m_3N,\ \frac{ds_b^*}{ds} = \frac{1}{\sqrt{h^2+1}}(k + h\tau).$$

Differentiating (4.35) with respect to s_b^* and using (2.1) and first relation of (3.30), we obtain

$$N_b^* = \frac{m_1 m_2 m_3}{\sqrt{1+f^2}}(T - fB), \; k_b^* = \frac{\sqrt{1+f^2}\sqrt{1+h^2}}{(1+hf)}. \tag{4.36}$$

Differentiating N_b^* in (4.36) with respect to s_b^* and using (2.1) and (4.35), we get

$$\frac{dN_b^*}{ds_b^*} = m_1 m_2 m_3 \frac{\sqrt{1+h^2}\big(k(1+f^2)^2 N - f'(B+fT)\big)}{k(1+hf)(1+f^2)^{3/2}}. \tag{4.37}$$

Using (4.35), (4.36) and (4.37) in second relation of (3.30), we obtain

$$B_b^* = -\frac{m_1 m_2 m_3}{\sqrt{1+f^2}}(B + fT), \; \tau_b^* = \frac{f'\sqrt{1+h^2}}{k(1+f^2)(1+hf)}. \tag{4.38}$$

Using (4.35), (4.36) and (4.38), we obtain (4.28). This completes the proof of the Theorem. □

Next, using Proposition 2.6, Proposition 2.7, Theorem 3.1, and Theorem 3.3, we have

Corollary 4.4. *Let α and α^* be homothetic Bertrand pair curves satisfying* (2.2)*, then*

$$\Gamma = \Gamma^* = \frac{\tau_t}{k_t} = \frac{\tau_b}{k_b} = \frac{\tau_t^*}{k_t^*} = \frac{\tau_b^*}{k_b^*}. \tag{4.39}$$

Using Proposition 2.6, Proposition 2.7, Theorem 4.1, and Theorem 4.3, we have

Corollary 4.5. *Let α and α^* be non-homothetic Bertrand pair curves satisfying* (2.2)*, then*

$$\Gamma^* = m_3 \frac{\tau_t^*}{k_t^*} = m_3 \frac{\tau_b^*}{k_b^*}, \tag{4.40}$$

$$\Gamma = \frac{\tau_t}{k_t} = \frac{\tau_b}{k_b}. \tag{4.41}$$

Theorem 4.6. *A non-helical and non-planar Bertrand pair curves satisfying* (2.2) *is a slant helix if and only if their tangent indicatrix or binormal indicatrix is a spherical helix.*

Proof. Let α be a slant helix. Therefore, from (4.41), we have

$$\Gamma = \frac{\tau_t}{k_t} = \frac{\tau_b}{k_b} = constant. \tag{4.42}$$

Conversely, let the tangent indicatrix or binormal indicatrix of the Bertrand curve be a spherical helix. Then from (4.39), we get Γ is constant. Hence, the proof is complete.

The proof is similar to the Bertrand mate curve. □

Proposition 4.7. *The Γ_t, Γ_b, Γ_t^* and Γ_b^* of a tangent indicatrices and binormal indicatrices of Bertrand pair curves α and α^* satisfying* (2.2) *are given by*

$$\Gamma_t = \Gamma_b = \frac{\big(1+H^2)^{3/2}(H''k^*(1+H^2) - 3k^*HH'^2 - H'k^{*\prime}(1+H^2)\big)}{({k^*}^2(1+H^2)^3 + H'^2)^{3/2}}, \tag{4.43}$$

$$\Gamma_t^* = \Gamma_b^* = \frac{(1+f^2)^{3/2}\big(f''k(1+f^2) - 3kff'^2 - f'k'(1+f^2)\big)}{\big(k^2(1+f^2)^3 + f'^2\big)^{3/2}}, \tag{4.44}$$

where $H' = \frac{dH}{ds^*}$, $H'' = \frac{d^2H}{ds^{*2}}$, ${k^*}' = \frac{dk^*}{ds^*}$, $f' = \frac{df}{ds}$, $f'' = \frac{d^2f}{ds^2}$, $k' = \frac{dk}{ds}$.

Proof. Using (2.13), we have

$$\Gamma_t = \left(\frac{\tau_t}{k_t}\right)' \frac{k_t^2}{(k_t^2+\tau_t^2)^{3/2}}. \tag{4.45}$$

Using k_t and τ_t from (3.2) or from (4.1) in (4.45), we obtain (4.43).

Now, again using (2.13), we have

$$\Gamma_b = \left(\frac{\tau_b}{k_b}\right)' \frac{k_b^2}{(k_b^2+\tau_b^2)^{3/2}}. \tag{4.46}$$

Using k_b and τ_b from (3.29) or from (4.27) in (4.46), we obtain (4.43).

Using (2.13), we have

$$\Gamma_t^* = \left(\frac{\tau_t^*}{k_t^*}\right)' \frac{k_t^{*2}}{(k_t^{*2}+\tau_t^{*2})^{3/2}}. \tag{4.47}$$

Using k_t^* and τ_t^* from (3.2) or from (4.2) in (4.47), we obtain (4.44).

Now, again using (2.13), we have

$$\Gamma_b^* = \left(\frac{\tau_b^*}{k_b^*}\right)' \frac{k_b^{*2}}{(k_b^{*2}+\tau_b^{*2})^{3/2}}. \tag{4.48}$$

Using k_b^* and τ_b^* from (3.29) or from (4.27) in (4.48), we obtain (4.44). Thus, proof is complete. □

Theorem 4.8. *The spherical image of tangent and binormal indicatrices α_t and α_b are spherical slant helix if and only if α is a slant helix.*

Proof. From (14) and (4.43), we get the relation

$$\Gamma_t = \Gamma_b = \frac{(1+H^2)\Gamma'}{(k^{*2}(1+H^2)^3 + H'^2)^{3/2}}. \tag{4.49}$$

Thus we have the proof of the Theorem. □

5. Example

Example 5.1. Let us consider the curve in $\mathbb{E}^3$ given by

$$\alpha(s) = \Big(\frac{-2}{\sqrt{3}} \sin\sqrt{3}s\ \cos s + \sin s\ \cos\sqrt{3}s, \frac{2}{\sqrt{3}} \cos\sqrt{3}s\ \cos s + \sin s\ \sin\sqrt{3}s, -\frac{\sqrt{2}}{\sqrt{3}} \cos s\Big),$$

with curvature $k = \sqrt{2}\cos s$ and torsion $\tau = \sqrt{2}\sin s$.

The Frenet frame of α is given by

$$\begin{cases} T = \frac{1}{\sqrt{3}}\big(-\sqrt{3}\cos s\ \cos\sqrt{3}s - \sin s\ \sin\sqrt{3}s, -\sqrt{3}\ \sin\sqrt{3}s\ \cos s + \sin s\ \cos\sqrt{3}s, \sqrt{2}\ \sin s\big), \\ N = \frac{1}{\sqrt{3}}\big(\sqrt{2}\ \sin\sqrt{3}s, -\sqrt{2}\ \cos\sqrt{3}s, 1\big), \\ B = \frac{1}{\sqrt{3}}\big(-\sin\sqrt{3}s\ \cos s + \sqrt{3}\ \sin s\ \cos\sqrt{3}s, \cos s\ \cos\sqrt{3}s + \sqrt{3}\ \sin s\ \sin\sqrt{3}s, \sqrt{2}\ \cos s\big). \end{cases}$$

Moreover, from (14), we find

$$\Gamma = \frac{-1}{\sqrt{2}}. \tag{5.1}$$

Using (2.2), the Bertrand mate curve α^* is given by

$$\alpha^* = \Big(\frac{-1}{\sqrt{3}} \sin\sqrt{3}s\ \big(2\cos s - s\ \csc s\big) + \sin s\ \cos\sqrt{3}s, \frac{1}{\sqrt{3}} \cos\sqrt{3}s\ \big(2\cos s - s\ \csc s\big) + \sin s\ \sin\sqrt{3}s, \frac{-\sqrt{2}}{\sqrt{3}}\Big(\frac{3}{2} + \cos s + s\ \csc s\Big)\Big),$$

where $u = -(s+\sin s)$, v$= -\frac{1}{\sqrt{2}}$ and $w = -(\cos s + s\ \cot s)$.

Computing the curvature and torsion of α^*, we get

$$k^* = \frac{\sqrt{2}\sin^3 s}{s - \cos s \sin s},\ \tau^* = \frac{\sqrt{2}\sin^2 s \cos s}{s - \cos s\ \sin s}.$$

Further, the Frenet frame of α^* is given by

$$\begin{cases} T^* & = \frac{1}{\sqrt{3}}\big(\sqrt{3}\ \sin s\ \cos\sqrt{3}s - \sin\sqrt{3}s\ \cos s, \sqrt{3}\ \sin s\ \sin\sqrt{3}s + \\ & \cos\sqrt{3}s\ \cos s, \sqrt{2}\ \cos s\big), \\ N^* & = \frac{1}{\sqrt{3}}\big(-\sqrt{2}\ \sin\sqrt{3}s, \sqrt{2}\ \cos\sqrt{3}s, -1\big), \\ B^* & = \frac{1}{\sqrt{3}}\big(-\sin s\ \sin\sqrt{3}s - \sqrt{3}\ \cos s\ \cos\sqrt{3}s, \sin s\ \cos\sqrt{3}s - \\ & \sqrt{3}\ \sin\sqrt{3}s\ \cos s, \sqrt{2}\ \sin s\big). \end{cases}$$

Also from (18), we have

$$\Gamma^* = \frac{1}{\sqrt{2}}. \tag{5.1}$$

Now, the tangent indicatrix of Bertrand curve α is $\alpha_t = T$.

Using (4.1), we get the curvature and torsion of α_t as follows:

$$\tau_t = \frac{1}{\sqrt{2}\cos s},\ k_t = \frac{-1}{\cos s}. \tag{5.2}$$

Now, the binormal indicatrix of Bertrand curve α is $\alpha_b = B$.

Using (4.27), we obtain the curvature and torsion of α_b as

$$\tau_b = \frac{-1}{\sqrt{2}\sin s},\ k_b = \frac{1}{\sin s}. \tag{5.3}$$

From (5.1), (5.2) and (5.3), we have

$$\frac{\tau_t}{k_t} = \frac{\tau_b}{k_b} = \Gamma. \tag{5.4}$$

Now, the tangent indicatrix of Bertrand mate curve α^* is $\alpha_t^* = T^*$.

Using (4.2), we obtain the curvature and torsion of α_t^* as

$$\tau_t^* = \frac{-1}{\sqrt{2}\sin s},\ k_t^* = \frac{-1}{\sin s}. \tag{5.5}$$

Now, the binormal indicatrix of Bertrand mate curve α^* is $\alpha_b^* = B^*$.

Using (4.28) the curvature and torsion of α_b^* as follows:

$$\tau_b^* = \frac{1}{\sqrt{2}\cos s},\ k_b^* = \frac{1}{\cos s}. \tag{5.6}$$

From (5.1), (5.5) and (5.6), we find

$$\frac{\tau_t^*}{k_t^*} = \frac{\tau_b^*}{k_b^*} = \Gamma^*. \tag{5.7}$$

This example also supports Theorem 4.6.

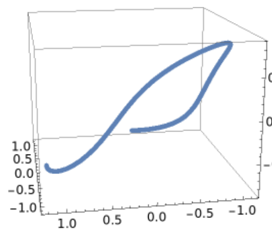

Figure 4: $Curve\ \alpha$

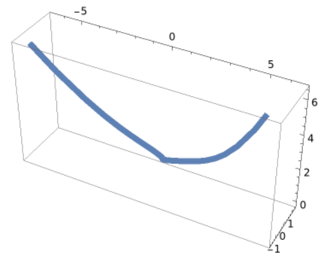

Figure 5: $Curve\,\alpha^*$

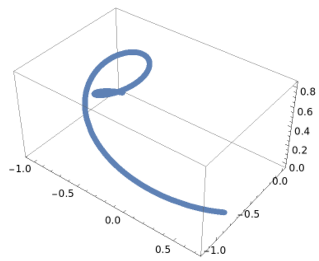

Figure 6: $Curve\ \alpha_t$

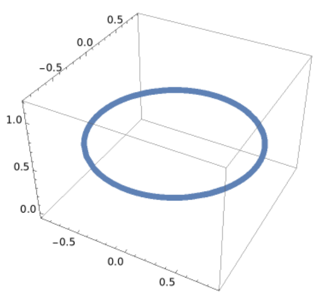

Figure 7: $Curve\ \alpha_n$

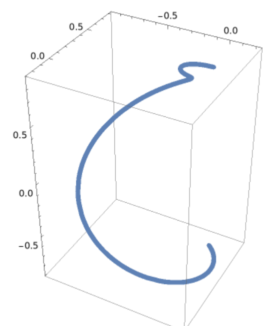

Figure 8: $Curve\ \alpha_b$

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References

[1] A. T. Ali and R. López, *Slant helices in Minkowski space* $\mathbb{E}_1^3$, J. Korean Math. Soc., **48(1)**(2011) 159–167.

[2] A. T. Ali, *Position vectors of slant helices in Euclidean 3-space*, J. Egypt. Math. Soc., **20**(2012), 1–6.

[3] B. Altunkaya, F. K. Aksoyak, L. kula and C. Aytekin, *On rectifying slant helices in Euclidean 3-space*, Konuralp J. Math., **4(2)**(2016), 17–24.

[4] I. Aydemir and F. Yerlikaya, *New representations of spherical indicatrices of Bertrand curves in Minkowski 3-space*, Geometry, Article ID 509058 (2015), 5 pages.

[5] M. Babaarslan and Y. Yayli, *On helices and Bertrand curves in Euclidean 3-space*, Math. Comput. Appl., **18(1)**(2013), 1–11.

[6] H. Balgetir, M. Bektas and M. Ergut, *Bertrand curve for non-null curves in 3-dimensional Lorentzian space*, Hadronic J., **27(2)**(2004), 229–236.

[7] J.M. Bertrand, *Mémoire sur la théorie des courbes á double courbure*, Comptes Rendus, **15**(1850), 332–350.

[8] C. Camci, L. Kula and M. Altinok, *On spherical slant helices in Euclidean 3-space*, arXiv:1308.5532 [math.DG], 2013.

[9] C. Camci, A. Ucum and K. Ilarslan, *A new approach to Bertrand curves in Euclidean 3-space*, J. Geom., **111(3)**(2020).

[10] Y. M. Cheng and C. C. Lin, *On the generalized Bertrand curves in Euclidean N-spaces*, Note Mat., **29(2)**(2009), 33–39.

[11] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces,* Prentice Hall, Englewood Cliffs(1976).

[12] McGraw-Hill, *Dictionary of Scientific and Technical Terms*, 6E(2003).

[13] K. Ilarslan and A. Aslan, *On spacelike Bertrand curve in Minkowski 3-space*, Konuralp J. Math., **5(1)**(2017), 214–222.

[14] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28**(2004), 531–537.

[15] L. Kula and Y. Yaylı, *On slant helix and its spherical indicatrix*, Appl. Math. Comput., **169(1)**(2005), 600–607.

[16] L. Kula, N. Ekmekci, Y. Yaylı and K. Ilarslan, *Characterizations of slant helices in Euclidean 3-space*, Turk. J. Math., **34(2)**(2010), 261–273.

[17] M. A. Lancret, *Mámoire sur les courbes à double courbure*, Mémoires présentés à l'Institut1.(1806), 416–454.

[18] H. Matsuda and S. Yorozu, *Notes on Bertrand curves*, Yokohama Math. J., **50**(2003), 41–58.

[19] T. Menninger, *Characterization of the slant helix as successor curve of the general helix*, Int. Electron. J. Geom., **7(2)**(2014), 84–91.

[20] J. Monterde, *Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion*, Comput. Aided Geom. Design, **26(3)**(2009), 271–278.

[21] Venant B. Saint, *Mémoire sur les lignes courbes non planes*, Journal de l'Ecole Polytechnique, **18**(1845), 1–76.

[22] E. Salkowski, *Zur transformation von raumkurven*, Mathematische Annalen, **66**(1909), 517–557.

[23] P. D. Scofield, *Curves of constant precession*, Amer. Math. Monthly, **102**(1995), 531–537.

[24] S. Tamta and R. S. Gupta, *New parametrization of Bertrand partner D-curves in* $\mathbb{E}^3$, Bol. Soc. Paran. Mat., (in press) doi:10.5269/bspm.63309.

[25] Y. Tuncer and S. Unal, *New representations of Bertrand pairs in Euclidean 3-space*, Appl. Math. Comput., **219(4)**(2012), 1833–1842.

[26] A. Ucum, K. Ilarslan, *On timelike Bertrand curve in Minkowski 3-space*, Honam Math. J., **38(3)**(2016), 467–477.

[27] A. Ucum, O. Kecilioglu and K. Ilarslan, *Generalized Bertrand curves with timelike (1,3)-normal plane in Minkowski Space-time*, Kuwait J. Sci., **42(3)**(2015), 10–27.

[28] X. Yang, *High accuracy approximation of helices by quintic curves*, Comput. Aided Geom. Des., **20**(2003), 303–317.

[29] B. Yilmaz and A. Has, *New approach to slant helix*, Int. Electron. J. Geom., **12(1)**(2019), 111–115.