KYUNGPOOK Math. J. 63(2023), 187-198
https://doi.org/10.5666/KMJ.2023.63.2.187
pISSN 1225-6951 eISSN 0454-8124

Quasinormal Subgroups in Division Rings Radical over Proper Division Subrings

Le Qui Danh*
Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
Vietnam National University, Ho Chi Minh City, Vietnam
University of Architecture Ho Chi Minh City, 196 Pasteur Str., Dist. 3, Ho Chi Minh City, Vietnam
e-mail : danh.lequi@uah.edu.vn

Trinh Thanh Deo
Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
Vietnam National University, Ho Chi Minh City, Vietnam
e-mail : ttdeo@hcmus.edu.vn

Abstract. The motivation for this study comes from a question posed by I.N. Herstein in the Israel Journal of Mathematics in 1978. Specifically, let D be a division ring with center F. The aim of this paper is to demonstrate that every quasinormal subgroup of the multiplicative group of D, which is radical over some proper division subring, is central if one of the following conditions holds: (i) D is weakly locally finite; (ii) F is uncountable; or (iii) D is the Mal'cev-Neumann division ring.

1. Introduction

The motivation of this study comes from a question posed by Herstein [13] in 1978: Is it true that every subnormal subgroup of the multiplicative group of a division ring D which is radical over the center F of D is central? Herstein himself showed that the conditions "subnormal" and "normal" are equivalent [14]. At the present, the question is affirmatively answered for the following particular cases:

* Corresponding Author.
Received August 5, 2022; revised February 12, 2023; accepted May 4, 2023.
2020 Mathematics Subject Classification: 16K20.
Key words and phrases: division ring, normal subgroup, quasinormal subgroup, Mal'cev-Neumann, radical.
This research is funded by University of Architecture Ho Chi Minh City under grant number KHCB 03 - NCKH 22.

- for periodic subgroups [13, Theorem 8];
- for a division ring with uncountable center [14, Theorem 2];
- for a division ring finite dimensional over the center [9, Theorem 1];
- for normal subgroups in a division ring of type 2, where the center F is replaced by an arbitrary proper division subring K of D [8, Theorem 3.2].

Motivated by the results obtained in [8, Theorem 3.2], the authors posed a more general question: For a division ring D, given a normal subgroup N of D^*, is N contained in the center F of D, provided it is radical over some proper division subring K of D? In this paper, we replace the assumption "normal" with "quasinormal". In fact, for a division ring D with center F, we show that every quasinormal subgroup of the multiplicative group of D which is radical over some proper division subring is central in case of one of the following conditions:

(1) D is weakly locally finite;

(2) F is uncountable;

(3) D is the Mal'cev-Neumann division ring.

Recall that, in the theory of division rings, there are several classical constructions of new division rings from given ones. One of such structures is the class of Mal'cev-Neumann division rings, were completely presented in [18] by Neumann who used Mal'cev's ideas in [17]. Mal'cev-Neumann division rings have a vast number of applications. For example, they were used to construct examples of noncrossed product division algebras [2, 10, 11], to describe the multiplicative group of group rings of ordered groups, [15, Corollary 14.24], etc. The problems describe the properties of Mal'cev Neumann division rings and their special cases have been studied in several papers. For instance, it was proved that there are free group algebras in the Mal'cev-Neumann division rings [19]; there are free symmetric group algebras in division rings generated by poly-orderable groups [6]. Also, Amitsur and Tignol determined abelian Galois subfields of the Mal'cev-Neumann division rings with G finite and D a field in [22]. In [4], the authors introduced weakly locally finite division rings. Recall that a division ring D is called *weakly locally finite* if for every finite subset S of D, the division subring of D generated by S is finite dimensional over its center. Also in [4], by using the general structure of the Mal'cev-Neumann division rings, they gave an example to prove that the class of weakly locally finite division rings is strictly contained in the class of division rings that is finite dimensional over its center.

The paper is organized as follows: In Section 2, we prove that N is central in the case where D has the uncountable center and that N is a normal subgroup of D^* which is radical over a proper division subring of D. This is an important result for replacing the "normal subgroup" by the "quasinormal subgroup" in the next sections. In Section 3, by considering the similar result of the Herstein conjecture and by considering D to be either a weakly locally finite division ring or a division ring with uncountable center, we prove that every quasinormal subgroup of D^* which radical over a proper division subring of D is central. In Section 4, we also give some results regarding quasinormal subgroups of D^* which is radical over a proper division subring of D, as an interesting example for the class of these subgroups. In Section 5, we investigate quasinormal subgroups of the multiplicative group $\mathcal{D}^*$ of the Mal'cev-Neumann division ring $\mathcal{D}$, we also show that every quasinormal subgroup of $\mathcal{D}^*$ which is radical over a proper division subring of $\mathcal{D}$ is central.

The symbols and notations we use in this paper are standard that can be found in the literature on division rings and in the cited items of the presented paper.

2. Normal Subgroups in Division Rings that are Radical over Proper Division Subrings

Recall that a division ring D with center F is said to be *a division ring of type* 2 if for every two elements $x, y \in D$, the division subring $F(x, y)$ generated by x, y over F is a finite dimensional vector space over F. An element $x \in D$ is *radical* over F if there exists some positive integer $n(x)$ depending on x such that $x^{n(x)} \in F$. A subset S of D is *radical* over F if every element of S is radical over F. If we replace F by some division subring L of D, then we have the notion of radicality over L.

In [8, Theorem 3.2], it was shown that every normal subgroup of the multiplicative group of a division ring of type 2 which is radical over its proper division subring is central. The proof of [8, Theorem 3.2] is based on the properties of a division subring generated by two elements over F. We use this idea to prove the following theorem.

Theorem 2.1. *Let D be a division ring with center F and N a normal subgroup of D^* which is radical over a proper division subring of D. Assume that S is a finite subset of D and $K = F(S)$ is the division subring of D generated by S over F. Then, $N \cap K$ is radical over the center $Z(K)$ of K.*

Proof. Let L be a proper division subring of D such that N is radical over L. If $|S| = 1$, then K is a field. Thus, let $|S| \geq 2$. We can assume that $S = \{x_1, \dots, x_k\}$, where $k \geq 2$. Suppose $a \in N \cap K$. We claim that, for any x in S, there exists a positive integer n so that $a^n x = x a^n$.

Case 1: $x \notin L$

Clearly, we can assume $a + x \neq 0$ and $x \neq \pm 1$. Consider the elements $\alpha = (a + x)a(a + x)^{-1}$ and $\beta = (x + 1)a(x + 1)^{-1}$. Since N is normal in D^* and radical over L, there exist some positive integers m_1 and m_2 such that

$$\alpha^{m_1} = (a + x)a^{m_1}(a + x)^{-1} \in L \text{ and } \beta^{m_2} = (x + 1)a^{m_2}(x + 1)^{-1} \in L.$$

With $n_1 = m_1 m_2$, we have

$$\alpha^{n_1} = (a + x)a^{n_1}(a + x)^{-1} \in L \text{ and } \beta^{n_1} = (x + 1)a^{n_1}(x + 1)^{-1} \in L.$$

There are two possible subcases for a given element $a \in N$.

Subcase 1.1: $a \in L$

Then,

$$\alpha^{n_1}(a+x) - \beta^{n_1}(x+1) = (a+x)a^{n_1} - (x+1)a^{n_1}. \tag{1}$$

This implies

$$\alpha^{n_1}a + \alpha^{n_1}x - \beta^{n_1}x - \beta^{n_1} = aa^{n_1} - a^{n_1}. \tag{2}$$

Hence,

$$(\alpha^{n_1} - \beta^{n_1})x = a^{n_1}(a-1) + \beta^{n_1} - \alpha^{n_1}a. \tag{3}$$

Since $x \notin L$ and $\alpha^{n_1} - \beta^{n_1}, a^{n_1}(a-1) + \beta^{n_1} - \alpha^{n_1}a \in L$, from the last equality we must have $\alpha^{n_1} - \beta^{n_1} = 0$. Now, from (3), we get $a^{n_1} = \alpha^{n_1} = \beta^{n_1}$. Recall that $\beta^{n_1} = (x+1)a^{n_1}(x+1)^{-1}$. This implies that $a^{n_1}x = xa^{n_1}$.

Subcase 1.2: $a \notin L$

Since a is radical over L, there exists a positive integer m such that $b := a^m \in L$. According to Subcase 1.1, there exists a positive integer n_1 such that $b^{n_1}x = xb^{n_1}$. Hence, $a^{mn_1}x = xa^{mn_1}$.

Case 2: $x \in L$

Take $c \in D \backslash L$. According to Case 1, there exist positive integers m_1 and m_2 such that $a^{m_1}c = ca^{m_1}$ and $a^{m_2}(x+c) = (x+c)a^{m_2}$. Hence, $a^{m_1m_2}x = xa^{m_1m_2}$.

Thus, the claim is proved.

Therefore, for each $i = 1, 2, ..., k$, there exists a positive integer n_i such that $a^{n_i}x_i = x_ia^{n_i}$. Now, if $n = n_1n_2 \dots n_k$, then $a^nx_i = x_ia^n$ for all $i = 1, 2, ..., n$. This implies that $a^n \in Z(K)$, that is, a is radical over $Z(K)$. □

Using Theorem 2.1, we can now extend some earlier results. In [14], Herstein proved that in a division ring D with uncountable center F, if a normal subgroup of D^* is radical over F then it is central. In the following theorem, we will show that F can be replaced by an arbitrary proper division subring of D.

Theorem 2.2. *Let D be a division ring with uncountable center and N a normal subgroup of D^*. If N is radical over a proper division subring of D, then it is central.*

Proof. Let F be the center of D. Assume that N is non-central. Take $x \in N$ and $y \in D$ such that $xy \neq yx$ and put $K = F(x, y)$. By Theorem 2.1, $N \cap K$ is radical over the center $Z(K)$ of K. Since $Z(K)$ contains F, $Z(K)$ is uncountable. By [14, Theorem 2], $N \cap K \subseteq Z(K)$. In particular, $xy = yx$, a contradiction. Hence, N is central. □

3. Herstein Conjecture for Quasinormal Subgroups

Based on the definition of the weakly locally finite division rings in [4], in this section, we will show a similar result to a conjecture of Herstein for quasinormal subgroups. First, we use the following result on the relationship between quasinormal subgroups, radical subgroups, and subnormal subgroups.

Recall that a subgroup H of a group G is called *subnormal* in G if there exists a series of $r+1$ subgroups

$$H = N_r \trianglelefteq N_{r-1} \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0 = G.$$

A *quasinormal subgroup* (or *permutable subgroup*) is a subgroup H of a group G that commutes (permutes) with every other subgroup K of G with respect to the product of subgroups, i.e. $HK = KH$.

Lemma 3.1. *Let G be a group and N a quasinormal subgroup of G. Then, either N is subnormal in G or G is radical over N.*

Proof. This lemma is [3, Lemma 6]. □

Next, combining [4, Theorem 11] and Theorem 2.2, we have the following result for the normal subgroup of the multiplicative group of a division ring.

Theorem 3.2. *Let D be a division ring with center F, L a proper division subring of D, and N a normal subgroup of D^* which is radical over L. If either F is uncountable or D is weakly locally finite, then N is central.*

Proof. The result is obtained from [4, Theorem 11] and Theorem 2.2. □

For a group G, the normal core $\mathrm{Core}_G(H)$ of a subgroup H in G is the largest normal subgroup of G that is contained in H (or equivalently, the intersection of all conjugates of H), i.e.,

$$\mathrm{Core}_G(H) := \bigcap_{x \in G} x^{-1}Hx.$$

To prove the main result of this section, we need the following results from [7, Theorem 1], [21, Theorem 4], and [5, Theorem B].

Lemma 3.3. *Let G be a group and H a subnormal subgroup of G. If H is a quasinormal subgroup of G, then $H/\mathrm{Core}_G(H)$ is a solvable group.*

Proof. This lemma is a part of [7, Theorem 1]. □

Lemma 3.4.([21, Theorem 4]) *Let D be a division ring. Then, all solvable subnormal subgroups of D^* are central.*

The following is an interesting result of Faith in [5] regarding a division ring radical over its proper division subring.

Lemma 3.5.([5, Theorem B]) *Let D be a division ring and L be a proper division subring of D. If D is radical over L, then D is commutative.*

Now we show the main result of this section.

Theorem 3.6. *Let D be either a weakly locally finite division ring or a division ring with uncountable center, and N a quasinormal subgroup of D^* which radical over a proper division subring of D. Then, N is central.*

Proof. Let F be the center of D and assume that L is a proper division subring of D such that N is radical over L. Since N is quasinormal in D^*, by Lemma 3.1, either N is a subnormal subgroup in D^* or D^* is radical over N.
Case 1: N is subnormal in D^*. Since $\mathrm{Core}_{D^*}(N) \subseteq N$, $\mathrm{Core}_{D^*}(N)$ is radical over L. Furthermore, $\mathrm{Core}_{D^*}(N)$ is a normal subgroup of D^*, so $\mathrm{Core}_{D^*}(N) \subseteq F$, by Theorem 3.2. In particular, $\mathrm{Core}_{D^*}(N)$ is solvable. On the other hand, due to Lemma 3.3, we get that $N/\mathrm{Core}_{D^*}(N)$ is also solvable, so N is solvable. Then, by Lemma 3.4, we have N is central.
Case 2: D^* is radical over N. Since N is radical over L, we obtain that D is radical over L. Then, according to Lemma 3.5, D is commutative. Consequently, N is central.

In both cases, N is central. The proof is now complete. □

4. Some More Results on Quasinormal Subgroups

In this section, we will give some results regarding quasinormal subgroups of D^* which are radical over a proper division subring of D, as an interesting example for the class of these subgroups. First, we have the following lemma.

Lemma 4.1. *Let D be a division ring with center F, and assume that N is a normal subgroup of D^*. If N is radical over F, then for any $a \in N$ with $a^2 \in F$, we have $a \in F$.*

Proof. Since $a \in N$, for any $x \in D^*$, there exists a positive integer n such that $(axa^{-1}x^{-1})^n \in F$. If $a^2 \in F$, by [13, Theorem 6], $a \in F$. □

Let D be a division ring with center F. An element a in D is called algebraic over F if a is a root of a nonzero polynomial over F. Additionally, the minimal polynomial of a, denoted $m_a(t)$, is the monic polynomial of of lowest degree such that $m_a(a) = 0$. For convenience of use, we give the following two lemmas which are based on similar ideas from the work [16] of M. Mahdavi-Hezavehi and S. Akbari-Feyzaabaadi with some slight modifications.

Lemma 4.2. *Let D be a division ring with center F, and assume that N is a normal subgroup of D^* which is radical over F. If the minimal polynomial of an element $a \in N$ has degree n, then $a^n \in F$.*

Proof. By replacing N by the subgroup generated by $N \cup F^*$, so without loss of generality, we assume that N contains F^*. Let $a \in N$. We assume that $a^m \in F$ for some positive m and n is the degree of $m_a(t)$. We must show that $a^n \in F$. Indeed, observe that $n = [F(a) : F]$ so that by [16, Lemma 1], there exists $c \in F(a)^* \cap D'$ such that $N_{F(a)/F}(a) = ca^n$. Then, $c = N_{F(a)/F}(a)a^{-n}$ which implies that $c^m = N_{F(a)/F}(a)^m a^{-mn} = \lambda \in F$. Since $N_{F(a)/F}(c) = 1$, we obtain that $1 = N_{F(a)/F}(c)^{mn} = \lambda^n = c^{mn}$. On the other hand, since N contains F, we have $c \in N$. By [13, Theorem 9], $c \in F$, it follows that $a^n = c^{-1}N_{F(a)/F}(a) \in F$. □

Lemma 4.3. *Let D be a division ring with center F, and assume that N is a normal subgroup of D^*. If N is radical over F, then for any $a \in N$ with $a^3 \in F$, we have $a \in F$.*

Proof. We can suppose that D is noncommutative. Assume that $a \in N$ such that $a^3 \in F$. Let $f(t)$ be the minimal polynomial of a over F. Then, the degree of $f(t)$ is less than or equal to 3. If the degree of $f(t)$ is 2, then by Lemma 4.2, $a^2 \in F$. Due to Lemma 4.1, $a \in F$. Now we assume that the degree of $f(t)$ is 3. By Lemma 4.2 again, the minimal polynomial of a is $f(t) = t^3 - \lambda$, with $\lambda \in F$. By Wedderburn's Theorem,

$$f(t) = (t-a)(t-bab^{-1})(t-dad^{-1}) \text{ for some } b, d \in D^*.$$

Thus, $a + bab^{-1} + dad^{-1} = 0$. Let $\alpha = bab^{-1}a^{-1}$ and $\beta = dad^{-1}a^{-1}$, leads to $1 + \alpha + \beta = 0$. By the fact that α and β are in N, it follows that α and β are radical over F. Let m be the degree of the minimal polynomial of α. Then, $m = [F(\alpha) : F]$. Because $1 + \alpha + \beta = 0$, $m = [F(\beta) : F]$. By Lemma 4.2, $\alpha^m \in F$ and $\beta^m \in F$, so we have $\alpha^m \in F$ and $(1+\alpha)^m \in F$. Moreover, since m is the degree of the minimal polynomial of α on F, we get $m = 1$ or $\mathrm{char}(D) = p > 0$ and $p \mid m$. If the latter happens, then by putting $m = pq$ we get $\alpha^m = (\alpha^q)^p \in F$, so by [14, Lemma 2] we obtain $\alpha^q \in F$, which contradicts the minimality of m. So $m = 1$, that is, $\alpha \in F$. We claim that $\alpha = 1$. Assume that $\alpha \neq 1$. By the fact that $bab^{-1} = \alpha a$, one has $a^3 = (bab^{-1})^3 = \alpha^3 a^3$, so $\alpha^3 = 1$. Therefore, $ab^{-3}a^{-1} = (ab^{-1}a^{-1})^3 = \alpha^3 b^{-3} = b^{-3}$, so we have $ab^3 = b^3 a$. Put $D_1 = F(a, b)$ the division subring of D generated by a, b over F. Observe that $\alpha \neq 1$, so that $ab \neq ba$. It implies that D_1 is noncommutative. Let F_1 be the center of D_1. Observe that $b^3 a = ab^3$, so $b^3 \in F_1$. Because $bab^{-1} = \alpha a$, and $\alpha, b^3 \in F_1$, every element in D_1 may be written as

$$\sum_{\text{finite}} \alpha_i a^{n_i} b^{m_i}, \text{ where } \alpha_i \in F_1, n_i \in \mathbb{N};\ 0 \leq m_i \leq 2.$$

Hence, D_1 is finite dimensional over the subfield $F_1(a)$ of D_1 generated by F_1. It is well known that D_1 satifies a polynomial identity and, as a corollary, D is centrally finite. The claim is shown. Since N is radical over F, it follows that $N \cap D_1$ is radical over F_1. Thus, according to [4, Theorem 10], $N \cap D_1 \subseteq F_1$. In particular, $a \in F_1$, i.e. $ab = ba$, a contradiction. The claim is shown, that is, $\alpha = 1$. Similarly, $\beta = 1$. Hence, $\mathrm{char}(D) = 3$. Since $a^3 \in F$, so by applying [13, Lemma 1] we obtain $a \in F$. □

Lemma 4.4. *Let D be a division ring with center F, and assume that N is a normal subgroup of D^* which is radical over a proper division subring of D. Then $C_D(a) = C_D(a^2) = C_D(a^3)$ for any $a \in N$.*

Proof. Clearly, $C_D(a) \subseteq C_D(a^2)$. Now let $b \in C_D(a^2)$, and put $K = F(a, b)$. By Theorem 2.1, $N \cap K$ is radical over the center $Z(K)$ of K. Observe that $a^2 \in Z(K)$, so $a \in Z(K)$ by Lemma 4.1. In particular, $ab = ba$. Hence, $C_D(a) = C_D(a^2)$.

Using Lemma 4.3, by the similar way, we can get $C_D(a) = C_D(a^3)$. □

Now we can prove an analogue of [16, Corollary 3] for a quasinormal subgroup which is radical over a proper division subring.

Theorem 4.5. *Let D be a division ring with center F and N a quasinormal subgroup of D^*. Assume that L is a proper division subring of D and for every $x \in N$, there exist positive integers m, n such that $x^{2^n 3^m} \in L$. Then, N is central.*

Proof. With similar reasons to those in the proof of Theorem 3.6, we only need to consider the case of N is a normal subgroup of D^*. It suffices to prove that $C_D(a) = D$ for any $a \in N$. For any $a \in N, b \in D$, by repeating the argument in the proof of Thereom 2.1, there exist positive integers m and n such that $a^{2^m 3^n} b = ba^{2^m 3^n}$. It implies that $b \in C_D(a^{2^n 3^m})$. By Lemma 4.4, $C_D(a) = C_D(a^{2^n 3^m})$, which implies that $b \in C_D(a)$. Thus, $C_D(a) = D$ for any $a \in N$. □

Theorem 4.6. *Let D be a division ring, L a proper division subring of D, and N a quasinormal subgroup of D^*. If there exists a positive integer d such that for every $x \in N$, $x^n \in L$ for some $n \leq d$, then N is central.*

Proof. With similar reasons to those in the proof of Theorem 3.6, we only need to consider the case of N is a normal subgroup of D^*. Let F be the center of D. For any $x, y \in N$, put $K = F(x, y)$ and $M = N \cap K$. Observe that for any $a \in M$, by the same argument as in the proof of Theorem 2.1, we can choose some $n \leq (d!)^3$ such that $a^n \in Z(K)$. By [1, Theorem 1], M is abelian, so $xy = yx$. Hence, N is abelian, and by [12], N is central. □

5. Quasinormal Subgroups of the Mal'cev-Neumann Division Rings Which are Radical over a Proper Division Ring

Throughout this section, we assume that D is a division ring with center F, G is a nontrivial ordered group, and $\Phi\colon G \to \mathrm{Aut}(D), x \mapsto \Phi_x$ is a homomorphism from G to the automorphism group $\mathrm{Aut}(D)$. Recall that the Mal'cev-Neumann division ring of Laurent series $\mathcal{D} = D((G, \Phi))$ is defined as a ring consisting of all Laurent series

$$\alpha = \sum_{g \in G} a_g g, \text{ where } a_g \in D \text{ and } \mathrm{supp}(\alpha) = \{g \in G \mid a_g \neq 0\} \text{ is a well-ordered set,}$$

with "addition" and "multiplication" defined respectively by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{u \in G} \sum_{gh = u} a_g \Phi_g(b_h) u$$

(see [15, page 229–235] or [18]). The aim of this section is to show that every quasinormal subgroup of $\mathcal{D}^*$ which is radical over a proper division subring of $\mathcal{D}$ is central. Note that in general, we do not know the dimension of $\mathcal{D}$ over its center $Z(\mathcal{D})$. There is a few information about this dimension in particular cases. For instance, if D is a field, G is nontrivial and Φ is injective then $\dim_{Z(\mathcal{D})} \mathcal{D}$ is infinite [15, Corollary 14.26]. We can also prove that if G is nontrivial with $Z(G) = 1$, then $\dim_{Z(\mathcal{D})} \mathcal{D} \geq \dim_F D$. In fact, for any $\alpha = \sum\limits_{g\in G} a_g g \in Z(\mathcal{D})$, one has $\alpha h = h\alpha$ for any $h \in G$, so $\alpha = a \in D$ since $Z(G) = 1$. Moreover, $ab = ba$ for any $b \in D$. Hence, $\alpha = a \in F$. This means $Z(\mathcal{D}) \subseteq F$, and it follows that $\dim_{Z(\mathcal{D})} \mathcal{D} \geq \dim_F \mathcal{D} = \dim_F D((G,\Phi)) \geq \dim_F D$.

For any element $\alpha = \sum\limits_{g\in G} a_g g \in G \in \mathcal{D}$, the minimal element of the set $\operatorname{supp}(\alpha)$ will be denoted by $\min(\alpha)$. Observe that $\min(\alpha\beta) = \min(\alpha).\min(\beta)$ for any $\alpha, \beta \in \mathcal{D}$. The following lemma is [15, Corollary 14.23].

Lemma 5.1.([15, Corollary 14.23]) *Let $\mathcal{D} = D((G,\Phi))$ be as above. For any $\alpha \in \mathcal{D}$, if $\min(\alpha) > 1$, then $\sum\limits_{i=0}^{\infty} a_i\alpha^i$ is an element of $\mathcal{D}$ for any $a_i \in D$.*

Lemma 5.2. *Let $\mathcal{D} = D((G,\Phi))$ be as above and α be an element of $\mathcal{D}$ such that $\min(\alpha) > 1$. If a subset $R \subseteq D$ has at least two elements, then the subset $S = \left\{ \sum\limits_{i=0}^{\infty} a_i\alpha^i \mid a_i \in R \right\}$ is uncountable.*

Proof. By Lemma 5.1, S is a subset of $\mathcal{D}$. Clearly, S is infinite. Assume that S is countable, that is, there exists a bijection $f\colon \mathbb{N} \to S$. For any $n \in \mathbb{N}$, $\beta_n = f(n) = \sum\limits_{i=0}^{\infty} a_{ni}\alpha^i$. For $i \geq 0$, choose b_i from $R\backslash\{a_{ii}\}$ and put $\beta = \sum\limits_{i=0}^{\infty} b_i\alpha^i \in S$. One has $\beta \neq \beta_n$ for any $n \in \mathbb{N}$. This implies that f is not a bijection, a contradiction. Hence, S is uncountable. □

As a corollary, $\mathcal{D} = D((G,\Phi))$ is uncountable if the group G is nontrivial. But the center of $\mathcal{D}$ is not necessarily uncountable. For example, if D is a countable field and Φ is injective then the center of $\mathcal{D}$ is a subfield of D, which is countable (see [15, Corollary 14.26]).

Now, we are ready to prove the main theorem of this section.

Theorem 5.3. *Let $\mathcal{D} = D((G,\Phi))$ be the Mal'cev-Neumann division ring of Laurent series as above. Then, every quasinormal subgroup of $\mathcal{D}^*$ which is radical over a proper division subring of $\mathcal{D}$ is central.*

Proof. Let $\mathcal{F}$ be the center of $\mathcal{D}$ and assume that N is a quasinormal subgroup of $\mathcal{D}^*$ which is radical over a proper division subring $\mathcal{H}$ of $\mathcal{D}$.

Case 1: N is a normal subgroup of $\mathcal{D}^*$

Assume that N is not central. Put

$$L = Z(D) \cap \{ a \in D \mid \Phi_g(a) = a \text{ for any } g \in G \}.$$

Now, we have two subcases.

Subcase 1.1: $\min(\alpha) = 1$ *for any* $\alpha \in \mathcal{F}^*$

It is clear that $L \subseteq \mathcal{F}$. Put $\alpha = \sum_{1 \leq g \in G} a_g g \in \mathcal{F}$. Then, for any $b \in D$, one has $\alpha b = b\alpha$. This implies that $a_1 b = b a_1$, so $a_1 \in Z(D)$. Moreover, for each $h \in G$, we have $h\alpha = \alpha h$, which implies that $\Phi_h(a_1) = a_1$. Hence, $a_1 \in L$. It follows that $a_1 \in \mathcal{F}$, and so, $\alpha - a_1 \in \mathcal{F}$. However, since $\min(\alpha - a_1) \neq 1$, we conclude that $\alpha = a_1 \in L$. Thus, $\mathcal{F} = L$.

Now, if $\min(\beta) = 1$ for any $\beta \in N \backslash \mathcal{H}$, then $\min(\delta) = 1$ for any $\delta \in \mathcal{H} \cap N$. In fact, if there exists $\delta \in \mathcal{H} \cap N$ such that $\min(\delta) \neq 1$, then $\delta\beta \in N \backslash \mathcal{H}$ and $\min(\delta\beta) \neq 1$, a contradiction. This implies that $\min(\beta) = 1$ for any $\beta \in N$, so N is a subset of D. Since G is nontrivial, by [20, 14.3.8, Page 439], we have $D \subseteq \mathcal{F}$. In particular, $N \subseteq \mathcal{F}$, a contradiction. Thus, there exists $\beta = \sum_{g \in G} b_g g \in N \backslash \mathcal{H}$ such that $\min(\beta) \neq 1$. Since $\min(\beta\beta^{-1}) = \min(1) = 1$, without loss of generality, we can assume that $\min(\beta) \geq 1$, that is, $g \geq 1$ for any $g \in \operatorname{supp}(\beta)$. The condition $\min(\beta) \neq 1$ implies that $\beta \notin L$, so there exists $a \in D$ and $h \in G$ such that $\beta a h \neq a h \beta$. Let $K = \mathcal{F}(b_1, \beta, ah)$ be the division subring of $\mathcal{D}$ generated by b_1, β, ah over $\mathcal{F}$. Then, by Theorem 2.1, K is noncommutative and $N \cap K$ is radical over the center $Z(K)$ of K. Hence, there exists a positive integer n such that $\beta^n \in Z(K)$. Since β^n commutes with ah, so is b_1^n. Put $\gamma = \beta^n - b_1^n$. Then, $\min(\gamma) > 1$, and by Lemma 5.1, $S = \{ \sum_{i=0}^{\infty} c_i \gamma^i \mid c_i \in L \}$ is a subset of $\mathcal{D}$. Let K_1 be a division subring of $\mathcal{D}$ generated by $\{b_1, ah, \beta\} \cup S$ over L. Put $H_1 = K_1 \cap \mathcal{H}$ and $N_1 = K_1 \cap N$. We have H_1 is a proper subdivision ring of K (because $\beta \notin H_1$) and N_1 is radical over H_1. It is easy to check that $\gamma \in Z(K_1)$, which implies that $S = \{ \sum_{i=0}^{\infty} c_i \gamma^i \mid c_i \in L \}$ is a subset of $Z(K_1)$. By Lemma 5.2, S is uncountable, so is $Z(K_1)$. Due to Theorem 2.2, N_1 is central. In particular, $\beta a h = a h \beta$, a contradiction.

Subcase 1.2: There exists $\alpha \in \mathcal{F}^*$ *such that* $\min(\alpha) \neq 1$

Since $\min(\alpha\alpha^{-1}) = \min(1) = 1$, without loss of generality, we can assume that $\min(\alpha) > 1$. Therefore, by Lemma 5.1, $\sum_{i=0}^{\infty} c_i \alpha^i, c_i \in L$, is an element of $\mathcal{D}$, and it belongs to $\mathcal{F}$. Hence, $\mathcal{F}$ contains the set $T = \{ \sum_{i=0}^{\infty} c_i \alpha^i \mid c_i \in L \}$. By Lemma 5.2, T is uncountable, so is $\mathcal{F}$. Now, by Theorem 2.2, N is central, a contradiction.

Case 2: N is a non-normal subgroup of $\mathcal{D}^*$

By the assumption, we have $\mathrm{Core}_{\mathcal{D}^*}(N)$ is a normal subgroup of $\mathcal{D}^*$. Therefore, according to Case 1 we obtain that $\mathrm{Core}_{\mathcal{D}^*}(N)$ is central. With similar reasons to those in the proof of Theorem 3.6 we get the result. □

Acknowledgments. The authors would like to thank the referee for his/her useful suggestions. This work was written while the second author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). The author would like to thank the institute for providing a fruitful research environment and working condition.

References

[1] M. H. Bien, *On normal subgroups of D^* whose elements are periodic modulo the center of D^* of bounded order*, Int. Electron. J. Algebra, **16**(2014), 66–71.

[2] C. Coyette, *Mal'cev-Neumann rings and noncrossed product division algebras*, J. Algebra Appl., **11(3)**(2012), 12 pp.

[3] L. Q. Danh and H. V. Khanh, *Locally solvable subnormal and quasinormal subgroups in division rings*, Hiroshima Math. J., **51(3)**(2021), 267–274.

[4] T. T. Deo, M. H. Bien and B. X. Hai, *On weakly locally finite division rings*, Acta Math. Vietnam., **44**(2019), 553–569.

[5] C. Faith, *Algebraic division ring extensions*, Proc. Amer. Math. Soc., **11(1)**(1960), 43–53.

[6] V. O. Ferreira, J. Z. Goncalves and J. Sanchez, *Free symmetric group algebras in division rings generated by poly-orderable groups*, J. Algebra, **392**(2013), 69–84.

[7] F. Gross, *Subnormal, core-free, quasinormal subgroups are solvable*, Bull. London Math. Soc., **7**(1975), 93–95.

[8] B. X. Hai, T. T. Deo and M. H. Bien, *On subgroups in division rings of type 2*, Studia Sci. Math. Hungar., **49(4)**(2012), 549–557.

[9] B. X. Hai and L. K. Huynh, *On subnormal subgroups of the multiplicative group of a division ring*, Vietnam J. Math., **32(1)**(2004), 21–24.

[10] T. Hanke, *A twisted Laurent series ring that is a noncrossed product*, Israel J. Math., **150**(2005), 199–203.

[11] T. Hanke and J. Sonn, *The location of noncrossed products in Brauer groups of Laurent series fields over global fields*, Math. Ann., **350(2)**(2011), 313–337.

[12] I. N. Herstein and W. R. Scott, *Subnormal subgroups of division rings*, Canadian J. Math., **15**(1963), 80–83.

[13] I. N. Herstein, *Multiplicative commutators in division rings*, Israel J. Math., **31(2)**(1978), 180–188.

[14] I. N. Herstein, *Multiplicative commutators in division rings II*, Rend. Circ. Mat. Palermo, **29(2)**(1980), 485–489.

[15] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, Berlin(1991).

[16] M. Mahdavi-Hezavehi and S. Akbari-Feyzaabaadi, *On a question of Herstein concerning commutators in division rings*, Rev. Roumaine Math. Pures Appl., **41**(1996), 201–204.

[17] A. I. Mal'cev, *On the embedding of group algebras in division algebras* (in Russian), Doklady Akad. Nauk SSSR (N.S.), **60**(1948), 1499–1501.

[18] B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc., **66**(1949), 202–252.

[19] J. Sánchez, *Free group algebras in Malcev-Neumann skew fields of fractions*, Forum Math., **26(2)**(2014), 443–466.

[20] W. R. Scott, *Group Theory*, 2^{nd} edition, Dover Publications Inc., New York(1987).

[21] C. J. Stuth, *A generalization of the Cartan-Brauer-Hua Theorem*, *Proc. Amer. Math. Soc.* **15**(1964), 211–217.

[22] J. P. Tignol and S. A. Amitsur, *Kummer subfields of Mal'cev-Neumann division algebras*, Israel J. Math., **50**(1985), 114–144.