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An Upper Bound for the Probability of Generating a Finite Nilpotent Group

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Abstract. Let G be a finite group and let $\nu(G)$ be the probability that two randomly selected elements of G produce a nilpotent group. In this article we show that for every positive integer $n > 0$, there is a finite group G such that $\nu(G) = \frac{1}{n}$. We also classify all groups G with $\nu(G) = \frac{1}{2}$. Further, we prove that if G is a solvable nonnilpotent group of even order, then $\nu(G) \leq \frac{p+3}{4p}$, where p is the smallest odd prime divisor of $|G|$, and that equality exists if and only if $\frac{G}{Z_\infty(G)}$ is isomorphic to the dihedral group of order $2p$ where $Z_\infty(G)$ is the hypercenter of G. Finally we find an upper bound for $\nu(G)$ in terms of $|G|$ where G ranges over all groups of odd square-free order.

1. Introduction

In the past 40 years, there has been a growing attention in the application of probability in finite groups (for example see [8, 16]). In this paper, we denote by $\nu(G)$ the probability that two randomly selected elements of G produce a nilpotent

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subgroup. In other words we have

$$\nu(G)=\frac{|\{(x,y)\in G\times G:\langle x,y\rangle\ is\ nilpotent\}|}{|G|^2}.$$

The notion $\nu(G)$ is introduced in [11] on the model of the commutativity degree, via

$$cp(G)=\frac{|\{(x,y)\in G\times G:\langle x,y\rangle\ is\ abelian\}|}{|G|^2}.$$

Note that for $x,y\in G$, we have $xy=yx$ if and only if $\langle x,y\rangle$ is abelian.

It is easy to see that $cp(G)=\frac{\sum_{x\in G}|C_G(x)|}{|G|^2}$ where $C_G(x)$ is the centralizer of x in G as $C_G(x)$ is a subgroup of G for any $x\in G$.

Similarly if

$$Nil_G(x)=\{y\in G|\langle x,y\rangle\ is\ nilpotent\},$$

then

$$\nu(G)=\frac{\sum_{x\in G}|Nil_G(x)|}{|G|^2}.$$

However, $Nil_G(x)$ is not necessarily a subgroup of G, and so it is difficult to glean information about a group G from $\nu(G)$.

A finite group G is nilpotent if and only if $\nu(G)=1$ (see Theorem 1 of [5]). On the other hand, Wilson [16] showed that in finite groups G the probability that two random elements of G produce a nilpotent group goes to 0 as the index of the Fitting subgroup of G goes to infinity.

Gustafson [8] proved that if G is a non-abelian group, then $cp(G)\le\frac{5}{8}$, and that equality holds if and only if $\frac{G}{Z(G)}$ is isomorphic to the Kelian four-group $Z_2\times Z_2$. Several authors determined the structure of a finite group G when $cp(G)$ is sufficiently large, see [2, 9, 12].

In [7] Guralnick and Wilson found that if G is a nonnilpotent group, then $\nu(G)\le\frac{1}{2}$. In this paper we classify groups G with $\nu(G)=\frac{1}{2}$ (see Proposition 2.6).

It is easy to see that $cp(A_5)=\nu(A_5)=\frac{1}{12}$ where A_5 is the alternating group of degree five. Dixon observed that $cp(G)\le\frac{1}{12}$ for any finite nonabelian simple group G. This was submitted by Dixon as a problem in Canadian Math. Bulletin, 13 (1970), with his own solution appearing in 1973. Guralnick and Robinson [6] extended this result to nonsolvable groups and determined precisely for which nonsolvable groups the equality happens. Recently in [10] the authors of the present paper showed that if G is a group such that $Nil_G(x)$ is a subgroup of G for every $x\in G$ and $\nu(G)>\frac{1}{12}$, then G is solvable.

Fulman et al. [5] proved that if G is a solvable nonnilpotent group and p is the smallest prime number that divides $|G|$, then $\nu(G)\le\frac{1}{p}$ and equality holds if and only if $p=2$ and $\frac{G}{Z_\infty(G)}$ is isomorphic to the dihedral group of order 6 (see [5]). Here $Z_\infty(G)$ is the hypercenter of G (i.e. the terminal term of the upper central series of G, see [3, 13]). In this article we improve this upper bound as follows.

Theorem 1.1. *Suppose that G is a solvable nonnilpotent group of even order. Then $\nu(G) \leq \frac{p+3}{4p}$ where p is the smallest odd prime number that divides $|G|$; equality holds if and only if $\frac{G}{Z_\infty(G(G))} \cong D_{2p}$ is the dihedral group of order $2p$.*

For a prime p we denote by Z_p^k the elementary abelian group of order p^k. We propose the following conjecture for every nonnilpotent group of odd order.

Conjecture *Let G be a finite solvable nonnilpotent group such that $|G| = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ where $2 < p_1 < \cdots < p_r$ are primes. Then*

$$\nu(G) \leq \frac{p_k^{t_k} + p_l^2 - 1}{p_k^{t_k} p_l^2} := max\{\frac{p_i^{t_i} + p_j^2 - 1}{p_i^{t_i} p_j^2} : \quad p_j | p_i^{t_i} - 1, 1 \leq j < i \leq r, 1 \leq t_i \leq n_i\}$$

for some $1 \leq l < k \leq r$ and equality holds if and only if $\frac{G}{Z_\infty(G)} \cong Z_{p_k}^{t_k} \rtimes Z_{p_l}$. We think that this conjecture is true for the class of $\mathcal{N}$-groups, introduced by Abdollahi and Zarrin in [1], which are the groups in which $Nil_G(x)$ is a nilpotent group for every $x \in G \setminus Z_\infty(G)$. We feel that the method used in proof of main theorem of [15] may be useful in proving this.

In Section 2 we compute $\nu(G)$ for Frobenius groups and Dihedral groups. We also prove that for any positive integer n, there is a group G such that $\nu(G) = \frac{1}{n}$. Finally we classify all groups G with $\nu(G) = \frac{1}{2}$. In Section 3 we verify Theorem 1.1 and, with Theorem 3.2, confirm the above conjecture for groups of square-free order.

In this article G is a finite group and $Z_\infty(G)$ is its hypercenter. Most notation we use is standard and follows [14].

2. Computing $\nu(G)$ for Certain Groups

The following lemmas are very useful in the sequel.

Lemma 2.1. *Suppose that G is a group. Then $\nu(G) = \nu(\frac{G}{Z_\infty(G)})$.*

Proof. See Corollary 3 of [5]. □

Lemma 2.2. *Suppose that G and H are finite groups. Then $\nu(G \times H) = \nu(G) \times \nu(H)$.*

Proof. The proof is not complicated. □

Proposition 2.3. *If $G = H \ltimes K$ is a Frobenius group where the Frobenius kernel is K and the complement is H, then $\nu(G) = \frac{1}{|H|^2}(1 - \frac{1}{|K|}) + \frac{\nu(H)}{|K|}$.*

Proof. By hypothesis, we have $C_G(h) \subseteq H$ for each $1 \neq h \in H$, $C_G(k) \subseteq K$ for each $1 \neq k \in K$ and $H \cap H^x = 1$ for each $x \in G \setminus H$. Now if $\langle h_1k_1, h_2k_2 \rangle$ is nilpotent such that $h_1, h_2 \in H$ and $k_1, k_2 \in K$, then $h_1 = h_2 = 1$ or $k_1 = k_2 = 1$. On the other hand $\{K, (H^x - 1) | x \in K\}$ is a partition of G and since K is nilpotent, we are done. □

Corollary 2.4. *Suppose that G is the dihedral group of order $2^r n$ where $r > 1$ and n is odd. Then $\nu(G) = \frac{n+3}{4n}$.*

Proof. Since $\frac{D_{2^r n}}{Z(D_{2^r n})} \cong D_{2^{r-1} n}$ for $r > 1$, by Lemma 2.1 we conclude that $\nu(D_{2^r n}) = \nu(D_{2^{r-1} n}) = \cdots = \nu(D_{2n})$. Since n is odd, D_{2n} is a Frobenious group with the cyclic kernel of order n and so we are done by Proposition 2.3. □

Corollary 2.5. *For any integer $n > 0$, there is a group G of even order such that $\nu(G) = \frac{1}{n}$.*

Proof. Our proof is by induction on n. If $n \in \{1, 2, 3\}$, then the result holds since $\nu(D_2) = 1$, $\nu(D_6) = \frac{1}{2}$ and $\nu(D_{18}) = \frac{1}{3}$. So assume that $n \geq 4$ and the that the result holds for all positive integers $m < n$. If n is even, then there is a group H where $\nu(H) = \frac{2}{n}$ by induction hypothesis and so $\nu(H \times D_6) = \frac{1}{n}$ by Lemma 2.2. Suppose that n is odd, then $n = 4m+1$ or $n = 4m+3$ for some positive integer m. It follows from Corollary 2.4 that $\nu(D_{2(4m+1)}) = \frac{m+1}{n}$ and $\nu(D_{2(12m+9)}) = \frac{m+1}{n}$. Since $m + 1 < n$, we are done by induction hypothesis and Lemma 2.2. This completes the proof. □

In the following we classify all groups G with $\nu(G) = \frac{1}{2}$.

Proposition 2.6. *Suppose that G is a finite group (not necessarily solvable). Then $\nu(G) = \frac{1}{2}$ if and only if $\frac{G}{Z_\infty(G)} \cong D_6$, the dihedral group of order 6.*

Proof. We get necessity by By Lemma 2.1. Conversely if $\nu(G) = \frac{1}{2}$, then the probability of solvability of G is equal or greater than $\frac{1}{2}$ and so G is solvable by [7]. By Theorem 5 of [5], we conclude that $\nu(G) \leq \frac{1}{2}$ and equality holds when $\frac{G}{Z_\infty(G)} \cong D_6$, as needed. □

3. Upper Bound for $\nu(G)$

S. Franciosi and F. Giovanni defined and studied a JNN group as a group all of whose proper quotients are nilpotent (see [4] and [5], Definition 1). It should be noted that a finite group G is a JNN group if and only if $G = L \ltimes A$ where A is an elementary abelian p-group and L is a nilpotent group such that p dose not divide the order of G and the action of L on A is faithful and irreducible (See Theorem 4 of [5] and what follows it).

Proof of Theorem 1.1.
If $p = 3$, then $\nu(G) = 1 - \nu_0(G) \leq \frac{1}{2}$ by Theorem 5 of [5] and equality holds if and only if $\frac{G}{Z_\infty(G)} \cong D_6$. So we assume that the smallest odd prime divisor of $|G|$ is greater than 3. It is enough to prove the result for JNN groups. For if G is a counterexample of minimal order, then there is a nontrivial normal subgroup K of G such that $\frac{G}{K}$ is nonnilpotent (since G is solvable). Suppose that r is the

smallest odd prime that divides $|\frac{G}{K}|$. If $\frac{G}{K}$ is of even order, then $\nu(G) \leq \nu(\frac{G}{K}) \leq \frac{r+3}{4r} \leq \frac{p+3}{4p}$ because $r \geq p$. Also if $\frac{G}{K}$ is of odd order, then by Theorem 5 of [5], $\nu(G) \leq \nu(\frac{G}{K}) \leq \frac{1}{r} \leq \frac{1}{4} \leq \frac{p+3}{4p}$ which gives a contradiction. So let us assume that G is a JNN group. Then $G = L \ltimes A$ where $L \cong P_k \times P_{k-1} \times \cdots \times P_1$, P_i's are the unique Sylow p_i-subgroups of L and A is an elementary abelian q-group. By setting $N = P_{k-1} \ltimes \cdots \ltimes (P_1 \ltimes A)$ we have $G = P_k \ltimes N$. We claim that if $q = 2$ and $1 \neq x_p \in P_k$, then $|C_G(x_p) \bigcap N| \leq |\frac{N}{4}|$.

Assume that $|A| = 2^t$ $(t \geq 2)$ and $H = C_G(x_p) \bigcap A$. If $|H| = 2^{t-1}$ and $a \in A \setminus H$, then $a^{x_p} = ah_1$ for some $h_1 \in H$. Hence $a^{x_p^2} = a^{x_p}h_1 = ah_1^2 = a$ and so $b^{x_p^2} = b$ for all $b \in A$. But P_k acts faithfully on A which implies that $x_p^2 = 1$, obviously absurd. Hence $|H| \leq 2^{t-2}$. If $M := P_{k-1} \times P_{k-2} \times \cdots \times P_1$, then $C_G(x_p) \cap N = M(C_G(x_p) \cap A) = MH$ and so $|C_G(x_p) \cap N| = |M||H| = \frac{|N|}{|A|}|H| \leq \frac{|N|}{4}$, as claimed.

Now we want to count the ordered pairs (x, y) in a fixed pair (a_1N, a_2N) for some $a_1, a_2 \in G$ where $\langle x, y\rangle$ is nilpotent. By page 14 of [5], the probability that a selected pair (x, y) from the coset pair (x_pN, y_pN) generates a nilpotent subgroup is not greater than $\frac{|C_G(x_p) \bigcap C_G(y_p) \bigcap N|}{|N|}$ and by our claim this probability is equal or less than $\frac{1}{4}$.

Now we continue by induction on the number k of prime divisors of $|L|$. Here our aim is showing that if the upper bound mentioned in the assertion is correct for N, it is correct for G too. As mentioned above if $q = 2$, then there is nothing to prove. So assume that $q \neq 2$. Since the action of P_k on A is faithful, in a similar way it can be seen that $|C_G(x_p) \bigcap N| \leq \frac{|N|}{q}$. If $q \geq 5$, then this probability is equal or less than $\frac{1}{5} \leq \frac{1}{4} \leq \frac{p+3}{4p}$ and by the assumption on N, we conclude that $\nu(G) \leq \frac{p+3}{4p}$. Also it is not hard to see that if $k \geq 2$, then the equality does not hold, since in this case N is not an elementary group and as mentioned above above in both cases, whether q is equal to 2 or not, the probability is less than $\frac{1}{4} < \frac{p+3}{4p}$. So it is enough to prove it for the base step of the induction. Assume that $G = R \ltimes A$ where $A = (Z_q)^n$, R is a Sylow r-subgroup and $|R| = r^m$. Then we investigate two cases:

Case 1: Assume that $q = 2$. Then $\nu(G) \leq \frac{2^{2n}+(2^{2n}r^{2m}-2^{2n})\times\frac{1}{4}}{2^{2n}r^{2m}}$. So $\nu(G) \leq \frac{r^{2m}+3}{4r^{2m}} < \frac{r+3}{4r}$. As one can see, equality cannot hold in this case.

Case 2: Suppose that $r = 2$. Then $\nu(G) \leq \frac{q^{2n}+(q^{2n}2^{2m}-q^{2n})\times\frac{1}{q}}{q^{2n}2^{2m}} = \frac{2^{2m}+q-1}{q2^{2m}}$ and since $q \neq 3$, we have $\frac{2^{2m}+q-1}{q2^{2m}} \leq \frac{q+3}{4q}$ and equality holds if and only if $m = 1$ and hence $G \cong Z_2 \ltimes (Z_q)^n$. Now we claim that $n = 1$.

Let $1 \neq a \in A$ and $1 \neq x \in R$. If $a^x = a$, then $\langle a^r\rangle = \langle a\rangle$ and since the action of R on A is irreducible, we have $\langle a\rangle = A$. Henceforth $G \cong Z_2 \ltimes Z_q \cong D_{2q}$. Otherwise, it can be assumed that $C_G(R) \bigcap A = 1$, which results that $G \cong Z_2 \ltimes (Z_q)^n$ is a Frobenius group. It follows that $\nu(G) = \frac{q^n+3}{4q^n}$ (see Proposition 2.1). This implies that the equality exists in our assertion if and only if $n = 1$ and $G \cong D_{2q}$, while G

is a JNN group.

Now if G is not a JNN group, so there is a normal subgroup N of G such that $\frac{G}{N}$ is a JNN because G is solvable. Let $\nu(G) = \frac{p+3}{4p}$ where p is the smallest odd prime that divides $|G|$ and $\frac{G}{N}$ is of even order and p_s be the smallest odd prime that divides the order of $\frac{G}{N}$. Then $\frac{p+3}{4p} = \nu(G) \leq \nu(\frac{G}{N}) \leq \frac{p_s+3}{4p_s}$ which implies that $p = p_s$ and $\frac{G}{N} \cong D_{2p}$. Now by an argument similar to that on page 16 of [5] it can be proved that $\frac{G}{Z_\infty(G)} \cong D_{2p}$. Let $\frac{G}{N}$ be of odd order and $p_s > 3$ be its smallest prime divisor. Then $\frac{p+3}{4p} = \nu(G) \leq \nu(\frac{G}{N}) \leq \frac{1}{p_s}$, our final contradiction. □

For an odd prime p, we denote by $\mathfrak{G}_p$ the set of all solvable nonnilpotent groups G of even order such that p is the smallest odd prime that divides the order of G.

Corollary 3.1. *Suppose that $G \in \mathfrak{G}_p$ where p is an odd prime . Then $\nu(G)$ is the largest value of ν on $\mathfrak{G}_p$ if and only if $\frac{G}{Z_\infty(G)} \cong D_{2p}$.*

Theorem 3.2. *Suppose that G is a finite group of odd order and $|G| = p_1p_2\cdots p_r$ where $p_1 < \cdots < p_r$ are primes. Then we have*

$$\nu(G) \leq \frac{p_k + p_l^2 - 1}{p_k p_l^2} := max\{\frac{p_i + p_j^2 - 1}{p_i p_j^2} : \quad p_j | p_i - 1, 1 \leq j < i \leq r\}$$

for some $1 \leq l < k \leq r$ and the equality holds if and only if $\frac{G}{Z_\infty(G)} \cong Z_{p_k} \rtimes Z_{p_l}$.

Proof. Similar to the proof of Theorem 1.1 we prove it for JNN groups. Let $G = L \ltimes A$ be a JNN group and let $p_1 < p_2 < \cdots < p_r$. Then $A \cong Z_{p_r}$ and $p_1p_2\cdots p_{r-1}|p_r - 1$ since L acts faithfully on A. We proceed by induction as it was done in the Theorem 1.1. Thus set $N \leq G$ such that $|N| = p_2p_3\cdots p_r$. It follows that $G = P_1 \ltimes N$. We claim that if the assertion is correct for N it will be correct for G too. It is not hard to see that the probability that a pair selected from the coset pair (x_pN, y_pN) for some $x_p, y_p \in P_1$ generates a nilpotent subgroup of G is bounded by $\frac{|C_G(x_p)\bigcap C_G(y_p)\bigcap N|}{|N|}$. But the action of P_1 on A is faithful and then if both x_p and y_p are not identity, then $\frac{|C_G(x_p)\bigcap C_G(y_p)\bigcap N|}{|N|} \leq \frac{1}{p_r}$. Now since $\frac{1}{p_r} < max\{\frac{p_i+p_j^2-1}{p_ip_j^2}|p_j|p_i - 1, 1 \leq i, j \leq r\}$, one can conclude that the bound is right and the equality does not hold when $r \geq 3$. Coming back to the base of induction, let $|G| = p_1p_2$ with $p_1 < p_2$. Then $G = Z_{p_1} \ltimes Z_{p_2}$ and $\nu(G) = \frac{p_2+p_1^2-1}{p_2p_1^2}$, as wanted. □

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