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Subordination Properties for Classes of Analytic Univalent Involving Linear Operator

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Abstract. In this paper, we use the use the linear operator $\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)$ and the concept of the subordination to analyse the general class of all analytic univalent functions. Our main results are implication properties between the classes of such functions and the application of these properties to special cases.

1. Introduction

Let ¥ be the class of all analytic univalent functions having power series extensions of the form

$$\mathfrak{f}(z) = z + \sum_{m=2}^{\infty} e_m z^m \tag{1.1}$$

in the open unit disk $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$ when normalized with the condition $\mathfrak{f}(0) = \mathfrak{f}'(z) - 1 = 0$.

The Hadamard product (or convolution) $(\mathfrak{f}_1 * \mathfrak{f}_2)(z)$ of the functions

$$\mathfrak{f}_\ell(z) = z + \sum_{m=2}^{\infty} e_{m,\ell} z^m \in ¥, \qquad (\ell = 1, 2) \tag{1.2}$$

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is given by

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(z) = z + \sum_{m=2}^{\infty} e_{m,1} e_{m,2}\, z^m = (\mathfrak{f}_2 * \mathfrak{f}_1)(z) \quad (z \in \mathfrak{U}). \tag{1.3}$$

For two functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ which are analytic in $\mathfrak{U}$, we say that the function $\mathfrak{f}_1$ is subordinate to $\mathfrak{f}_2$, and denote this by

$$\mathfrak{f}_1(z) \prec \mathfrak{f}_2(z), \quad (z \in \mathfrak{U}),$$

if there exists a Schwarz function $\zeta(z)$ analytic in $\mathfrak{U}$ with $\zeta(0) = 0$ and $|\zeta(z)| < 1\,(z \in \mathfrak{U})$ such that $\mathfrak{f}_1(z) = \mathfrak{f}_2(\zeta(z))$, for $z \in \mathfrak{U}$. See [3]. In particular, if $\mathfrak{f}_2(z)$ is univalent in $\mathfrak{U}$, we get the following equivalence

$$\begin{array}{c} \mathfrak{f}_1(z) \prec \mathfrak{f}_2(z) \Longleftrightarrow \mathfrak{f}_1(0) = \mathfrak{f}_2(0) \\ \text{and} \\ \mathfrak{f}_1(\mathfrak{U}) \subset \mathfrak{f}_2(\mathfrak{U}). \end{array} \tag{1.4}$$

Now, for function $\mathfrak{f}(z)$ of the form (1.1) we define the series expansion of the linear operator $\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)$ of the function as

$$\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(v + y)\Gamma(u + my)}{\Gamma(u + y)\Gamma(v + my)} \left(1 + \frac{\tau(m - 1)}{\sigma + 1}\right)^x e_m\, z^m, \tag{1.5}$$

for $x \in \mathbb{Z}$, $\sigma > -1$, $\tau > 0$, $y > 0$ and $\mathrm{Re}(u) > \mathrm{Re}(v) > -y$.

The operator $\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)$ was introduced in [9] generalizes some known operators as follows

(i) $\mathfrak{Z}^x_{1,0}(u, u, y) = \mathfrak{D}^x\, (x \in \mathbb{N}_0 = \{0, 1, ...\})$(Salagean [10]),
(ii) $\mathfrak{Z}^x_{\tau,0}(u, u, y) = \mathfrak{D}^x_\tau\, (x \in \mathbb{N}_0)$(Al-Oboudi [1]),
(iii) $\mathfrak{Z}^0_{\tau,\sigma}(u, v + \mathfrak{t}, 1) = \mathfrak{G}^{\mathfrak{t}}_u\, (\mathfrak{t} > 0,\ u > -1)$(Gao et al. [4],Jung et al.[5]),
(iv) $\mathfrak{Z}^x_{\tau,\sigma}(u, 0, 1) = \mathfrak{J}^x\, (\tau, u, \sigma)(x \in \mathbb{N}_0)$(Catas [2]),
(v) $\mathfrak{Z}^{-x}_{1,u}(u, u, y) = \mathfrak{L}^x_{u+1}\, (x \in \mathbb{N}_0,\ u \geq 0)$(Komatu [6]),
(vi) $\mathfrak{Z}^x_{\tau,0}(u, v, 1) = \mathfrak{D}^x_\tau\, (u + 1, v + 1)(x \in \mathbb{N}_0)$(Selvaraj and Karthikeyan [11]).

From (1.5), it is easy to show that

$$z\left(\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)\right)' = \left(\tfrac{u}{y} + 1\right) \mathfrak{Z}^x_{\tau,\sigma}(u + 1, v, y)\mathfrak{f}(z) - \tfrac{u}{y}\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z), \tag{1.6}$$

$$z\left(\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)\right)' = \tfrac{\sigma+1}{\tau}\mathfrak{Z}^{x+1}_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) - \left(\tfrac{\sigma+1}{\tau} - 1\right) \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z), \tag{1.7}$$

$$z\left(\mathfrak{Z}^x_{\tau,\sigma}(u, v + 1, y)\mathfrak{f}(z)\right)' = \left(\tfrac{v}{y} + 1\right) \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z) - \tfrac{v}{y}\mathfrak{Z}^x_{\tau,\sigma}(u, v + 1, y)\mathfrak{f}(z). \tag{1.8}$$

Using the concept of subordination in (1.4) and the operator $\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)$, we investigate the subclass of ¥ defined as follows.

Definition 1.1. A function $\mathfrak{f} \in ¥$ is in the class $\mathfrak{T}\mathfrak{R}^{x}_{\tau,\sigma}(u,v,y;\gamma;\mathcal{V},\mathcal{W})$ if it satisfies

$$\left(\frac{z\left(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)\right)'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}\right) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},\ (z \in \mathfrak{U}), \tag{1.9}$$

for $\mathcal{V},\mathcal{W} \in \mathbb{R}$ with $-1 \le \mathcal{V} < \mathcal{W} \le 1$ and $0 \le \gamma < 1$.

The following lemmas will be useful in deriving our results.

Lemma 1.2. ([8]) *If* $-1 \le \mathcal{V} < \mathcal{W} \le 1,\ \lambda > 0$ *and* $\eta \in \mathbb{C}$ *is restricted by*

$$\operatorname{Re}\eta \ge -\frac{1-\mathcal{V}}{1-\mathcal{W}}\lambda,$$

then the differential equation

$$\mathfrak{g}(z) + \frac{z\mathfrak{g}'(z)}{\lambda\mathfrak{g}(z)+\eta} = \frac{1+\mathcal{V}z}{1+\mathcal{W}z},\ (z \in \mathfrak{U})$$

has a univalent solution given by

$$\mathfrak{g}(z) = \begin{cases} \dfrac{z^{\lambda+\eta}(1+\mathcal{W}z)^{\frac{\lambda(\mathcal{V}-\mathcal{W})}{\mathcal{W}}}}{\lambda\int\limits_0^z t^{\lambda+\eta-1}(1+\mathcal{W}t)^{\frac{(\mathcal{V}-\mathcal{W})}{\mathcal{W}}}dt} - \dfrac{\eta}{\lambda}, & \mathcal{W} \ne 0 \\[2ex] \dfrac{z^{\lambda+\eta}e^{\lambda\mathcal{V}z}}{\lambda\int\limits_0^z t^{\lambda+\eta-1}e^{\lambda\mathcal{V}z}dt} - \dfrac{\eta}{\lambda}, & \mathcal{W} = 0. \end{cases}$$

If ψ *is regular in* $\mathfrak{U}$ *and satisfies the differential subordination*

$$\psi(z) + \frac{z\psi'(z)}{\lambda\psi(z)+\eta} \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},$$

then $\psi(z) \prec \mathfrak{g}(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}$ *and* $\mathfrak{g}$ *is the best dominant of the above subordination.*

Lemma 1.3.([7]) *Let* $\mathfrak{g}$ *be univalent in* $\mathfrak{U}$ *and* Ψ, ϑ *be analytic functions in a domain* D *containing* $\mathfrak{g}(D)$ *with* $\Psi(\rho) \ne 0$ *for* $\rho \in \mathfrak{g}(D)$. *Set* $\mathcal{T}(z) = z\mathfrak{g}'(z)\Psi(\mathfrak{g}(z))$ *and* $g(z) = \vartheta(\mathfrak{g}(z)) + \mathcal{T}(z)$. *Suppose that*

(i) $\mathcal{T}$ *is starlike univalent in* $\mathfrak{U}$,

(ii) $\operatorname{Re}\frac{zg'(z)}{\mathcal{T}(z)} > 0,\ z \in \mathfrak{U}$.

If h *is analytic in* $\mathfrak{U}$ *for* $h(0) = \mathfrak{g}(0),\ h(\mathfrak{U}) \subseteq D$ *and*

$$\vartheta(h(z)) + zh'(z)\Psi(h(z)) \prec \vartheta(\mathfrak{g}(z)) + z\mathfrak{g}'(z)\Psi(\mathfrak{g}(z)),$$

then

$$h(z) \prec \mathfrak{g}(z),$$

and $\mathfrak{g}$ *is the best dominant.*

2. Main Results

For $x \in \mathbb{Z}$, $\sigma > -1$, $\tau > 0$, $y > 0$, $-1 \le \mathcal{V} < \mathcal{W} \le 1$ and $0 \le \gamma < 1$, the following theorems are obtained.

Theorem 2.1. *Let* $\mathfrak{f} \in \mathfrak{TR}^{x}_{\tau,\sigma}(u+1, v, y;\gamma;\mathcal{V},\mathcal{W})$ *such that* $\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z) \neq 0$, $\forall z \in \mathfrak{U}^{*} = \mathfrak{U}\backslash\{0\}$ *and* $(1-\gamma)(\mathcal{V}-1) \le (1-\mathcal{W})\left(\gamma + \frac{u}{y}\right)$, *then*

$$\left(\frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}\right) \prec \mathfrak{g}_1(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}, \ (z \in \mathfrak{U}) \tag{2.1}$$

where

$$\mathfrak{g}_1(z) = \frac{1}{1-\gamma}\left[\frac{1}{\hbar(z)} - \gamma - \frac{u}{y}\right], \tag{2.2}$$

and

$$\hbar(z) = \begin{cases} \int\limits_0^1 t^{\frac{u}{y}} \left(\frac{1+\mathcal{W}zt}{1+\mathcal{W}z}\right)^{(1-\gamma)\left(\frac{\mathcal{V}}{\mathcal{W}}-1\right)} dt, & \mathcal{W} \neq 0 \\ \int\limits_0^1 t^{\frac{u}{y}} e^{(1-\gamma)(t-1)\mathcal{V}z} dt, & \mathcal{W} = 0. \end{cases} \tag{2.3}$$

Further, $\mathfrak{g}_1(z)$ *is the best dominant of* (2.1) *.*

Proof. Since $\mathfrak{f} \in \mathfrak{TR}^{x}_{\tau,\sigma}(u+1, v, y;\gamma;\mathcal{V},\mathcal{W})$, if we consider the function

$$\psi(z) = \frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}, \ (z \in \mathfrak{U}) \tag{2.4}$$

then $\psi(z)$ is analytic in $\mathfrak{U}$ and $\psi(0) = 1$. Using identity (1.6),yields

$$(1-\gamma)\psi(z) + \gamma + \frac{u}{y} = \left(\frac{u}{y} + 1\right)\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z)}{\mathfrak{Z}^{x}_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)} \tag{2.5}$$

then differentiating (2.5) with respect to z and multiplying by z, we have

$$\psi(z) + \frac{z\psi'(z)}{(1-\gamma)\psi(z) + \gamma + \frac{u}{y}}$$

$$= \frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma} \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}.$$

Applying Lemma 1.2 for $\lambda = 1-\gamma$ and $\eta = \gamma + \frac{u}{y}$, we get

$$\psi(z) \prec \mathfrak{g}_1(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},$$

where $\mathfrak{g}_1(z)$ defined in (2.2) and it is the best dominant. Hence, the proof is completed. $\square$

Theorem 2.2. *If* $\mathfrak{f} \in \mathfrak{TR}^{x+1}_{\tau,\sigma}(u,v,y;\gamma;\mathcal{V},\mathcal{W})$ *is such that* $\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z) \neq 0$, $\forall z \in \mathfrak{U}^* = \mathfrak{U}\backslash\{0\}$ *and* $(1-\gamma)(\mathcal{V}-1) \leq (1-\mathcal{W})\left(\frac{\sigma+1}{\tau} - 1 + \gamma\right)$, *then*

$$\left(\frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}\right) \prec \mathfrak{g}_2(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}, \ (z \in \mathfrak{U}) \tag{2.6}$$

such that

$$\mathfrak{g}_2(z) = \frac{1}{1-\gamma}\left[\frac{1}{\chi(z)} - \frac{\sigma+1}{\tau} + 1 - \gamma\right], \tag{2.7}$$

and

$$\chi(z) = \begin{cases} \int\limits_0^1 t^{\frac{\sigma+1}{\tau}-1}\left(\frac{1+\mathcal{W}zt}{1+\mathcal{W}z}\right)^{(1-\gamma)\left(\frac{\mathcal{V}}{\mathcal{W}}-1\right)} dt, & \mathcal{W} \neq 0 \\ \int\limits_0^1 t^{\frac{\sigma+1}{\tau}-1} e^{(1-\gamma)(t-1)\mathcal{V}z} dt, & \mathcal{W} = 0. \end{cases} \tag{2.8}$$

Further, $\mathfrak{g}_2(z)$ *is the best dominant of* (2.6).

Proof. Suppose that $\mathfrak{f} \in \mathfrak{TR}^{x+1}_{\tau,\sigma}(u,v,y;\gamma;\mathcal{V},\mathcal{W})$, define

$$\psi(z) = \frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}, \ (z \in \mathfrak{U}). \tag{2.9}$$

Thus, the function $\psi(z)$ is analytic in $\mathfrak{U}$ and $\psi(0) = 1$. From (1.7), the equation (2.9) becomes

$$(1-\gamma)\psi(z) + \frac{\sigma+1}{\tau} - 1 + \gamma = \frac{\sigma+1}{\tau}\frac{\mathfrak{Z}^{x+1}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}, \tag{2.10}$$

then differentiating (2.10) with respect to z and multiplying by z, we obtain

$$\psi(z) + \frac{z\psi'(z)}{(1-\gamma)\psi(z) + \frac{\sigma+1}{\tau} - 1 + \gamma}$$

$$= \frac{z(\mathfrak{Z}^{x+1}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x+1}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma} \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}.$$

By make use of Lemma 1.2 for $\lambda = 1-\gamma$ and $\eta = \frac{\sigma+1}{\tau} - 1 + \gamma$, we get

$$\psi(z) \prec \mathfrak{g}_2(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},$$

where $\mathfrak{g}_2(z)$ defined in (2.7) and it is the best dominant. Thus, the proof is completed. □

Theorem 2.3. *If* $\mathfrak{f} \in \mathfrak{T}\mathfrak{R}^{x}_{\tau,\sigma}(u,v,y;\gamma;\mathcal{V},\mathcal{W})$ *such that* $\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z) \neq 0$, $\forall z \in \mathfrak{U}^* = \mathfrak{U}\backslash\{0\}$ *and* $(1-\gamma)(\mathcal{V}-1) \leq (1-\mathcal{W})\left(\frac{v}{y}+\gamma\right)$, *then*

$$\left(\frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma}\right) \prec \mathfrak{g}_3(z) \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},\ (z \in \mathfrak{U}) \tag{2.11}$$

where

$$\mathfrak{g}_3(z) = \frac{1}{1-\gamma}\left[\frac{1}{\phi(z)} - \frac{v}{y} - \gamma\right], \tag{2.12}$$

and

$$\phi(z) = \begin{cases} \int_0^1 t^{\frac{v}{y}} \left(\frac{1+\mathcal{W}zt}{1+\mathcal{W}z}\right)^{(1-\gamma)\left(\frac{\mathcal{V}}{\mathcal{W}}-1\right)} dt, & \mathcal{W} \neq 0 \\ \int_0^1 t^{\frac{v}{y}} e^{(1-\gamma)(t-1)\mathcal{V}z} dt, & \mathcal{W} = 0. \end{cases} \tag{2.13}$$

Further, $\mathfrak{g}_3(z)$ *is the best dominant of* (2.11).

Proof. Suppose that $\mathfrak{f} \in \mathfrak{T}\mathfrak{R}^{x}_{\tau,\sigma}(u,v,y;\gamma;\mathcal{V},\mathcal{W})$ and consider that the function

$$\psi(z) = \frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma},\ (z \in \mathfrak{U}). \tag{2.14}$$

Subsequently, $\psi(z)$ is analytic in $\mathfrak{U}$ together $\psi(0) = 1$. Applying (1.8), yields

$$(1-\gamma)\psi(z) + \frac{v}{y} + \gamma = \left(\frac{v}{y}+1\right)\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{\mathfrak{Z}^{x}_{\tau,\sigma}(u,v+1,y)\mathfrak{f}(z)}. \tag{2.15}$$

By differentiation (2.15) with respect to z and multiplying by z, we have

$$\psi(z) + \frac{z\psi'(z)}{(1-\gamma)\psi(z) + \frac{v}{y} + \gamma}$$

$$= \frac{z(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z))'}{(1-\gamma)\mathfrak{Z}^{x+1}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - \frac{\gamma}{1-\gamma} \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z}.$$

Thus, from Lemma 1.2 for $\lambda = 1 - \gamma$ and $\eta = \frac{v}{y} + \gamma$, we obtain

$$\psi(z) \prec \mathfrak{g}_3(z) \prec \frac{1 + \mathcal{V}z}{1 + \mathcal{W}z},$$

where $\mathfrak{g}_3(z)$ defined in (2.12) and it is the best dominant. □

Theorem 2.4. *Suppose that* $\mathfrak{g}(z)$ *is a univalent function in* $\mathfrak{U}$ *with* $\mathfrak{g}(0) = 1$ *and*

$$\text{Re}\left(1 + \frac{z\mathfrak{g}''(z)}{\mathfrak{g}'(z)} - \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)}\right) > 0, \ (z \in \mathfrak{U}). \tag{2.16}$$

Let $\mathfrak{f} \in ¥$ *satisfy the condition*

$$\frac{\theta \mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) + \kappa \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)}{z(\theta + \kappa)} \neq 0,$$

where $\theta, \ \kappa \in \mathbb{C}$ *and* $\theta + \kappa \neq 0$. *If*

$$\delta\left(\frac{\theta z\left(\mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z)\right)' + \kappa z\left(\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)\right)'}{\theta \mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) + \kappa \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)} - 1\right) \prec \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)}, \tag{2.17}$$

then

$$\left(\frac{\theta \mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) + \kappa \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)}{z(\theta + \kappa)}\right)^{\delta} \prec \mathfrak{g}(z), \tag{2.18}$$

for $\delta \in \mathbb{C}\backslash\{0\}$, *and* $\mathfrak{g}(z)$ *is the best dominant of* (2.17).

Proof. From Lemma 1.3, we can prove the results above for

$$\Psi(\rho) = \frac{1}{\rho}, \ \vartheta(\rho) = 0, \ \mathcal{T}(z) = z\mathfrak{g}'(z)\Psi(\mathfrak{g}(z)) = \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)}, \ g(z) = \mathcal{T}(z), \ \rho \in \mathbb{C}.$$

Since $\mathcal{T}'(0) = \mathfrak{g}'(0) \neq 0$, from (2.16) the function $\mathcal{T}$ is a starlike univalent in $\mathfrak{U}$ and

$$\text{Re}\,\frac{zg'(z)}{\mathcal{T}(z)} = \text{Re}\left(1 + \frac{z\mathfrak{g}''(z)}{\mathfrak{g}'(z)} - \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)}\right) > 0, \ (z \in \mathfrak{U}).$$

Hence, both (i) and (ii) of Lemma 1.3 are satisfied. Consider the function h be defined by

$$h(z) = \left(\frac{\theta \mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) + \kappa \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)}{z(\theta + \kappa)}\right)^{\delta}, \ (z \in \mathfrak{U}) \tag{2.19}$$

subsequently, the function h is analytic in $\mathfrak{U}$, $h(0) = \mathfrak{g}(0) = 1$, and

$$\frac{zh'(z)}{h(z)} = \delta\left(\frac{\theta z\left(\mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z)\right)' + \kappa z\left(\mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)\right)'}{\theta \mathfrak{Z}^x_{\tau,\sigma}(u+1, v, y)\mathfrak{f}(z) + \kappa \mathfrak{Z}^x_{\tau,\sigma}(u, v, y)\mathfrak{f}(z)} - 1\right), \tag{2.20}$$

from (2.20), (2.17) becomes

$$\frac{zh'(z)}{h(z)} \prec \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)},$$

equivalent to

$$\vartheta(h(z)) + zh'(z)\Psi(h(z)) \prec \vartheta(\mathfrak{g}(z)) + z\mathfrak{g}'(z)\Psi(\mathfrak{g}(z)).$$

Thus, applying Lemma 1.3 it follows that $h(z) \prec \mathfrak{g}(z)$ and that $\mathfrak{g}$ is the best dominant of (2.17). Taking $\theta = 0$, $\kappa = 1$ and $\mathfrak{g}(z) = \dfrac{1+\mathcal{V}z}{1+\mathcal{W}z}$ $(-1 \leq \mathcal{V} < \mathcal{W} \leq 1)$, it is easy to show

$$\mathrm{Re}\left(1 + \frac{z\mathfrak{g}''(z)}{\mathfrak{g}'(z)} - \frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)}\right) = 1 - \mathrm{Re}\left(\frac{\mathcal{V}z}{1+\mathcal{V}z} + \frac{\mathcal{W}z}{1+\mathcal{W}z}\right)$$

$$> 1 - \left(\frac{|\mathcal{V}|}{1+|\mathcal{V}|} + \frac{|\mathcal{W}|}{1+|\mathcal{W}|}\right) = \frac{1-|\mathcal{V}|\,|\mathcal{W}|}{(1+|\mathcal{V}|)\,(1+|\mathcal{W}|)} > 0, \ (z \in \mathfrak{U}).$$

□

From Theorem 2.4, we obtain the following result.

Corollary 2.5. *Let* $\mathfrak{f} \in ¥$ *satisfy the condition*

$$\frac{\mathfrak{Z}^x_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{z} \neq 0, \ (z \in \mathfrak{U}).$$

If

$$\delta\left(\frac{z\left(\mathfrak{Z}^x_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)\right)'}{\mathfrak{Z}^x_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)} - 1\right) \prec \frac{(\mathcal{V}-\mathcal{W})z}{(1+\mathcal{V}z)(1+\mathcal{W}z)}, \tag{2.21}$$

then

$$\left(\frac{\mathfrak{Z}^x_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{z}\right)^{\delta} \prec \frac{1+\mathcal{V}z}{1+\mathcal{W}z},$$

with $\delta \in \mathbb{C}\backslash\{0\}$ *and* $\dfrac{1+\mathcal{V}z}{1+\mathcal{W}z}$ *is the best dominant of* (2.21).

We took $\mathcal{V} = 1$ and $\mathcal{W} = -1$ in Corollary 2.5 as a special case and we got:

Corollary 2.6. *Let* $\mathfrak{f} \in ¥$ *satisfies the condition*

$$\frac{\mathfrak{Z}^x_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{z} \neq 0, \ (z \in \mathfrak{U}).$$

If

$$\delta\left(\frac{z\left(\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)\right)'}{\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}-1\right)\prec\frac{2z}{1-z^{2}}, \tag{2.22}$$

then

$$\left(\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u,v,y)\mathfrak{f}(z)}{z}\right)^{\delta}\prec\frac{1+z}{1-z}.$$

For $\delta\in\mathbb{C}\backslash\{0\}$ *and* $\frac{1+z}{1-z}$ *is the best dominant of* (2.22).

Taking $\theta=1$ and $\kappa=0$ and $\mathfrak{g}(z)=\frac{1+\mathcal{V}z}{1+\mathcal{W}z}$ $(-1\le\mathcal{V}<\mathcal{W}\le1)$ in Theorem 2.4, it follows that.

Corollary 2.7. *Suppose that* $\mathfrak{f}\in$¥ *satisfies the condition*

$$\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}{z}\neq0,\ \ (z\in\mathfrak{U}).$$

If

$$\delta\left(\frac{z\left(\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)\right)'}{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}-1\right)\prec\frac{(\mathcal{V}-\mathcal{W})z}{(1+\mathcal{V}z)(1+\mathcal{W}z)}, \tag{2.23}$$

then

$$\left(\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}{z}\right)^{\delta}\prec\frac{1+\mathcal{V}z}{1+\mathcal{W}z}.$$

For $\delta\in\mathbb{C}\backslash\{0\}$ *and* $\frac{1+\mathcal{V}z}{1+\mathcal{W}z}$ *is the best dominant of* (2.23).

We took $\mathcal{V}=1$ and $\mathcal{W}=-1$ in Corollary 2.7 as a special case and we got:

Corollary 2.8. *Let* $\mathfrak{f}\in$¥ *satisfies the condition*

$$\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}{z}\neq0,\ \ (z\in\mathfrak{U}).$$

If

$$\delta\left(\frac{z\left(\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)\right)'}{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}-1\right)\prec\frac{2z}{1-z^{2}}, \tag{2.24}$$

then

$$\left(\frac{\mathfrak{Z}^{x}_{\tau,\sigma}(u+1,v,y)\mathfrak{f}(z)}{z}\right)^{\delta}\prec\frac{1+z}{1-z}$$

For $\delta\in\mathbb{C}\backslash\{0\}$ *and* $\frac{1+z}{1-z}$ *is the best dominant of* (2.24).

3. Conclusion

In the present work, we were able to obtain the best results, or best dominants of the subordination. Our main results give an interesting process for the study of many analytic univalent classes earlier defined by several authors. These classes expand and generalize many of those defined by many specialists in this field. Furthermore, the general subordination theorems lead us to some special cases that were used to determine new results connected with the classes we investigated.

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