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Geometry of (p, f)-bienergy variations between Riemannian manifolds

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Abstract. In this paper, we extend the definition of the Jacobi operator of smooth maps, and biharmonic maps via the variation of bienergy between two Riemannian manifolds. We construct an example of (p, f)-biharmonic non (p, f)-harmonic map. We also prove some Liouville type theorems for (p, f)-biharmonic maps.

1. Introduction

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The p-energy of φ is defined by

$$E_p(\varphi; D) = \frac{1}{p}\int_D |d\varphi|^p v_g \qquad (p \geq 2), \tag{1.1}$$

where D is a compact subset of M. We say that φ is a p-harmonic map if it is a critical point of the p-energy functional, that is to say, if it satisfies the following Euler-Lagrange equation (see [1, 2, 3, 6, 7])

$$\tau_p(\varphi) \equiv \operatorname{div}^M(|d\varphi|^{p-2} d\varphi) = 0. \tag{1.2}$$

Let f be a smooth positive function on M. The (p, f)-energy of φ is defined by

$$E_{p,f}(\varphi; D) = \frac{1}{p}\int_D f|d\varphi|^p v_g. \tag{1.3}$$

The (p, f)-energy functional (1.3) includes as a special case ($f = 1$) the p-energy functional, and a special case ($p = 2$) the f-energy functional (see [4, 5, 13]). We call (p, f)-harmonic (or generalized p-harmonic) a smooth map φ which is a critical

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point of the (p,f)-energy functional for any compact domain D.

Theorem 1.1. (The first variation of the (p,f)-energy [14]) *Let φ be a smooth map from Riemannian manifold (M,g) to Riemannian manifold (N,h), and $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ a smooth variation of φ to support in $D\subset M$. Then*

$$\frac{d}{dt}E_{p,f}(\varphi_t;D)\Big|_{t=0} = -\int_D h(v,\tau_{p,f}(\varphi))v_g, \tag{1.4}$$

where $\tau_{p,f}(\varphi)$ is the (p,f)-tension field of φ given by

$$\tau_{p,f}(\varphi) \equiv \operatorname{div}^M(f|d\varphi|^{p-2}d\varphi) = f\tau_p(\varphi) + |d\varphi|^{p-2}d\varphi(\operatorname{grad}^M f), \tag{1.5}$$

and $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$.

The extension of p-harmonic and f-harmonic maps between Riemannian manifolds has been studied by several authors (see for example, P. Baird [2], N. Course [4], A. M. Cherif, M. Djaa, R. Nasri, S. Ouakkas and K. Zagga [5, 13]). In this paper, we extend the definition of p-biharmonic maps between Riemannian manifolds. Liouville type theorems for p-harmonic and p-biharmonic maps between complete smooth Riemannian manifolds have been done by many authors. J. Liu, D. J. Moon, H. Liu, S. D. Jung [9, 11] and N. Nakauchi [12] proved the Liouville type theorem for p-harmonic maps. A. M. Cherif [10] also proved the Liouville type theorem for p-biharmonic maps. The purpose of this article is also to provide a proof of Liouville's type theorem for (p,f)-biharmonic maps from compact orientable Riemannian manifold without boundary (resp. from complete non-compact Riemannian manifold) into a Riemannian manifold with non-positive sectional curvature.

2. Main Results

In the following, we will compute the second variation formula of the (p,f)-harmonic maps.

Theorem 2.1. (The second variation of the (p,f)-energy) *Let φ be an (p,f)-harmonic map from Riemannian manifold (M,g) to Riemannian manifold (N,h), and $\{\varphi_{t,s}\}_{(t,s)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ a smooth variation of φ to support in $D\subset M$. We set*

$$v = \frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \quad w = \frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \tag{2.1}$$

Then we have

$$\frac{\partial^2}{\partial t\partial s}E_{p,f}(\varphi_{t,s};D)\Big|_{(t,s)=(0,0)} = \int_D h(J^\varphi_{p,f}(v),w)v_g, \tag{2.2}$$

where $J^\varphi_{p,f}(v)$ is the generalized Jacobi operator of φ given by

$$\begin{aligned} J^\varphi_{p,f}(v) &= -f|d\varphi|^{p-2}\operatorname{trace}_g R^N(v,d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f|d\varphi|^{p-2}\nabla^\varphi v \\ &\quad -(p-2)\operatorname{trace}_g \nabla\langle\nabla^\varphi v, d\varphi\rangle f|d\varphi|^{p-4}\,d\varphi. \end{aligned} \tag{2.3}$$

Proof. Let $\phi : M \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \to N$ be a smooth map defined by

$$\phi(x,t,s) = \varphi_{t,s}(x), \quad \forall (x,t,s) \in M \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon).$$

We have $\phi(x,0,0) = \varphi(x)$, and the variation vectors fields v, w associated to the variation $\{\varphi_{t,s}\}_{(t,s)\in(-\epsilon,\epsilon)^2}$ are given by

$$v(x) = d_{(x,0,0)}\phi(\frac{\partial}{\partial t}), \quad w(x) = d_{(x,0,0)}\phi(\frac{\partial}{\partial s}), \;\; \forall x \in M. \tag{2.4}$$

Let $\{e_i\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_j} e_i = 0$ at $x \in M$, for all $i, j = 1, ..., m$. We compute

$$\frac{\partial^2}{\partial t \partial s} E_{p,f}(\varphi_{t,s}; D)\Big|_{t=s=0} \;=\; \frac{1}{p}\int_D f \frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p\Big|_{t=s=0} v_g. \tag{2.5}$$

First, note that

$$\begin{aligned}
\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{p}{2}\left(|d\varphi_{t,s}|^2\right)^{\frac{p}{2}-1}\frac{\partial}{\partial s}(|d\varphi_{t,s}|^2)\right) \\
&= \frac{\partial}{\partial t}\left(p|d\varphi_{t,s}|^{p-2} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0), d\phi(e_i,0,0))\right).
\end{aligned}$$

So that

$$\begin{aligned}
\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p &= p\frac{\partial}{\partial t}(|d\varphi_{t,s}|^2)^{\frac{p-2}{2}} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0), d\phi(e_i,0,0)) \\
&\quad + p|d\varphi_{t,s}|^{p-2} h(\nabla^\phi_{\frac{\partial}{\partial s}} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0,0), d\phi(e_i,0,0)) \\
&\quad + p|d\varphi_{t,s}|^{p-2} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0,0)).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p &= p(p-2)|d\varphi_{t,s}|^{p-4} h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_j,0,0), d\phi(e_j,0,0)) \\
&\quad\; h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0), d\phi(e_i,0,0)) \\
&\quad + p|d\varphi_{t,s}|^{p-2} h(\nabla^\phi_{\frac{\partial}{\partial s}} \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0,0), d\phi(e_i,0,0)) \\
&\quad + p|d\varphi_{t,s}|^{p-2} h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0), \nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0,0)).
\end{aligned}$$

By (2.4), and the definition of the curvature tensor of (N,h), with

$$\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0,0) = \nabla^\phi_{(e_i,0,0)} d\phi(\frac{\partial}{\partial t}), \quad \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i,0,0) = \nabla^\phi_{(e_i,0,0)} d\phi(\frac{\partial}{\partial s}),$$

we obtain the following equation

$$\begin{aligned}\frac{1}{p}f\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p\Big|_{t=s=0} &= (p-2)f|d\varphi|^{p-4}h(\nabla^{\varphi}_{e_j}v,d\varphi(e_j))h(\nabla^{\varphi}_{e_i}w,d\varphi(e_i))\\ &\quad +f|d\varphi|^{p-2}h(R^N(w,d\varphi(e_i))v,d\varphi(e_i))\\ &\quad +f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t})\Big|_{t=s=0},d\varphi(e_i))\\ &\quad +f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}w,\nabla^{\varphi}_{e_i}v).\end{aligned}\tag{2.6}$$

The first term on the left-hand side of (2.6) is given by

$$\begin{aligned}(p-2)f|d\varphi|^{p-4}h(\nabla^{\varphi}_{e_j}v,d\varphi(e_j))h(\nabla^{\varphi}_{e_i}w,d\varphi(e_i)) &= \operatorname{div}^M\omega_1\\ &\quad -(p-2)h(w,\operatorname{trace}_g\nabla\langle\nabla^{\varphi}v,d\varphi\rangle f|d\varphi|^{p-4}d\varphi),\end{aligned}\tag{2.7}$$

where $\langle\nabla^{\varphi}v,d\varphi\rangle = h(\nabla^{\varphi}_{e_j}v,d\varphi(e_j))$, and $\omega_1\in\Gamma(T^*M)$ is defined by

$$\omega_1(X) = (p-2)h(w,\langle\nabla v,d\varphi\rangle f|d\varphi|^{p-4}d\varphi(X)),\quad \forall X\in\Gamma(TM).$$

By the (p,f)-harmonicity condition of φ, the third term on the left-hand side of (2.6) is given by

$$f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t})\Big|_{t=s=0},d\varphi(e_i)) = \operatorname{div}^M\omega_2,\tag{2.8}$$

where $\omega_2\in\Gamma(T^*M)$ is defined by

$$\omega_2(X) = h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\phi(\frac{\partial}{\partial t})\Big|_{t=s=0},f|d\varphi|^{p-2}d\varphi(X)),\quad \forall X\in\Gamma(TM).$$

The fourth term on the left-hand side of (2.6) is given by

$$\begin{aligned}f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}w,\nabla^{\varphi}_{e_i}v) &= e_i\left(h(w,f|d\varphi|^{p-2}\nabla^{\varphi}_{e_i}v)\right)-h(w,\nabla^{\varphi}_{e_i}f|d\varphi|^{p-2}\nabla^{\varphi}_{e_i}v)\\ &= \operatorname{div}^M\omega_3-h(w,\operatorname{trace}_g\nabla^{\varphi}f|d\varphi|^{p-2}\nabla^{\varphi}v),\end{aligned}\tag{2.9}$$

where $\omega_3(X) = h(w,f|d\varphi|^{p-2}\nabla^{\varphi}_X v)$, for all $X\in\Gamma(TM)$. Substituting (2.7), (2.8) and (2.9) in (2.6), we obtain

$$\begin{aligned}\frac{1}{p}f\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^p\Big|_{t=s=0} &= \operatorname{div}^M\omega_1-(p-2)h(w,\operatorname{trace}_g\nabla\langle\nabla^{\varphi}v,d\varphi\rangle f|d\varphi|^{p-4}d\varphi)\\ &\quad -f|d\varphi|^{p-2}h(R^N(v,d\varphi(e_i))d\varphi(e_i),w)+\operatorname{div}^M\omega_2\\ &\quad +\operatorname{div}^M\omega_3-h(w,\operatorname{trace}_g\nabla^{\varphi}f|d\varphi|^{p-2}\nabla^{\varphi}v).\end{aligned}\tag{2.10}$$

By the equations (2.5), (2.10), and the divergence Theorem, we get

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}E_{p,f}(\varphi_{t,s};D)\Big|_{t=s=0} &= \int_D h\Big(-f|d\varphi|^{p-2}\operatorname{trace}_g R^N(v,d\varphi)d\varphi\\ &\quad -(p-2)\operatorname{trace}_g\nabla\langle\nabla^{\varphi}v,d\varphi\rangle f|d\varphi|^{p-4}d\varphi\\ &\quad -\operatorname{trace}_g\nabla^{\varphi}f|d\varphi|^{p-2}\nabla^{\varphi}v,w\Big)v_g.\end{aligned}$$

This completes the proof of the theorem. □

In [8], G. Y. Jiang calculated the first variation formula of the bienergy functional, thereby finding the biharmonic maps between two Riemannian manifolds. A. M. Cherif [10], introduced the notion of p-biharmonic maps. In the following, we extend the definition of p-biharmonic maps between Riemannian manifolds. The (p,f)-bienergy of a smooth map $\varphi:(M,g)\to(N,h)$ between two Riemannian manifolds is defined by

$$E_{2,p,f}(\varphi;D)=\frac{1}{2}\int_D |\tau_{p,f}(\varphi)|^2 v_g \quad (p\geq 2),$$

where f is a smooth positive function on M, and D a compact subset of M. A map is called (p,f)-biharmonic (or generalized p-biharmonic), if it is a critical point of the (p,f)-bienergy functional over any compact subset D of M. Under the above notation, we obtain the following result.

Theorem 2.2. (The first variation of the (p,f)-bienergy) *Let $\varphi:(M,g)\to(N,h)$ be a smooth map between two Riemannian manifolds, and $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ a smooth variation of φ to support in $D\subset M$. Then*

$$\frac{d}{dt}E_{2,p,f}(\varphi_t;D)\Big|_{t=0}=-\int_D h(v,\tau_{2,p,f}(\varphi))v_g, \tag{2.11}$$

where $\tau_{2,p,f}(\varphi)$ is the (p,f)-bitension field of φ given by

$$\begin{aligned}\tau_{2,p,f}(\varphi) &= -f|d\varphi|^{p-2}\operatorname{trace}_g R^N(\tau_{p,f}(\varphi),d\varphi)d\varphi-\operatorname{trace}_g\nabla^\varphi f|d\varphi|^{p-2}\nabla^\varphi\tau_{p,f}(\varphi)\\ &\quad -(p-2)\operatorname{trace}_g\nabla\langle\nabla^\varphi\tau_{p,f}(\varphi),d\varphi\rangle f|d\varphi|^{p-4}\,d\varphi,\end{aligned} \tag{2.12}$$

and $v=\frac{d\varphi_t}{dt}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$.

Proof. Let $\phi: M\times(-\epsilon,\epsilon)\to N$ be a smooth map defined by $\phi(x,t)=\varphi_t(x)$ for all $(x,t)\in M\times(-\epsilon,\epsilon)$, we have $\phi(x,0)=\varphi(x)$, and the variation vector field v associated to the variation $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ is given by $v(x)=d_{(x,0)}\phi(\frac{\partial}{\partial t})$, for all $x\in M$. Let $\{e_i\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_j}e_i=0$ at $x\in M$ for all $i,j=1,...,m$. We compute

$$\frac{d}{dt}E_{2,p,f}(\varphi_t;D)\Big|_{t=0}=\int_D h(\nabla^\phi_{\frac{\partial}{\partial t}}\tau_{p,f}(\varphi_t),\tau_{p,f}(\varphi_t))\Big|_{t=0}v_g, \tag{2.13}$$

By using (1.5), we have

$$\nabla^\phi_{\frac{\partial}{\partial t}}\tau_{p,f}(\varphi_t)=\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)}f|d\varphi_t|^{p-2}d\phi(e_i,0).$$

From the definition of curvature tensor of (N,h), we get

$$\begin{aligned}\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)}f|d\varphi_t|^{p-2}d\phi(e_i,0) &= R^N(d\phi(\frac{\partial}{\partial t}),d\phi(e_i,0))f|d\varphi_t|^{p-2}d\phi(e_i,0)\\ &\quad+\nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}}f|d\varphi_t|^{p-2}d\phi(e_i,0).\end{aligned} \tag{2.14}$$

By the compatibility of ∇^ϕ with the metric h, we obtain

$$h(\nabla^\phi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0),\tau_{p,f}(\varphi_t))$$

$$= e_i\big(h(\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0),\tau_{p,f}(\varphi_t))\big) -h(\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0),\nabla^\phi_{(e_i,0)}\tau_{p,f}(\varphi_t)). \tag{2.15}$$

By using the property

$$\nabla^\phi_X d\phi(Y) = \nabla^\phi_Y d\phi(X) + d\phi([X,Y]),$$

(see [1]), with $X = \frac{\partial}{\partial t}$, and $Y = f|d\varphi_t|^{p-2}(e_i,0)$, and by a simple calculation, we get

$$\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0) = f|d\varphi_t|^{p-2}\nabla^\phi_{(e_i,0)}d\phi(\frac{\partial}{\partial t}) +(p-2)f|d\varphi_t|^{p-4}h(\nabla^\phi_{(e_j,0)}d\phi(\frac{\partial}{\partial t}),d\phi(e_j,0))d\phi(e_i,0).$$

Substituting the last equation in (2.15), we obtain

$$h(\nabla^\phi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0),\tau_{p,f}(\varphi_t))\Big|_{t=0}$$

$$= \operatorname{div}^M\eta_1 + \operatorname{div}^M\eta_2 - f|d\varphi|^{p-2}h(\nabla^\varphi_{e_i}v,\nabla^\varphi_{e_i}\tau_{p,f}(\varphi)) -(p-2)f|d\varphi|^{p-4}h(\nabla^\varphi_{e_j}v,d\varphi(e_j))h(d\varphi(e_i),\nabla^\varphi_{e_i}\tau_{p,f}(\varphi)), \tag{2.16}$$

where η_1 and η_2 are defined by

$$\eta_1(X) = f|d\varphi|^{p-2}h(\nabla^\varphi_X v,\tau_{p,f}(\varphi)),$$
$$\eta_2(X) = (p-2)f|d\varphi|^{p-4}\langle\nabla^\varphi_X v,d\varphi\rangle h(d\varphi(X),\tau_{p,f}(\varphi)),$$

for all $X\in\Gamma(TM)$. The equation (2.16) is equivalent to the following

$$h(\nabla^\phi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}} f|d\varphi_t|^{p-2}d\phi(e_i,0),\tau_{p,f}(\varphi_t))\Big|_{t=0}$$

$$= \operatorname{div}^M\eta_1 + \operatorname{div}^M\eta_2 - \operatorname{div}^M\eta_3 + h(v,\nabla^\varphi_{e_i} f|d\varphi|^{p-2}\nabla^\varphi_{e_i}\tau_{p,f}(\varphi)) -\operatorname{div}^M\eta_4 + (p-2)h(v,\nabla^\varphi_{e_j} f|d\varphi|^{p-4}\langle d\varphi,\nabla^\varphi\tau_{p,f}(\varphi)\rangle d\varphi(e_j)). \tag{2.17}$$

where η_3 and η_4 are defined by

$$\eta_3(X) = f|d\varphi|^{p-2}h(v,\nabla^\varphi_X\tau_{p,f}(\varphi)),$$
$$\eta_4(X) = (p-2)f|d\varphi|^{p-4}\langle d\varphi,\nabla^\varphi\tau_{p,f}(\varphi)\rangle h(v,d\varphi(X)),$$

for all $X\in\Gamma(TM)$. By equations (2.13), (2.14), (2.17), and the divergence Theorem, the Theorem 2.2. follows. □

Remark 2.3. For any smooth map $\varphi : (M, g) \to (N, h)$ between two Riemannian manifolds, we have

$$\tau_{2,p,f}(\varphi) = J^{\varphi}_{p,f}(\tau_{p,f}(\varphi)).$$

From Theorem 2.2., the Euler-Lagrange equation for the (p, f)-bienergy functional is $\tau_{2,p,f}(\varphi) = 0$. We deduce:

Corollary 2.4. *A smooth map $\varphi : (M, g) \to (N, h)$ between two Riemannian manifolds is (p, f)-biharmonic if and only if $\tau_{2,p,f}(\varphi) = 0$.*

Remark 2.5. If the (p, f)-tension field of a smooth map $\varphi : (M, g) \to (\mathbb{R}^n, \langle\, , \,\rangle_{\mathbb{R}^n})$ is parallel along φ (that is, the components of $\tau_{p,f}(\varphi)$ are constants), then φ is (p, f)-biharmonic map.

Example 2.6. The smooth map

$$\begin{array}{rcl} \varphi : (\mathbb{R}^2\backslash\{(0,0)\} \times \mathbb{R}, dx^2 + dy^2 + dz^2) & \to & (\mathbb{R}^2, du^2 + dv^2), \\ (x, y, z) & \mapsto & (\sqrt{x^2 + y^2}, z) \end{array}$$

is (p, f)-biharmonic non (p, f)-harmonic, where

$$f(x, y, z) = 2^{-\frac{p}{2}} c\,\sqrt{x^2 + y^2}, \quad \forall (x, y, z) \in \mathbb{R}^2\backslash\{(0,0)\} \times \mathbb{R},$$

for some constant $c > 0$, and $p \geq 2$. Indeed; we have $g = dx^2 + dy^2 + dz^2$ and $h = du^2 + dv^2$. We set $\varphi_1(x, y, z) = \sqrt{x^2 + y^2}$ and $\varphi_2(x, y, z) = z$, for all $(x, y, z) \in \mathbb{R}^2\backslash\{(0,0)\} \times \mathbb{R}$. We compute

$$\begin{array}{rcl} |d\varphi|^2 & = & g^{ij}(h_{\alpha\beta} \circ \varphi)\dfrac{\partial \varphi_\alpha}{\partial x_i}\dfrac{\partial \varphi_\beta}{\partial x_j} \\ & = & \delta_{ij}\delta_{\alpha\beta}\dfrac{\partial \varphi_\alpha}{\partial x_i}\dfrac{\partial \varphi_\beta}{\partial x_j} \\ & = & \left(\dfrac{\partial \varphi_\alpha}{\partial x_i}\right)^2 \\ & = & 2. \end{array}$$

Here, $x_1 = x$, $x_2 = y$, and $x_3 = z$. Thus, the p-tension field of φ is given by

$$\begin{array}{rcl} \tau_p(\varphi) & = & 2^{\frac{p-2}{2}}\dfrac{\partial^2 \varphi_\alpha}{\partial x_i^2}\dfrac{\partial}{\partial u_\alpha} \\ & = & \dfrac{2^{\frac{p-2}{2}}}{\sqrt{x^2 + y^2}}\dfrac{\partial}{\partial u_1}, \end{array}$$

where $u_1 = u$ and $u_2 = v$. From equation (1.5), we get

$$\begin{aligned} \tau_{p,f}(\varphi) &= \frac{2^{\frac{p-2}{2}} f}{\sqrt{x^2+y^2}} \frac{\partial}{\partial u_1} + 2^{\frac{p-2}{2}} \frac{\partial f}{\partial x_i} \frac{\partial \varphi_\alpha}{\partial x_i} \frac{\partial}{\partial u_\alpha} \\ &= \frac{c}{2} \frac{\partial}{\partial u_1} + \frac{c}{2}\left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right] \frac{\partial}{\partial u_1}. \end{aligned}$$

Therefore, the components of $\tau_{p,f}(\varphi) = (c, 0)$ are constants. According to Remark 2.4., the map φ is (p, f)-biharmonic.

Using the similar technique of Theorem 5 and Theorem 6 in [10], we have the following Liouville type theorems for (p, f)-biharmonic maps.

Theorem 2.7. (Liouville type theorem for (p, f)-biharmonic maps: compact case) *Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h) a Riemannian manifold with non-positive sectional curvature. Then, every (p, f)-biharmonic map from (M, g) to (N, h) is (p, f)-harmonic.*

Proof. Let $\varphi : (M, g) \to (N, h)$ be a smooth (p, f)-biharmonic map, fix a point $x \in M$, and let $\{e_i\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_j} e_i = 0$ at $x \in M$, for all $i, j = 1, ..., m$. We have

$$\begin{aligned} 0 \;=\; & -f|d\varphi|^{p-2} \operatorname{trace}_g R^N(\tau_{p,f}(\varphi), d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f|d\varphi|^{p-2}\nabla^\varphi \tau_{p,f}(\varphi) \\ & -(p-2) \operatorname{trace}_g \nabla \langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4} d\varphi. \end{aligned}$$

Calculating at x, we get

$$\begin{aligned} & f|d\varphi|^{p-2} h(R^N(\tau_{p,f}(\varphi), d\varphi(e_i))d\varphi(e_i), \tau_{p,f}(\varphi)) \\ &\quad = -h(\nabla^\varphi_{e_i} f|d\varphi|^{p-2}\, \nabla^\varphi_{e_i} \tau_{p,f}(\varphi), \tau_{p,f}(\varphi)) \\ &\qquad -(p-2)h(\nabla^\varphi_{e_i}\langle \nabla^\varphi\, \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4} d\varphi(e_i), \tau_{p,f}(\varphi)). \end{aligned} \tag{2.18}$$

Let θ_1, θ_2 $\in \Gamma(T^*M)$ defined by

$$\begin{aligned} \theta_1(X) &= f|d\varphi|^{p-2} h(\nabla^\varphi_X\, \tau_{p,f}(\varphi),\ \tau_{p,f}(\varphi)), \\ \theta_2(X) &= (p-2)\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4} h(d\varphi(X), \tau_{p,f}(\varphi)), \end{aligned}$$

where $X \in \Gamma(TM)$. So that

$$\begin{aligned} \operatorname{div}^M \theta_1 &= h(\nabla^\varphi_{e_i} f|d\varphi|^{p-2}\nabla^\varphi_{e_i}\, \tau_{p,f}(\varphi),\ \tau_{p,f}(\varphi)) \\ &\quad + f|d\varphi|^{p-2} h(\nabla^\varphi_{e_i}\, \tau_{p,f}(\varphi),\ \nabla^\varphi_{e_i} \tau_{p,f}(\varphi)), \\ \operatorname{div}^M \theta_2 &= (p-2)h(\nabla^\varphi_{e_i}\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4} d\varphi(e_i), \tau_{p,f}(\varphi)) \\ &\quad + (p-2)\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4} h(d\varphi(e_i), \nabla^\varphi_{e_i} \tau_{p,f}(\varphi)). \end{aligned} \tag{2.19}$$

By (2.18) and (2.19), we obtain

$$\begin{aligned} f|d\varphi|^{p-2}h(R^N(\tau_{p,f}(\varphi),\, d\varphi(e_i))d\varphi(e_i),\, \tau_{p,f}(\varphi)) \\ = \; -\operatorname{div}^M \theta_1 + f|d\varphi|^{p-2}h(\nabla^\varphi_{e_i}\tau_{p,f}(\varphi), \nabla^\varphi_{e_i}\tau_{p,f}(\varphi)) - \operatorname{div}^M \theta_2 \\ +(p-2)\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}h(d\varphi(e_i), \nabla^\varphi_{e_i}\tau_{p,f}(\varphi)). \end{aligned}$$

The last equation is equivalent to the following

$$\begin{aligned} f|d\varphi|^{p-2}h(R^N(\tau_{p,f}(\varphi),\, d\varphi(e_i))d\varphi(e_i),\, \tau_{p,f}(\varphi)) \\ = \; -\operatorname{div}^M \theta_1 + f|d\varphi|^{p-2}|\nabla^\varphi \tau_{p,f}(\varphi)|^2 - \operatorname{div}^M \theta_2 \\ +(p-2)\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle^2 f|d\varphi|^{p-4}. \end{aligned}$$

Since the sectional curvature of (N, h) is non-positive, we conclude that

$$-\operatorname{div}^M \theta_1 + f|d\varphi|^{p-2}|\nabla^\varphi \tau_{p,f}(\varphi)|^2 - \operatorname{div}^M \theta_2 \leq 0. \tag{2.20}$$

By using the Green Theorem, and equation (2.20), we have

$$\int_M f|d\varphi|^{p-2}|\nabla^\varphi \tau_{p,f}(\varphi)|^2 v_g = 0. \tag{2.21}$$

Consequently, $\nabla^\varphi_X \tau_{p,f}(\varphi) = 0$ for all $X \in \Gamma(TM)$. Thus, $\tau_{p,f}(\varphi) = 0$. □

We also get the following result.

Theorem 2.8. (Liouville type theorem for (p, f)-biharmonic maps: non-compact case) *Let (M, g) be a complete non-compact Riemannian manifold, (N, h) a Riemannian manifold with non-positive sectional curvature and $p \geq 2$. Then, every (p, f)-biharmonic map φ from (M, g) to (N, h) satisfying*

$$\int_M f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2 v_g < \infty, \quad \int_M f|d\varphi|^{p-2} v_g = \infty, \tag{2.22}$$

is (p, f)-harmonic.

Proof. Let $\varphi : (M, g) \to (N, h)$ be an (p, f)-biharmonic map. We have at x

$$\begin{aligned} 0 \;=\; & -f|d\varphi|^{p-2}R^N(\tau_{p,f}(\varphi), d\varphi(e_i))d\varphi(e_i) - \nabla^\varphi_{e_i} f|d\varphi|^{p-2}\nabla^\varphi_{e_i}\tau_{p,f}(\varphi) \\ & -(p-2)\nabla^\varphi_{e_i}\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}d\varphi(e_i), \end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on (M, g) such that $\nabla^M_{e_j} e_i = 0$ at x for all $i, j = 1, ..., m$. Let ρ be a smooth function with compact support on M. Since the sectional curvature R^N is non-positive, we obtain

$$\begin{aligned} 0 \;\geq\; & -h(\nabla^\varphi_{e_i} f|d\varphi|^{p-2}\nabla^\varphi_{e_i}\tau_{p,f}(\varphi), \rho^2\tau_{p,f}(\varphi)) \\ & -(p-2)h(\nabla^\varphi_{e_i}\langle \nabla^\varphi \tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}d\varphi(e_i), \rho^2\tau_{p,f}(\varphi)). \end{aligned}$$

Let $\beta_1, \beta_2 \in \Gamma(T^*M)$ defined by

$$\begin{aligned}\beta_1(X) &= h(f|d\varphi|^{p-2}\nabla^{\varphi}_X\tau_{p,f}(\varphi), \rho^2\tau_{p,f}(\varphi)),\\ \beta_2(X) &= h(\langle\nabla^{\varphi}\tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}d\varphi(X), \rho^2\tau_{p,f}(\varphi)),\end{aligned}$$

for all $X \in \Gamma(TM)$. So, the last inequality is equivalent to the following

$$\begin{aligned}0 \geq\; & -\operatorname{div}^M \beta_1 + \rho^2 f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi), \nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi))\\ & +2\rho e_i(\rho)f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi), \tau_{p,f}(\varphi)) - (p-2)\operatorname{div}^M \beta_2\\ & +(p-2)\rho^2\langle\nabla^{\varphi}\tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}h(d\varphi(e_i), \nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi))\\ & +2(p-2)\rho e_i(\rho)\langle\nabla^{\varphi}\tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}h(d\varphi(e_i), \tau_{p,f}(\varphi)).\end{aligned} \tag{2.23}$$

By Young's inequality, we have

$$\begin{aligned}&-2\rho e_i(\rho)f|d\varphi|^{p-2}h(\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi), \tau_{p,f}(\varphi))\\ &\qquad\leq \frac{1}{2}\rho^2 f|d\varphi|^{p-2}|\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi)|^2 + 2e_i(\rho)^2 f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2,\\ &-2\rho e_i(\rho)\langle\nabla^{\varphi}\tau_{p,f}(\varphi), d\varphi\rangle f|d\varphi|^{p-4}h(d\varphi(e_i), \tau_{p,f}(\varphi))\\ &\qquad\leq \rho^2\langle\nabla^{\varphi}\tau_{p,f}(\varphi), d\varphi\rangle^2 f|d\varphi|^{p-4} + e_i(\rho)^2 f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2.\end{aligned}$$

According to the last two inequalities, (2.23) becomes

$$\begin{aligned}pe_i(\rho)^2 f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2 \geq\; & -\operatorname{div}^M \beta_1 + \frac{1}{2}\rho^2 f|d\varphi|^{p-2}|\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi)|^2\\ & -(p-2)\operatorname{div}^M \beta_2.\end{aligned} \tag{2.24}$$

Let $\rho = \rho_R : M \to [0,1]$ be a smooth cut-off function with $\rho = 1$ on $B_R(x)$, $\rho = 0$ off $B_{2R}(x)$ and $|\operatorname{grad}\rho| \leq \frac{2}{R}$. From the inequality (2.24), and the divergence Theorem, we find that

$$\frac{4p}{R^2}\int_{B_{2R}(x)} f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2 v_g \geq \frac{1}{2}\int_{B_R(x)} f|d\varphi|^{p-2}|\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi)|^2 v_g. \tag{2.25}$$

Since $\int_M f|d\varphi|^{p-2}|\tau_{p,f}(\varphi)|^2 v_g < \infty$, when $R \to \infty$, we get

$$\int_M f|d\varphi|^{p-2}|\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi)|^2 v_g = 0. \tag{2.26}$$

Therefore, $|\nabla^{\varphi}_{e_i}\tau_{p,f}(\varphi)|^2 = 0$ for all $i = 1, ..., m$, that is the function $|\tau_{p,f}(\varphi)|^2$ is constant on M. By the assumption (2.22), we conclude that $\tau_{p,f}(\varphi) = 0$, that is the map φ is (p,f)-harmonic. □

References

[1] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannain manifolds*, Clarendon Press Oxford(2003).

[2] P. Baird and S. Gudmundsson, *p-Harmonic maps and minimal submanifolds*, Math. Ann., **294**(1992), 611–624.

[3] B. Bojarski and T. Iwaniec, *p-Harmonic equation and quasiregular mappings*, Banach Center Publ., **19(1)**(1987), 25–38.

[4] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math., **13**(2007), 423–435.

[5] M. Djaa, A. Mohammed Cherif, K. Zagga and S. Ouakkas, *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom., **5(1)**(2012), 90–100.

[6] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86**(1964), 109–160.

[7] A. Fardoun, *On equivariant p-harmonic maps*, Ann. Inst. Henri. Poincaré, **15**(1998), 25–72.

[8] G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A., **7(4)**(1986), 389–402.

[9] J. Liu, *Liouville-type Theorems of p-harmonic Maps with free Boundary Values,* Hiroshima Math., **40**(2010), 333–342.

[10] A. Mohammed Cherif, *On the p-harmonic and p-biharmonic maps*, J. Geom., **109**(2018), 11p.

[11] D. J. Moon, H. Liu and S. D. Jung, *Liouville type theorems for p-harmonic maps*, J. Math. Anal. Appl., **342**(2008), 354–360.

[12] N. Nakauchi, *A Liouville type theorem for p-harmonic maps* , Osaka J. Math., **35**(1998), 303–312.

[13] S. Ouakkas, R. Nasri and M. Djaa, *On the f-harmonic and f-biharmonic maps*, J. P. Journal of Geom. and Top., **10(1)**(2010), 11–27.

[14] E. Remli and A. Mohammed Cherif, *On the Generalized of p-harmonic and f-harmonic Maps*, Kyungpook Math. J., **61**(2021), 169–179.