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PRERESOLVING SUBCATEGORIES IN EXTRIANGULATED CATEGORIES

Songsong Liu and Jiaqun Wei

Abstract. In this paper, we introduce and study preresolving subcategories in an extriangulated category $\mathscr{C}$. Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. We give the characterization of $\mathcal{Z}$-proper $\mathcal{Y}$-resolution dimension of an object in $\mathscr{C}$. Next, for an object A in $\mathscr{C}$, if the $\mathcal{Z}$-proper $\mathcal{Y}$-resolution dimension of A is at most n, then all "n-$\mathcal{X}$-syzygies" of A are objects in $\mathcal{Y}$. Finally, we prove that A has a $\mathcal{Z}$-proper $\mathcal{X}$-resolution if and only if A has a $\mathcal{Z}$-proper $\mathcal{Y}$-resolution. As an application, we introduce $(\mathcal{X},\mathcal{Z})$-Gorenstein subcategory $\mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$ of $\mathscr{C}$ and prove that $\mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$ is both $\mathcal{Z}$-resolving subcategory and $\mathcal{Z}$-coresolving subcategory of $\mathscr{C}$.

1. Introduction

Extriangulated categories were introduced by Nakaoka and Palu [11], which is formulated by extracting those properties of Ext^1 on triangulated categories and exact categories. Extriangulated categories share much in common with exact categories and triangulated categories but also differs considerably. Exact categories and triangulated categories are examples of extriangulated categories, but there are extriangulated categories which are neither exact categories nor triangulated categories, see [11, 14].

Let $\mathscr{A}$ be a triangulated category with a proper class ξ' of triangles. Asadollahi and Salarian [1] introduced and studied ξ'-$\mathcal{G}$projective and ξ'-$\mathcal{G}$injective objects, and developed a relative homological algebra in $\mathscr{A}$. In [12], Ren and Liu further studied Gorenstein homological dimensions for triangulated categories. Hu, Zhang and Zhou [3] introduced the notion of a proper class of $\mathbb{E}$-triangles ξ in extriangulated category $\mathscr{C}$ and demonstrated that $(\mathscr{C}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ is a new extriangulated category. Moreover, they introduced and studied the ξ-Gorenstein

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projective objects in $\mathscr{C}$. In [4], they introduced and studied the Gorenstein category in an extriangulated category $\mathscr{C}$, and proved that the stability of the Gorenstein category $\mathcal{GW}(\xi)$ in $\mathscr{C}$. In [5], they studied Gorenstein homological dimensions and given some characterizations of ξ-$\mathcal{G}$projective dimension by using derived functors in an extriangulated category $\mathscr{C}$. Their series of studies provided a theoretical basis for the subsequent studies of extriangulated categories. In [2], Auslander and Bridger introduced the notion of resolving subcategories $\mathcal{X}$ of abelian category $\mathcal{B}$. In [15], Zhu introduced and studied the $\mathcal{X}$-resolution dimensions and special $\mathcal{X}$-precovers for resolving subcategory $\mathcal{X}$ of $\mathcal{B}$. In [6], Huang introduced relative preresolving subcategories and precoresolving subcategories of an abelian category and defined homological dimensions and codimensions relative to these subcategories. Moreover, Huang studied the properties of these homological dimensions and codimensions and unified some important properties possessed by some known homological dimensions. Recently, Huang [7] further studied the homological dimension of preresolving subcategories of an abelian category. In [10], Ma, Zhao and Huang introduced and studied (pre)resolving subcategories of a triangulated category and the homological dimension relative to these subcategories. Inspired by the above, in this paper, we introduce and study preresolving subcategories of an extriangulated category $\mathscr{C}$.

In this paper, we get the following three main results. Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. First, we give the characterization of $\mathcal{Z}$-proper $\mathcal{Y}$-resolution dimension of an object in $\mathscr{C}$. Based on this, we can calculate the $\mathcal{Z}$-proper $\mathcal{Y}$-resolution dimension of an object in $\mathscr{C}$ (See Theorem 3.9). Second, assume $\mathcal{X} \subseteq \mathcal{Z}$ and $\mathcal{Y}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations. For an object C in $\mathscr{C}$, if $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) \leqslant n$, then all "$n$-$\mathcal{X}$-syzygies" of C are objects in $\mathcal{Y}$ (See Theorem 3.16). Third, assume $\mathcal{Y}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations. For an object A in $\mathscr{C}$, we prove that A has a $\mathcal{Z}$-proper $\mathcal{X}$-resolution if and only if A has a $\mathcal{Z}$-proper $\mathcal{Y}$-resolution (See Theorem 3.18). As an application, we introduce $(\mathcal{X}, \mathcal{Z})$-Gorenstein subcategory $\mathcal{GX}_{\mathcal{Z}}(\xi)$ of $\mathscr{C}$ and prove that $\mathcal{GX}_{\mathcal{Z}}(\xi)$ is a $\mathcal{Z}$-resolving and $\mathcal{Z}$-coresolving subcategory of $\mathscr{C}$ (See Theorem 4.3). As a generalization, we consider the $\mathcal{GP}(\xi)$ and $\mathcal{GI}(\xi)$ of $\mathscr{C}$ and get some good results.

The paper is organized as follows. In Section 2, we recall some definitions of extriangulated categories and outline some properties that will be used. In Section 3, we introduce and study preresolving subcategories in $\mathscr{C}$. Moreover, we prove the above three main results. In Section 4, we prove the above application.

2. Preliminaries

Let us briefly recall some definitions and basic properties of extriangulated categories. We omit some details here, but the reader can find them in [11].

Let $\mathscr{C}$ be an additive category equipped with a biadditive functor

$$\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab,$$

where Ab is the category of abelian groups. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. The zero element $0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$-*extension*.

For any morphism $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \text{ and } \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We simply denote them as $a_*\delta$ and $c^*\delta$. A morphism $(a, c)\colon \delta \to \delta'$ of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ satisfying the equality $a_*\delta = c^*\delta'$.

Definition 2.1 ([11, Definition 2.9]). Let $\mathfrak{s}$ be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$$

to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition $(\star)$. In this case, we say that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ, whenever it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

$(\star)$ Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \text{ and } \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then, for any morphism $(a, c) : \delta \to \delta'$, there exists $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C'. \end{tikzcd}$$

Definition 2.2 ([11, Definition 2.12]). We call the triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ an *extriangulated category* if it satisfies the following conditions:

(ET1) $\mathbb{E}$: $\mathscr{C}^{op} \times \mathscr{C} \to Ab$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realized as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"'] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

there exists a morphism $(a, c)\colon \delta \to \delta'$ which is realized by (a, b, c).

(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be $\mathbb{E}$-extensions realized by $A \xrightarrow{f} B \xrightarrow{f'} D$ and $B \xrightarrow{g} C \xrightarrow{g'} F$, respectively. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"'] & D \arrow[d, "d"] \\ A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"'] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $\mathbb{E}(F, f')(\delta')$,

(ii) $\mathbb{E}(d, A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

In addition, we assume the following condition for the rest of the paper (see [11, Condition 5.8]).

Condition 2.3. (WIC) (1) Let $f : X \to Y$ and $g : Y \to Z$ be any composable pair of morphisms in $\mathscr{C}$. If gf is an inflation, then f is an inflation.

(2) Let $f : X \to Y$ and $g : Y \to Z$ be any composable pair of morphisms in $\mathscr{C}$. If gf is a deflation, then g is a deflation.

Remark 2.4. If $\mathscr{C}$ is an extriangulated category, then by [8, Proposition 2.7], the condition (WIC) is equivalent to $\mathscr{C}$ being weakly idempotent complete.

Definition 2.5 ([11, Definitions 2.15 and 2.19])**.** (1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation.*

(2) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} .$$

We usually do not write this "δ" if it is not used in the argument.

(3) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c) : \ \delta \longrightarrow \delta'$, then we write it as

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"'] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"'] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

and call (a,b,c) a *morphism* of $\mathbb{E}$-triangles. If a,b,c above are isomorphisms, then $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ are said to be isomorphic.

Lemma 2.6 ([11, Proposition 3.15])**.** *Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. Then the following hold*:

(1) *Let C be any object, and let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$ and $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. Then there is a commutative diagram in $\mathscr{C}$*

$$\begin{tikzcd}
 & A_2 \arrow[r, equal] \arrow[d, "m_2"'] & A_2 \arrow[d, "x_2"] & \\
A_1 \arrow[r, "m_1"] \arrow[d, equal] & M \arrow[r, "e_1"] \arrow[d, "e_2"'] & B_2 \arrow[r, dashed] \arrow[d, "y_2"] & {} \\
A_1 \arrow[r, "x_1"] & B_1 \arrow[r, "y_1"] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}$$

which satisfies $\mathfrak{s}(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$ and $\mathfrak{s}(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_1} B_1]$.

(2) *Let A be any object, and let $A \xrightarrow{x_1} B_1 \xrightarrow{y_1} C_1 \overset{\delta_1}{\dashrightarrow}$ and $A \xrightarrow{x_2} B_2 \xrightarrow{y_2} C_2 \overset{\delta_2}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. Then there is a commutative diagram in $\mathscr{C}$*

$$\begin{tikzcd}
A \arrow[r, "x_1"] \arrow[d, "x_2"'] & B_1 \arrow[r, "y_1"] \arrow[d, "m_2"] & C_1 \arrow[r, dashed] \arrow[d, equal] & {} \\
B_2 \arrow[r, "m_1"] \arrow[d, "y_2"'] & M \arrow[r, "e_1"] \arrow[d, "e_2"] & C_1 \arrow[r, dashed] & {} \\
C_2 \arrow[r, equal] \arrow[d, dashed] & C_2 \arrow[d, dashed] & & \\
{} & {} & &
\end{tikzcd}$$

which satisfies $\mathfrak{s}(x_{2*}\delta_1) = [B_2 \xrightarrow{m_1} M \xrightarrow{e_1} C_1]$ and $\mathfrak{s}(x_{1*}\delta_2) = [B_1 \xrightarrow{m_2} M \xrightarrow{e_2} C_2]$.

In [3], Hu, Zhang and Zhou gave the following definitions.

A class of $\mathbb{E}$-triangles ξ is *closed under base change* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \in \xi$$

and any morphism $c : C' \to C$, then any $\mathbb{E}$-triangle $A \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{c^*\delta}{\dashrightarrow}$ belongs to ξ.

Dually, a class of $\mathbb{E}$-triangles ξ is *closed under cobase change* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \in \xi$$

and any morphism $a : A \to A^{'}$, then any $\mathbb{E}$-triangle $A^{'} \xrightarrow{x^{'}} B^{'} \xrightarrow{y^{'}} C \overset{a_*\delta}{\dashrightarrow}$ belongs to ξ.

A class of $\mathbb{E}$-triangles ξ is called *saturated* if in the situation of Lemma 2.6(1), whenever $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow}$ and $A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2 \overset{y_2^*\delta_1}{\dashrightarrow}$ belong to ξ, then the $\mathbb{E}$-triangle $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$ belongs to ξ.

An $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ is called *split* if $\delta = 0$. It is easy to see that it is split if and only if x is section or y is retraction.

The full subcategory consisting of the split $\mathbb{E}$-triangles will be denoted by Δ_0.

Definition 2.7 ([3, Definition 3.1])**.** Let ξ be a class of $\mathbb{E}$-triangles which is closed under isomorphisms. ξ is called a *proper class* of $\mathbb{E}$-triangles if the following conditions hold:

(1) ξ is closed under finite coproducts and $\Delta_0 \subseteq \xi$.
(2) ξ is closed under base change and cobase change.
(3) ξ is saturated.

Definition 2.8 ([3, Definition 4.1])**.** An object $P \in \mathscr{C}$ is called ξ-*projective* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

in ξ, the induced sequence of abelian groups

$$0 \longrightarrow \mathscr{C}(P, A) \longrightarrow \mathscr{C}(P, B) \longrightarrow \mathscr{C}(P, C) \longrightarrow 0$$

is exact. Dually, we have the definition of ξ-*injective object.*

We denote $\mathcal{P}(\xi)$ (resp., $\mathcal{I}(\xi)$) the class of ξ-projective (resp., ξ-injective) objects of $\mathscr{C}$. It follows from the definition that these subcategories $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are full, additive, closed under isomorphisms and direct summands.

An extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is said to have *enough ξ-projectives* (resp., *enough ξ-injectives*) provided that for each object A, there exists an $\mathbb{E}$-triangle

$$K \longrightarrow P \longrightarrow A \dashrightarrow \ \text{(resp., } A \longrightarrow I \longrightarrow K \dashrightarrow)$$

in ξ with $P \in \mathcal{P}(\xi)$ (resp., $I \in \mathcal{I}(\xi)$).

Definition 2.9 ([3, Definition 4.4])**.** A ξ-*exact complex* $\mathbf{X}$ is a diagram

$$\mathbf{X} := \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in $\mathscr{C}$, such that for each integer n, there exists an $\mathbb{E}$-triangle $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$ in ξ and $d_n = g_{n-1}f_n$.

Definition 2.10 ([3, Definition 4.5]). Let $\mathcal{W}$ be a class of objects in $\mathscr{C}$. An $\mathbb{E}$-triangle

$$A \longrightarrow B \longrightarrow C \dashrightarrow$$

in ξ is called to be $\mathscr{C}(-,\mathcal{W})$-*exact* (resp., $\mathscr{C}(\mathcal{W},-)$-*exact*) if for any $W \in \mathcal{W}$, the induced sequence of abelian group $0 \longrightarrow \mathscr{C}(C,W) \longrightarrow \mathscr{C}(B,W) \longrightarrow \mathscr{C}(A,W) \longrightarrow 0$ (resp., $0 \longrightarrow \mathscr{C}(W,A) \longrightarrow \mathscr{C}(W,B) \longrightarrow \mathscr{C}(W,C) \longrightarrow 0$) is exact in *Ab*.

Definition 2.11 ([3, Definition 4.6]). Let $\mathcal{W}$ be a class of objects in $\mathscr{C}$. A complex $\mathbf{X}$ is called $\mathscr{C}(-,\mathcal{W})$-*exact* (resp., $\mathscr{C}(\mathcal{W},-)$-*exact*) if it is a ξ-exact complex

$$\mathbf{X} := \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in $\mathscr{C}$, such that there exists an $\mathscr{C}(-,\mathcal{W})$-exact (resp., $\mathscr{C}(\mathcal{W},-)$-exact) $\mathbb{E}$-triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$$

in ξ and $d_n = g_{n-1}f_n$ for each integer n.

3. Basic results

From now on to the end of this paper, if not otherwise specified, we will always assume that $\mathscr{C} = (\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category and ξ is a fixed proper class of $\mathbb{E}$-triangles in $\mathscr{C}$. We also assume the extriangulated category $\mathscr{C}$ has enough ξ-projectives and ξ-injectives, and it satisfies Condition (WIC).

Definition 3.1. Let $\mathcal{Y}, \mathcal{Z}$ be subcategories of $\mathscr{C}$ and M be an object in $\mathscr{C}$.

(1) A $\mathcal{Y}$-*resolution* of M is a ξ-exact complex

$$\cdots \longrightarrow Y_i \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M$$

in $\mathscr{C}$ with all $Y_i \in \mathcal{Y}$.

(2) A $\mathcal{Y}$-resolution of M is called $\mathcal{Z}$-*proper* $\mathcal{Y}$-*resolution* if it is $\mathscr{C}(\mathcal{Z},-)$-exact.

Dually, one can define the notice of a ($\mathcal{Z}$-*coproper*) $\mathcal{Y}$-*coresolution*. We use *res* $\mathcal{Y}_{\mathcal{Z}}$ (resp., *cores* $\widetilde{\mathcal{Y}_{\mathcal{Z}}}$) to denote the subcategory of objects of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper $\mathcal{Y}$-resolution (resp., $\mathcal{Z}$-coproper $\mathcal{Y}$-coresolution).

Definition 3.2. Let $\mathcal{Y}, \mathcal{Z}$ be subcategories of $\mathscr{C}$ and M be an object in $\mathscr{C}$.

(1) The $\mathcal{Z}$-*proper* $\mathcal{Y}$-*resolution dimension* of M, written $\mathcal{Y}_{\mathcal{Z}}$-*res.*dim$(M)$, is defined by

$$\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(M) := \inf\{n \geqslant 0 \mid \text{there exists a } \mathscr{C}(\mathcal{Z},-)\text{-exact } \xi\text{-exact complex } Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M \text{ in } \mathscr{C} \text{ with all } Y_i \in \mathcal{Y}\}.$$

(2) The $\mathcal{Z}$-*proper* $\mathcal{Y}$-*coresolution dimension* of M, written $\mathcal{Y}_{\mathcal{Z}}$-*cores.*dim$(M)$, is defined by

$$\mathcal{Y}_{\mathcal{Z}}\text{-cores.dim}(M) := \inf\{n \geqslant 0 \mid \text{there exists a } \mathscr{C}(\mathcal{Z},-)\text{-exact } \xi\text{-exact complex } M \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_n \text{ in } \mathscr{C} \text{ with all } Y_i \in \mathcal{Y}\}.$$

Dually, $\widetilde{\mathcal{Y}_{\mathcal{Z}}}$-cores.dim$(M)$ and $\widetilde{\mathcal{Y}_{\mathcal{Z}}}$-res.dim$(M)$ are defined.

We use $(\mathcal{Y}_{\mathcal{Z}})^{\wedge}$ (resp., $(\widetilde{\mathcal{Y}_{\mathcal{Z}}})^{\vee}$) to denote the subcategory of $\mathscr{C}$ consisting of objects having finite $\mathcal{Z}$-proper $\mathcal{Y}$-resolution (resp., $\mathcal{Z}$-coproper $\mathcal{Y}$-coresolution) dimension, and use $(\mathcal{Y}_{\mathcal{Z}})^{\wedge}_n$ (resp., $(\widetilde{\mathcal{Y}_{\mathcal{Z}}})^{\vee}_n$) to denote the subcategory of $\mathscr{C}$ consisting of objects having $\mathcal{Z}$-proper $\mathcal{Y}$-resolution (resp., $\mathcal{Z}$-coproper $\mathcal{Y}$-coresolution) dimension at most n.

For a ξ-exact complex

$$\cdots \xrightarrow{f_{n+1}} Y_n \longrightarrow \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M$$

$$(\text{resp., } M \xrightarrow{f_0} Y_0 \xrightarrow{f_1} Y_1 \xrightarrow{f_2} \cdots \longrightarrow Y_n \xrightarrow{f_{n+1}} \cdots)$$

in $\mathscr{C}$ with all $Y_i \in \mathcal{Y}$, there are $\mathbb{E}$-triangles

$$K_1 \xrightarrow{g_0} Y_0 \xrightarrow{f_0} M \dashrightarrow, \text{ and } K_{i+1} \xrightarrow{g_i} Y_i \xrightarrow{h_i} K_i \dashrightarrow$$

$$(\text{resp., } M \xrightarrow{f_0} Y_0 \xrightarrow{g_0} K_1 \dashrightarrow, \text{ and } K_i \xrightarrow{h_i} Y_i \xrightarrow{g_i} K_{i+1} \dashrightarrow)$$

in ξ with $f_i = g_{i-1}h_i$ (resp., $f_i = h_i g_{i-1}$) for each $i > 0$. The object K_i is called an *i-$\mathcal{Y}$-syzygy* (resp., *i-$\mathcal{Y}$-cosyzygy*) of M, denoted by $\Omega^i_{\mathcal{Y}}(M)$ (resp., $\Sigma^i_{\mathcal{Y}}(M)$).

Definition 3.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{C}$ with $\mathcal{X} \subseteq \mathcal{Y}$.

(1) $\mathcal{X}$ is called a *ξ-generator* (resp., *ξ-cogenerator*) of $\mathcal{Y}$ if for any object Y in $\mathcal{Y}$, there exists an $\mathbb{E}$-triangle $Y' \longrightarrow X \longrightarrow Y \dashrightarrow$ (resp., $Y \longrightarrow X \longrightarrow Y' \dashrightarrow$) in ξ with $X \in \mathcal{X}$ and $Y' \in \mathcal{Y}$ (See [13, Definition 2.24]).

(2) Let $\mathcal{Z}$ be a subcategory of $\mathcal{C}$. $\mathcal{X}$ is called a *$\mathcal{Z}$-proper ξ-generator* (resp., *$\mathcal{Z}$-coproper ξ-cogenerator*) of $\mathcal{Y}$ if for any object Y in $\mathcal{Y}$, there exists an $\mathcal{C}(\mathcal{Z},-)$-exact (resp., $\mathcal{C}(-,\mathcal{Z})$-exact) $\mathbb{E}$-triangle $Y' \longrightarrow X \longrightarrow Y \dashrightarrow$ (resp., $Y \longrightarrow X \longrightarrow Y' \dashrightarrow$) in ξ with $X \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.

Definition 3.4. Let $\mathcal{Y}, \mathcal{Z}$ be subcategories of $\mathscr{C}$. $\mathcal{Y}$ is called *$\mathcal{Z}$-preresolving* in $\mathscr{C}$ if the following conditions are satisfied.

(1) $\mathcal{Y}$ admits a $\mathcal{Z}$-proper ξ-generator.

(2) $\mathcal{Y}$ is closed under $\mathcal{Z}$-proper ξ-extensions, that is, for any $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in ξ, if $A, C \in \mathcal{Y}$, then $B \in \mathcal{Y}$.

A $\mathcal{Z}$-preresolving subcategory $\mathcal{Y}$ of $\mathscr{C}$ is called *$\mathcal{Z}$-resolving* if the following condition is satisfied.

(3) $\mathcal{Y}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations, that is, for any $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in ξ, if $B, C \in \mathcal{Y}$, then $A \in \mathcal{Y}$.

Dually, the *$\mathcal{Z}$-precoresolving* and *$\mathcal{Z}$-coresolving* are defined.

Lemma 3.5. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$, and let*

$$M \longrightarrow Y_1 \xrightarrow{f} Y_0 \longrightarrow A \tag{3.1}$$

be a ξ-exact complex in $\mathscr{C}$ with $Y_0, Y_1 \in \mathcal{Y}$. Then there exists a ξ-exact complex

$$M \longrightarrow Y \longrightarrow X \longrightarrow A \tag{3.2}$$

in $\mathscr{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Moreover, if the ξ-exact complex (3.1) *is $\mathscr{C}(\mathcal{Z},-)$-exact, then so is* (3.2).

Proof. Since $M \longrightarrow Y_1 \xrightarrow{f} Y_0 \longrightarrow A$ is a ξ-exact complex in $\mathscr{C}$ with $Y_0, Y_1 \in \mathcal{Y}$, there are two $\mathbb{E}$-triangles

$$M \longrightarrow Y_1 \xrightarrow{g} K \dashrightarrow \text{ and } K \xrightarrow{h} Y_0 \longrightarrow A \dashrightarrow$$

in ξ such that $f = hg$. Since $\mathcal{X}$ is a $\mathcal{Z}$-proper generator of $\mathcal{Y}$ and $Y_0 \in \mathcal{Y}$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$Y^{'} \longrightarrow X \longrightarrow Y_0 \dashrightarrow$$

in ξ with $X \in \mathcal{X}$ and $Y^{'} \in \mathcal{Y}$. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
Y^{'} \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & K \arrow[r, dashed] \arrow[d] & {} \\
Y^{'} \arrow[r] & X \arrow[r] \arrow[d] & Y_0 \arrow[r, dashed] \arrow[d] & {} \\
& A \arrow[r, equal] \arrow[d, dashed] & A \arrow[d, dashed] & \\
& {} & {} &
\end{tikzcd} \tag{3.3}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Moreover, by Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
& M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
Y^{'} \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Y_1 \arrow[r, dashed] \arrow[d] & {} \\
Y^{'} \arrow[r] & W \arrow[r] \arrow[d, dashed] & K \arrow[r, dashed] \arrow[d, dashed] & {} \\
& {} & {} &
\end{tikzcd} \tag{3.4}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the second row in (3.3) is $\mathscr{C}(\mathcal{Z}, -)$-exact, by [4, Lemma 2], we know that the first row in (3.3) and the second row in (3.4) are $\mathscr{C}(\mathcal{Z}, -)$-exact. Hence, $Y \in \mathcal{Y}$ since $Y^{'}, Y_1 \in \mathcal{Y}$ and $\mathcal{Y}$ is closed under $\mathcal{Z}$-proper ξ-extensions. Connecting the second columns in the above diagrams, we obtain that the ξ-exact complex $M \longrightarrow Y \longrightarrow X \longrightarrow A$ in $\mathscr{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Moreover, if (3.1) is $\mathscr{C}(\mathcal{Z}, -)$-exact, then so are the third columns in the above two diagrams. By [4, Lemma 2] and the snake lemma, we know that the second columns in the above two diagrams are $\mathscr{C}(\mathcal{Z}, -)$-exact. □

We can immediately obtain the following corollary.

Corollary 3.6. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$, and let*

$$Y_1 \longrightarrow Y_0 \longrightarrow A \dashrightarrow \ (\textit{resp.}, \ A \longrightarrow Y_0 \longrightarrow Y_1 \dashrightarrow) \tag{3.5}$$

be an $\mathbb{E}$-triangle in ξ with $Y_0, Y_1 \in \mathcal{Y}$. Then there exists an $\mathbb{E}$-triangle

$$Y \longrightarrow X \longrightarrow A \dashrightarrow \ (\textit{resp.}, \ A \longrightarrow Y \longrightarrow X \dashrightarrow) \tag{3.6}$$

where $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$. Moreover, if the $\mathbb{E}$-triangle (3.5) *is $\mathscr{C}(\mathcal{Z}, -)$-exact, then so is* (3.6)*.*

The following proposition gives the relationship between n-$\mathcal{Y}$-cosyzygy and n-$\mathcal{X}$-cosyzygy.

Proposition 3.7. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$, and let*

$$M \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow A$$

be a ξ-exact complex in $\mathscr{C}$ with all $Y_i \in \mathcal{Y}$ for $n \geqslant 1$. Then there exists a ξ-exact complex

$$N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A$$

in $\mathscr{C}$ and a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$Y \longrightarrow N \longrightarrow M \dashrightarrow$$

in ξ with all $X_i \in \mathcal{X}$ and $Y \in \mathcal{Y}$. In particular, if an object in $\mathscr{C}$ is an n-$\mathcal{Y}$-cosyzygy, then it is an n-$\mathcal{X}$-cosyzygy.

Proof. We proceed by induction on n. If $n = 1$, we have an $\mathbb{E}$-triangle

$$M \longrightarrow Y_0 \longrightarrow A \dashrightarrow$$

in ξ. Since $\mathcal{X}$ is a $\mathcal{Z}$-proper generator of $\mathcal{Y}$ and $Y_0 \in \mathcal{Y}$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$Y \longrightarrow X_0 \longrightarrow Y_0 \dashrightarrow$$

in ξ with $X_0 \in \mathcal{X}$ and $Y \in \mathcal{Y}$. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & M \arrow[r, dashed] \arrow[d] & {} \\
Y \arrow[r] & X_0 \arrow[r] \arrow[d] & Y_0 \arrow[r, dashed] \arrow[d] & {} \\
& A \arrow[r, equal] \arrow[d, dashed] & A \arrow[d, dashed] & \\
& {} & {} &
\end{tikzcd} \tag{3.7}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. By [4, Lemma 2], we know that the first row in (3.7) is $\mathscr{C}(\mathcal{Z},-)$-exact. Hence, the first row and the second column in (3.7) are what we need.

We assume that $n \geqslant 2$. By assumption, there are two ξ-exact complexes

$$M \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow K$$

and

$$K \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow A$$

in $\mathscr{C}$ with all $Y_i \in \mathcal{Y}$. By Lemma 3.5, we have a ξ-exact complex

$$K \longrightarrow Y^{'} \longrightarrow X_0 \longrightarrow A$$

with $X_0 \in \mathcal{X}$ and $Y^{'} \in \mathcal{Y}$. Moreover, we obtain two $\mathbb{E}$-triangles

$$K \longrightarrow Y^{'} \longrightarrow K_1 \dashrightarrow \text{ and } K_1 \longrightarrow X_0 \longrightarrow A \dashrightarrow (\star)$$

in ξ. Hence, we have a ξ-exact complex

$$M \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y \longrightarrow K_1.$$

By the induction hypothesis, there exist a ξ-exact complex

$$N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow K_1 \tag{3.8}$$

and a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$Y \longrightarrow N \longrightarrow M \dashrightarrow$$

in ξ with all $X_i \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Connecting the $\mathbb{E}$-triangle $(\star)$ and the ξ-exact complex (3.8), we obtain that the ξ-exact complex

$$N \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A.$$

□

Corollary 3.8. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$, and let M be an object in $\mathscr{C}$. If $\mathcal{Y}$-cores.dim$(M) = n(\geqslant 1)$, then there exists a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle*

$$Y \longrightarrow N \longrightarrow M \dashrightarrow$$

in ξ with $\mathcal{X}$-cores.dim$(N) \leqslant n$ and $Y \in \mathcal{Y}$.

Proof. If $\mathcal{Y}$-cores.dim$(M) = n$, there exists a ξ-exact complex

$$M \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow 0.$$

Note that the $\mathbb{E}$-triangle $Y_0 \longrightarrow Y_0 \longrightarrow 0 \dashrightarrow \in \xi$ by Definition 2.7. Applying Lemma 3.7 with $A = 0$. Clearly, this corollary is hold. □

The following result gives a criterion to calculate the $\mathcal{Z}$-proper $\mathcal{Y}$-resolution dimension of an object in $\mathscr{C}$.

Theorem 3.9. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. For any object M in $\mathscr{C}$ and any positive integer n, the following are equivalent*:

(1) $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(M) \leqslant n$.

(2) *For any integer k with $1 \leqslant k \leqslant n$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex*

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that $T_i \in \mathcal{X}$ if $0 \leqslant i < k$ and $T_j \in \mathcal{Y}$ if $j \geqslant k$.

(3) *For any integer k with $0 \leqslant k \leqslant n-1$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex*

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that $T_k \in \mathcal{X}$ and other $T_i \in \mathcal{Y}$.

(4) *For any integer k with $0 \leqslant k \leqslant n-1$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex*

$$Y_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

with $Y_n \in \mathcal{Y}$, such that $T_k \in \mathcal{Y}$ and other $T_i \in \mathcal{X}$.

Proof. (2) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (1), (4) $\Longrightarrow$ (1) are obvious.

(1) $\Longrightarrow$ (2): We proceed by induction on n. If $n = 1$, the assertion is true by Corollary 3.6. Now suppose $n \geqslant 2$. Then there exist two $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complexes

$$T_n^{'} \longrightarrow T_{n-1}^{'} \longrightarrow \cdots \longrightarrow T_2^{'} \longrightarrow B \tag{3.9}$$

and

$$B \longrightarrow T_1^{'} \longrightarrow T_0^{'} \longrightarrow M \tag{3.10}$$

with all $T_i^{'} \in \mathcal{Y}$. By Lemma 3.5, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$B \longrightarrow T_1^{''} \longrightarrow T_0 \longrightarrow M$$

with $T_1^{''} \in \mathcal{Y}$ and $T_0 \in \mathcal{X}$. Moreover, we obtain two $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangles

$$B \longrightarrow T_1^{''} \longrightarrow K \dashrightarrow (\clubsuit) \text{ and } K \longrightarrow T_0 \longrightarrow M \dashrightarrow (\spadesuit)$$

in ξ. Hence, connecting the $\mathbb{E}$-triangle ($\clubsuit$) and the ξ-exact complex (3.9), we have a new ξ-exact complex

$$T_n^{'} \longrightarrow T_{n-1}^{'} \longrightarrow \cdots \longrightarrow T_2^{'} \longrightarrow T_1^{''} \longrightarrow K. \tag{3.11}$$

If $k = 1$, the above ξ-exact complex (3.11) is what we need. By the induction hypothesis, for any integer k with $2 \leqslant k \leqslant n$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow K \tag{3.12}$$

such that $T_i \in \mathcal{X}$ if $1 \leqslant i < k$ and $T_j \in \mathcal{Y}$ if $j \geqslant k$. Note that $T_0 \in \mathcal{X} \subseteq \mathcal{Y}$. Connecting the $\mathbb{E}$-triangle (♠) and the ξ-exact complex (3.12), we have a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M.$$

(1) $\Longrightarrow$ (3): We also proceed by induction on n. Let $n = 1$, the assertion is true by Corollary 3.6. Now suppose $n \geqslant 2$. Then there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M$$

such that $T_i \in \mathcal{Y}$. We have a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex $D \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M$, by Lemma 3.5, we get a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex $D \longrightarrow Y_1^{'} \longrightarrow T_0 \longrightarrow M$ with $Y_1^{'} \in \mathcal{Y}$ and $T_0 \in \mathcal{X}$, which yields a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1^{'} \longrightarrow T_0^{'} \longrightarrow M. \tag{3.13}$$

If $k = 1$, the above ξ-exact complex (3.13) is what we need. Consider the $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1^{'} \longrightarrow E.$$

It follows that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(E) \leqslant n - 1$. By the induction hypothesis, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow E \tag{3.14}$$

which one of T_k is in $\mathcal{X}$ for $1 \leqslant k \leqslant n-1$ and other $T_i \in \mathcal{Y}$. Note that $T_0 \in \mathcal{X} \subseteq \mathcal{Y}$. Now one can paste the above $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex (3.14) and the $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle $E \longrightarrow T_0 \longrightarrow M \dashrightarrow$ together to obtain the $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that $T_k \in \mathcal{X}$ and other $T_i \in \mathcal{Y}$.

(1) $\Longrightarrow$ (4): The prove is similar to (1) $\Longrightarrow$ (3). □

Dually, we have the following.

Theorem 3.10. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-preresolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. For any object M in $\mathscr{C}$ and any positive integer n, the following are equivalent*:

(1) $\mathcal{Y}_{\mathcal{Z}}$-cores.dim$(M) \leqslant n$.

(2) *For any integer k with $1 \leqslant k \leqslant n$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex*

$$M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

such that $T_i \in \mathcal{Y}$ if $0 \leqslant i < k$ and $P_j \in \mathcal{X}$ if $j \geqslant k$.

(3) *For any integer k with $1 \leqslant k \leqslant n$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex*

$$M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

such that $T_k \in \mathcal{X}$ *and other* $T_i \in \mathcal{Y}$.

(4) *For any integer* k *with* $1 \leqslant k \leqslant n$, *there exists a* $\mathscr{C}(\mathcal{Z},-)$*-exact* ξ*-exact complex*

$$M \longrightarrow Y_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

with $Y_0 \in \mathcal{Y}$, *such that* $T_k \in \mathcal{Y}$ *and other* $T_i \in \mathcal{X}$.

The following two results generalize [4, Theorems 3 and 4]. The arguments here are similar to that in [4], so we omit them.

Proposition 3.11. *Let* $\mathcal{X} \subseteq \mathcal{Z}$ *be subcategories of* $\mathscr{C}$, *and let*

$$C_1 \longrightarrow C_0 \longrightarrow C \dashrightarrow \tag{3.15}$$

be an $\mathbb{E}$*-triangle in* ξ. *Let*

$$\cdots \longrightarrow X_0^i \longrightarrow \cdots \longrightarrow X_0^1 \longrightarrow X_0^0 \longrightarrow C_0 \tag{3.16}$$

and

$$\cdots \longrightarrow X_1^i \longrightarrow \cdots \longrightarrow X_1^1 \longrightarrow X_1^0 \longrightarrow C_1 \tag{3.17}$$

be $\mathcal{Z}$*-proper* $\mathcal{X}$*-resolutions of* C_0 *and* C_1 *with all* $X_0^i, X_1^i \in \mathcal{X}$. *Then*

(1) *If* (3.15)*is* $\mathscr{C}(\mathcal{Z},-)$*-exact, there exists a* $\mathcal{Z}$*-proper* $\mathcal{X}$*-resolution of* C

$$\cdots \longrightarrow X_0^i \oplus X_1^{i-1} \longrightarrow \cdots \longrightarrow X_0^1 \oplus X_1^0 \longrightarrow X_0^0 \longrightarrow C \tag{3.18}$$

(2) *If* (3.15), (3.16), (3.17) *are* $\mathscr{C}(-,\mathcal{Z})$*-exact, then so is* (3.18).

Proposition 3.12. *Let* $\mathcal{X} \subseteq \mathcal{Z}$ *be subcategories of* $\mathscr{C}$, *and let*

$$M \longrightarrow M^0 \longrightarrow M^1 \dashrightarrow \tag{3.19}$$

be an $\mathbb{E}$*-triangle in* ξ. *Let*

$$M^0 \longrightarrow X_0^0 \longrightarrow \cdots \longrightarrow X_1^0 \longrightarrow X_n^0 \longrightarrow \cdots \tag{3.20}$$

and

$$M^1 \longrightarrow X_0^1 \longrightarrow \cdots \longrightarrow X_1^1 \longrightarrow X_n^1 \longrightarrow \cdots \tag{3.21}$$

be $\mathcal{Z}$*-coproper* $\mathcal{X}$*-coresolutions of* M^0 *and* M^1 *with all* $X_i^0, X_i^1 \in \mathcal{X}$. *Then*

(1) *If* (3.19) *is* $\mathscr{C}(-,\mathcal{Z})$*-exact, there exists a* $\mathcal{Z}$*-coproper* $\mathcal{X}$*-coresolution of* M

$$M \longrightarrow X_0^0 \longrightarrow X_1^0 \oplus X_0^1 \longrightarrow X_2^0 \oplus X_1^1 \longrightarrow \cdots \longrightarrow X_i^0 \oplus X_{i-1}^1 \longrightarrow \cdots \tag{3.22}$$

(2) *If* (3.19), (3.20), (3.21) *are* $\mathscr{C}(\mathcal{Z},-)$*-exact, then so is* (3.22).

Proposition 3.13. *Let* $\mathcal{Y}$ *be a* $\mathcal{Z}$*-resolving subcategory of* $\mathscr{C}$ *admitting a* $\mathcal{Z}$*-proper* ξ*-generator* $\mathcal{X}$. *If* $A \longrightarrow B \longrightarrow C \dashrightarrow$ *is a* $\mathscr{C}(\mathcal{Z},-)$*-exact* $\mathbb{E}$*-triangle in* ξ *with* $C \in \mathcal{Y}$, *then* $\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(A) = \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(B)$.

Proof. Since $\mathcal{Y}$ is a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$, $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(A) = 0$ if and only if $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(B) = 0$.

Now, assume that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(B) = m \geqslant 1$. Then there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle $K_1^B \longrightarrow Y_0^{'} \longrightarrow B \dashrightarrow$ in ξ with $Y_0^{'} \in \mathcal{Y}$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^B)$ $\leqslant m-1$. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
K_1^B \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & A \arrow[r, dashed] \arrow[d] & {} \\
K_1^B \arrow[r] & Y_0^{'} \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d] & {} \\
 & C \arrow[r, equal] \arrow[d, dashed] & C \arrow[d, dashed] & \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the second row and the third column are $\mathscr{C}(\mathcal{Z}, -)$-exact, by the snake lemma and [4, Lemma 2], we know that the first row and the second column are $\mathscr{C}(\mathcal{Z}, -)$-exact. It is clear that $M \in \mathcal{Y}$ since $\mathcal{Y}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations. Hence,

$$\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(A) \leqslant \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(K_1^B) + 1 \leqslant m = \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(B).$$

Conversely, assume that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(A) = n \geqslant 1$. Then there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle $K_1^A \longrightarrow Y_0 \longrightarrow A \dashrightarrow$ in ξ with $Y_0 \in \mathcal{Y}$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^A) \leqslant n - 1$. Since $\mathcal{X}$ is a $\mathcal{Z}$-proper ξ-generator of $\mathcal{Y}$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$K_1^C \longrightarrow X_0 \longrightarrow C \dashrightarrow$$

in ξ with $X_0 \in \mathcal{X}$ and $K_1^C \in \mathcal{Y}$. By [4, Lemma 4], we have the following commutative diagram:

$$\begin{tikzcd}
K_1^A \arrow[r] \arrow[d] & K_1^B \arrow[r] \arrow[d] & K_1^C \arrow[r, dashed] \arrow[d] & {} \\
Y_0 \arrow[r] \arrow[d] & Y_0 \oplus X_0 \arrow[r] \arrow[d] & X_0 \arrow[r, dashed] \arrow[d] & {} \\
A \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangles in ξ. Since $\mathcal{X} \subseteq \mathcal{Y}$, we have $Y_0 \oplus X_0 \in \mathcal{Y}$. Note that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^B) \leqslant \mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^A)$, the proof is similar to $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(A) \leqslant \mathcal{Y}_{\mathcal{Z}}$-res.dim$(B)$. By the induction hypothesis, we have

$$\begin{aligned}\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(B) &\leqslant \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(K_1^B)+1\\ &\leqslant \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(K_1^A)+1\\ &\leqslant n=\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(A).\end{aligned}$$ □

Corollary 3.14. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. For any object C in $\mathscr{C}$ and any non-negative integer n. If $\mathcal{Y}_{\mathcal{Z}}$-*res.dim$(C) = n < \infty$, *then $\mathcal{Y}_{\mathcal{Z}}$-*res.dim$(C \oplus Y) = n$ *for any object Y in $\mathcal{Y}$.*

Proof. By Definition 2.7, we have a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$C \xrightarrow{\binom{1}{0}} C\oplus Y \xrightarrow{(0\ 1)} Y \dashrightarrow$$

in ξ. By Proposition 3.13, we know that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) = \mathcal{Y}_{\mathcal{Z}}$-res.dim$(C \oplus Y)$. □

Proposition 3.15. *Let $\mathcal{Y}$ be a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. If $A \longrightarrow B \longrightarrow C \dashrightarrow$ is a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle in ξ with $A \in \mathcal{Y}$ and neither B nor C in $\mathcal{Y}$, then $\mathcal{Y}_{\mathcal{Z}}$-*res.dim$(B) = \mathcal{Y}_{\mathcal{Z}}$-res.dim$(C)$.

Proof. Assume that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(B) = n \geqslant 1$. Then there exists a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle $K_1^B \longrightarrow Y_0' \longrightarrow B \dashrightarrow$ in ξ with $Y_0' \in \mathcal{Y}$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^B) \leqslant n-1$. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
K_1^B \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & A \arrow[r, dashed] \arrow[d] & {} \\
K_1^B \arrow[r] & Y_0' \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d] & {} \\
 & C \arrow[r, equal] \arrow[d, dashed] & C \arrow[d, dashed] & \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the second row and the third column are $\mathscr{C}(\mathcal{Z},-)$-exact, by [4, Lemma 2] and the snake lemma, we know that the first row and the second column are $\mathscr{C}(\mathcal{Z},-)$-exact. By Proposition 3.13, we have

$$\begin{aligned}\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(C) &\leqslant \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(M)+1=\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(K_1^B)+1\\ &\leqslant n=\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(B).\end{aligned}$$

Conversely, assume that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) = m \geqslant 1$. Then there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle $K_1^C \longrightarrow Y_0 \longrightarrow C \dashrightarrow$ in ξ with $Y_0 \in \mathcal{Y}$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^C) \leqslant m-1$. By [3, Theorem 3.2] and Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d] & A \arrow[d] & \\
K_1^C \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d] & {} \\
K_1^C \arrow[r] & Y_0 \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the third row and the third column are $\mathscr{C}(\mathcal{Z}, -)$-exact, by [4, Lemma 2], we know that the second row and the second column are $\mathscr{C}(\mathcal{Z}, -)$-exact. Since $\mathcal{Y}$ is closed under $\mathcal{Z}$-proper ξ-extensions, then $W \in \mathcal{Y}$. Hence, $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(B) \leqslant \mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^C) + 1 \leqslant m = \mathcal{Y}_{\mathcal{Z}}$-res.dim$(C)$. $\square$

The following result gives a sufficient condition such that for an object C in $\mathscr{C}$, if $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) \leqslant n$, then all "$n$-$\mathcal{X}$-syzygies" of C are objects in $\mathcal{Y}$.

Theorem 3.16. *Let $\mathcal{X} \subseteq \mathcal{Z}$ be subcategories of $\mathscr{C}$, and let $\mathcal{Y}$ be a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. For any object C in $\mathscr{C}$ and any positive integer n. If $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) \leqslant n < \infty$, we have the following.*

(1) *For any $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex*

$$K_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C$$

in $\mathscr{C}$ with $X_i \in \mathcal{X}$, we have that $K_n \in \mathcal{Y}$.

(2) *If $A \longrightarrow B \longrightarrow C \dashrightarrow$ is a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle in ξ with $B \in \mathcal{X}$, then $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(A) \leqslant n-1$.*

Proof. Assume that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) \leqslant n$. By Theorem 3.9(1) $\Longrightarrow$ (2) and take $k = n$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$Y_n \longrightarrow X'_{n-1} \longrightarrow \cdots \longrightarrow X'_1 \longrightarrow X'_0 \longrightarrow C \tag{3.23}$$

in $\mathscr{C}$ with $X'_i \in \mathcal{X}$ and $Y_n \in \mathcal{Y}$.

(1) Let

$$K_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C$$

be a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex in $\mathscr{C}$ with $X_i \in \mathcal{X}$. Applying Proposition 3.11 to the $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle $C \longrightarrow C \longrightarrow 0 \dashrightarrow$ in ξ, we get the following $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$K_n \longrightarrow Y_n \oplus X_{n-1} \longrightarrow \cdots \longrightarrow X'_2 \oplus X_1 \longrightarrow X'_1 \oplus X_0 \longrightarrow X'_0.$$

Since $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations, we have $K_n \in \mathcal{Y}$.

(2) By (3.23), we have a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle $K_1^C \longrightarrow X_0^{'} \longrightarrow C \dashrightarrow$ in ξ with $X_0^{'} \in \mathcal{X}$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(K_1^C) \leqslant n-1$. By [3, Theorem 3.2] and Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d] & A \arrow[d] & \\
K_1^C \arrow[r] \arrow[d, equal] & W \arrow[r, "f"] \arrow[d] & B \arrow[r, dashed] \arrow[d] & {} \\
K_1^C \arrow[r] & X_0^{'} \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the third row and the third column are $\mathscr{C}(\mathcal{Z},-)$-exact, by [4, Lemma 2], we know that the second row and the second column are $\mathscr{C}(\mathcal{Z},-)$-exact. Note that $B \subseteq \mathcal{X} \subseteq \mathcal{Z}$. Hence, $\mathscr{C}(B,f)$ is an epimorphism. So, the second row is split and $W \cong B \oplus K_1^C$. Then the second column yields a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle $A \longrightarrow B \oplus K_1^C \longrightarrow X_0^{'} \dashrightarrow$ in ξ. Since $X_0^{'} \subseteq \mathcal{X} \subseteq \mathcal{Y}$, by Proposition 3.13 and Corollary 3.14, we know that

$$\mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(A) = \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(B \oplus K_1^C) = \mathcal{Y}_{\mathcal{Z}}\text{-res.dim}(K_1^C) \leqslant n-1. \qquad \square$$

Corollary 3.17. *Let $\mathcal{X} \subseteq \mathcal{Z}$ be subcategories of $\mathscr{C}$, and let $\mathcal{Y}$ be a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. If $\mathcal{Y}$ is closed under direct summands, then so is $(\mathcal{Y}_{\mathcal{Z}})_n^{\wedge}$ for any non-negative integer n.*

Proof. We proceed by induction on n. The case for $n = 0$ follows from assumption. Now, let $n \geqslant 1$, and let C be an object in $\mathscr{C}$ with $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C) \leqslant n$ and $C = C_1 \oplus C_2$. By Theorem 3.9(1) $\Longrightarrow$ (2) and take $k = n$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$Y_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C \tag{3.24}$$

in $\mathscr{C}$ with $X_i \in \mathcal{X}$ and $Y_n \in \mathcal{Y}$. Hence, we have a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$K_1^C \longrightarrow X_0 \longrightarrow C \dashrightarrow$$

in ξ with $X_0 \in \mathcal{X}$. Note that both

$$C_1 \xrightarrow{\binom{1}{0}} C \xrightarrow{(0\ 1)} C_2 \dashrightarrow \tag{3.25}$$

and

$$C_2 \xrightarrow{\binom{1}{0}} C \xrightarrow{(0\ 1)} C_1 \dashrightarrow \tag{3.26}$$

are two $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangles in ξ. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
K_1^C \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & C_1 \arrow[r, dashed] \arrow[d] & {} \\
K_1^C \arrow[r] & X_0 \arrow[r] \arrow[d] & C \arrow[r, dashed] \arrow[d] & {} \\
 & C_2 \arrow[r, equal] \arrow[d, dashed] & C_2 \arrow[d, dashed] & \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the second row and the third column are $\mathscr{C}(\mathcal{Z},-)$-exact, by [4, Lemma 2] and the snake lemma, we know that the first row and the second column are $\mathscr{C}(\mathcal{Z},-)$-exact. Hence, we get a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$M \longrightarrow X_0 \longrightarrow C_2 \dashrightarrow \tag{3.27}$$

in ξ. Similarly, we get a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$N \longrightarrow X_0 \longrightarrow C_1 \dashrightarrow \tag{3.28}$$

in ξ. By Proposition 3.11, (3.24), (3.25) and (3.28), we get the following $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$T_1 \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_2. \tag{3.29}$$

By Proposition 3.11, (3.24), (3.26) and (3.27), we get the following $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$T_1^{'} \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_1. \tag{3.30}$$

By Proposition 3.11, (3.24), (3.26) and (3.29), we get the following $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$T_2 \longrightarrow X_2 \oplus X_1 \oplus X_0 \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_1.$$

By Proposition 3.11, (3.24), (3.25) and (3.30), we get the following $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$T_2^{'} \longrightarrow X_2 \oplus X_1 \oplus X_0 \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_2.$$

Continuing this procedure, we finally get the following two $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complexes

(3.31)

$$G_n \longrightarrow \bigoplus_{i=0}^{n-1} X_i \longrightarrow \bigoplus_{i=0}^{n-2} X_i \longrightarrow \cdots \longrightarrow X_2 \oplus X_1 \oplus X_0 \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_1$$

and
(3.32)

$$H_n \longrightarrow \bigoplus_{i=0}^{n-1} X_i \longrightarrow \bigoplus_{i=0}^{n-2} X_i \longrightarrow \cdots \longrightarrow X_2 \oplus X_1 \oplus X_0 \longrightarrow X_1 \oplus X_0 \longrightarrow X_0 \longrightarrow C_2.$$

Put $U_j = \bigoplus_{i=0}^{j} X_i \subseteq \mathcal{X}$. By [4, Lemma 4], we get the following $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$G_n \oplus H_n \longrightarrow U_{n-1} \oplus U_{n-1} \longrightarrow \cdots \longrightarrow U_2 \oplus U_2 \longrightarrow U_1 \oplus U_1 \longrightarrow U_0 \oplus U_0 \longrightarrow C.$$

By Theorem 3.16, we know that $G_n \oplus H_n \in \mathcal{Y}$. Since $\mathcal{Y}$ is closed under direct summands, we have $G_n \in \mathcal{Y}$ and $H_n \in \mathcal{Y}$. By (3.31), (3.32), Theorem 3.9 (2) $\Longrightarrow$ (1) and take $k = n$, we know that $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C_1) \leqslant n$ and $\mathcal{Y}_{\mathcal{Z}}$-res.dim$(C_2) \leqslant n$. □

The following result gives a necessary and sufficient condition such that for an object A in $\mathscr{C}$, A has a $\mathcal{Z}$-proper $\mathcal{X}$-resolution if and only if A has a $\mathcal{Z}$-proper $\mathcal{Y}$-resolution.

Theorem 3.18. *Let $\mathcal{X}$ and $\mathcal{Z}$ be subcategories of $\mathscr{C}$, and let $\mathcal{Y}$ be a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$ admitting a $\mathcal{Z}$-proper ξ-generator $\mathcal{X}$. Then $res\ \mathcal{X}_{\mathcal{Z}} = res\ \mathcal{Y}_{\mathcal{Z}}$.*

Proof. It is clear that $res\ \mathcal{X}_{\mathcal{Z}} \subseteq res\ \mathcal{Y}_{\mathcal{Z}}$. We only prove that $res\ \mathcal{Y}_{\mathcal{Z}} \subseteq res\ \mathcal{X}_{\mathcal{Z}}$. Assume that $A \in res\ \mathcal{Y}_{\mathcal{Z}}$. Then there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$B \longrightarrow Y \longrightarrow A \dashrightarrow$$

in ξ with $Y \in \mathcal{Y}$ and $B \in res\ \mathcal{Y}_{\mathcal{Z}}$. Since $\mathcal{X}$ is a $\mathcal{Z}$-proper ξ-generator of $\mathcal{Y}$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$Y_1 \longrightarrow X_1 \longrightarrow Y \dashrightarrow$$

in ξ with $Y_1 \in \mathcal{Y}$. By [3, Theorem 3.2] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d] & {} \\
Y_1 \arrow[r] & X_1 \arrow[r] \arrow[d] & Y \arrow[r, dashed] \arrow[d] & {} \\
& A \arrow[r, equal] \arrow[d, dashed] & A \arrow[d, dashed] & \\
& {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the second row and the third column are $\mathscr{C}(\mathcal{Z}, -)$-exact, by the snake lemma and [4, Lemma 2], the first row and the second column are $\mathscr{C}(\mathcal{Z}, -)$-exact. Since $B \in res\ \mathcal{Y}_{\mathcal{Z}}$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$N \longrightarrow Y_2 \longrightarrow B \dashrightarrow$$

in ξ with $Y_2 \in \mathcal{Y}$ and $N \in res\ \mathcal{Y}_{\mathcal{Z}}$. By [3, Theorem 3.2] and Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
Y_1 \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & Y_2 \arrow[r, dashed] \arrow[d] & {} \\
Y_1 \arrow[r] & M \arrow[r] \arrow[d, dashed] & B \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the third row and the third column are $\mathscr{C}(\mathcal{Z}, -)$-exact, by [4, Lemma 2], the second row and the second column are $\mathscr{C}(\mathcal{Z}, -)$-exact. Since $Y_1, Y_2 \in \mathcal{Y}$ and $\mathcal{Y}$ is closed under $\mathcal{Z}$-proper ξ-extensions, we have $W \in \mathcal{Y}$. Consider the second column in the second diagram and repeat the above process. We can get a $\mathscr{C}(\mathcal{Z}, -)$-exact $\mathbb{E}$-triangle

$$T \longrightarrow X_2 \longrightarrow M \dashrightarrow$$

in ξ with $X_2 \in \mathcal{X}$. Continuing this process, we get a $\mathscr{C}(\mathcal{Z}, -)$-exact ξ-exact complex

$$\cdots \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow A.$$

Hence, $A \in res\ \mathcal{X}_{\mathcal{Z}}$. $\square$

4. Applications

In this section, we introduce $(\mathcal{X}, \mathcal{Z})$-Gorenstein subcategory $\mathcal{GX}_{\mathcal{Z}}(\xi)$ of $\mathscr{C}$ and prove that $\mathcal{GX}_{\mathcal{Z}}(\xi)$ is both $\mathcal{Z}$-resolving subcategory and $\mathcal{Z}$-coresolving subcategory of $\mathscr{C}$. Moreover, we give some applications of the results obtained in Section 3. First, we give the following definition.

Definition 4.1. Let $\mathcal{X}$ and $\mathcal{Z}$ be subcategories of $\mathscr{C}$, and let A be an object of $\mathscr{C}$. A *complete $\mathcal{X}_{\mathcal{Z}}(\xi)$-resolution* of A is both $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact ξ-exact complex

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

in $\mathscr{C}$ with all $X_i, X^i \in \mathcal{X}$ such that

$$K_1 \longrightarrow X_0 \longrightarrow A \dashrightarrow \text{ and } A \longrightarrow X^0 \longrightarrow W^1 \dashrightarrow$$

are corresponding $\mathbb{E}$-triangles in ξ.

The $(\mathcal{X},\mathcal{Z})$-*Gorenstein subcategory* $\mathcal{GX}_{\mathcal{Z}}(\xi)$ of $\mathscr{C}$ is defined as

$$\mathcal{GX}_{\mathcal{Z}}(\xi)=\{A\in\mathscr{C}\mid A \text{ admits a complete } \mathcal{X}_{\mathcal{Z}}(\xi)\text{-resolution}\}.$$

Remark 4.2. (1) If $\mathcal{X}=\mathcal{Z}=\mathcal{P}(\xi)$ (resp., $\mathcal{X}=\mathcal{Z}=\mathcal{I}(\xi)$), then $\mathcal{GX}_{\mathcal{Z}}(\xi)$ coincides with $\mathcal{GP}(\xi)$ (resp., $\mathcal{GI}(\xi)$), where $\mathcal{GP}(\xi)$ (resp., $\mathcal{GI}(\xi)$) is the full subcategory of $\mathcal{C}$ consisting of ξ-Gorenstein projective (resp., injective) objects (see [3, Definition 4.8]).

(2) If $\mathcal{X}=\mathcal{Z}$, then $\mathcal{GX}_{\mathcal{Z}}(\xi)$ coincides with $\mathcal{GX}(\xi)$, where $\mathcal{GX}(\xi)$ is the Gorenstein subcategories of $\mathscr{C}$ (see [4, Definition 7]).

Theorem 4.3. *Let $\mathcal{X}\subseteq\mathcal{Z}$ be subcategories of $\mathscr{C}$. Then we have*
(1) *$\mathcal{GX}_{\mathcal{Z}}(\xi)$ is a $\mathcal{Z}$-resolving subcategory of $\mathscr{C}$.*
(2) *$\mathcal{GX}_{\mathcal{Z}}(\xi)$ is a $\mathcal{Z}$-coresolving subcategory of $\mathscr{C}$.*

Proof. (1) For any object A in $\mathcal{GX}_{\mathcal{Z}}(\xi)$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle

$$K_1^A\longrightarrow X_0^A\longrightarrow A\dashrightarrow$$

in ξ with $X_0^A\in\mathcal{X}$ and $K_1^A\in\mathcal{GX}_{\mathcal{Z}}(\xi)$. Hence, $\mathcal{X}$ is a $\mathcal{Z}$-proper ξ-generator for $\mathcal{GX}_{\mathcal{Z}}(\xi)$.

Let

$$A\xrightarrow{f}B\xrightarrow{g}C\dashrightarrow$$

be a $\mathscr{C}(\mathcal{Z},-)$-exact $\mathbb{E}$-triangle in ξ. Assume that $\mathcal{Z}\in\mathcal{GX}_{\mathcal{Z}}(\xi)$. Then there exists a $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact $\mathbb{E}$-triangle

$$K_1^C\longrightarrow X_0^C\longrightarrow C\dashrightarrow$$

in ξ with $X_0^C\in\mathcal{X}$ and $K_1^C\in\mathcal{GX}_{\mathcal{Z}}(\xi)$. By [3, Theorem 3.2] and Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
 & K_1^C \arrow[r, equal] \arrow[d] & K_1^C \arrow[d] & \\
A \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & X_0^C \arrow[r, dashed] \arrow[d] & {} \\
A \arrow[r] & B \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd} \tag{4.1}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the third row is $\mathscr{C}(\mathcal{Z},-)$-exact, by [4, Lemma 2], the second row is $\mathscr{C}(\mathcal{Z},-)$-exact. Note that $X_0^C\in\mathcal{X}\subseteq\mathcal{Z}$. Hence, the second row is split and $\mathscr{C}(-,\mathcal{Z})$-exact. Since the second row and the third column are both $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact, by [4, Lemma 3], all rows and columns are both $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact in (4.1).

Claim 1: If $A,C\in\mathcal{GX}_{\mathcal{Z}}(\xi)$, then $B\in\mathcal{GX}_{\mathcal{Z}}(\xi)$.

Since $A, C \in \mathcal{GX}_{\mathcal{Z}}(\xi)$, there exist $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangles

$$K_1^A \longrightarrow X_0^A \longrightarrow A \dashrightarrow, \ A \longrightarrow X_A^0 \longrightarrow K_A^1 \dashrightarrow$$

$$K_1^C \longrightarrow X_0^C \longrightarrow C \dashrightarrow, \ C \longrightarrow X_C^0 \longrightarrow K_C^1 \dashrightarrow$$

with $X_0^A, X_A^0, X_0^C, X_C^0 \in \mathcal{X}$ and $K_1^A, K_A^1, K_1^C, K_C^1 \in \mathcal{GX}_{\mathcal{Z}}(\xi)$. By [4, Lemma 4], we have the following commutative diagram:

$$\begin{tikzcd}
K_1^A \arrow[r] \arrow[d] & K_1^B \arrow[r] \arrow[d] & K_1^C \arrow[r, dashed] \arrow[d] & {} \\
X_0^A \arrow[r] \arrow[d] & X_0^A \oplus X_0^C \arrow[r] \arrow[d] & X_0^C \arrow[r, dashed] \arrow[d] & {} \\
A \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed] \arrow[d, dashed] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where all rows and columns are both $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangles in ξ. By [4, Lemma 5], we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r, dashed] \arrow[d] & {} \\
X_A^0 \arrow[r] \arrow[d] & X_A^0 \oplus X_C^0 \arrow[r] \arrow[d] & X_C^0 \arrow[r, dashed] \arrow[d] & {} \\
K_A^1 \arrow[r] \arrow[d, dashed] & K_B^1 \arrow[r] \arrow[d, dashed] & K_C^1 \arrow[r, dashed] \arrow[d, dashed] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where all rows and columns are both $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangles in ξ. Continuing this process, we get a complete $\mathcal{X}_{\mathcal{Z}}(\xi)$-resolution of B. Hence, $B \in \mathcal{GX}_{\mathcal{Z}}(\xi)$.

Claim 2: If $B, C \in \mathcal{GX}_{\mathcal{Z}}(\xi)$, then $A \in \mathcal{GX}_{\mathcal{Z}}(\xi)$.

Since $B, C \in \mathcal{GX}_{\mathcal{Z}}(\xi)$, B and C both have a $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathcal{X}$-coresolution. By Proposition 3.12, we have a $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact ξ-exact complex

$$A \longrightarrow X_A^0 \longrightarrow X_A^1 \longrightarrow \cdots \longrightarrow X_A^n \longrightarrow \cdots . \tag{4.2}$$

Since $B \in \mathcal{GX}_{\mathcal{Z}}(\xi)$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangle $K_1^B \xrightarrow{u} X_0^B \xrightarrow{v} B \overset{\beta}{\dashrightarrow}$ in ξ with $X_0^B \in \mathcal{X}$ and $K_1^B \in \mathcal{GX}_{\mathcal{Z}}(\xi)$. By [4, Lemma

3] and (ET4)$^{\mathrm{op}}$, we have the following commutative diagram:

$$
\begin{tikzcd}
K_1^B \arrow[r, "e"] \arrow[d, equal] & M \arrow[r, "h"] \arrow[d, "s"] & A \arrow[r, dashed, "f^*\beta"] \arrow[d, "f"] & {} \\
K_1^B \arrow[r, "u"'] & X_0^B \arrow[r, "v"'] \arrow[d, "t"] & B \arrow[r, dashed, "\beta"] \arrow[d, "g"] & {} \\
 & C \arrow[r, equal] \arrow[d, dashed, "\gamma"] & C \arrow[d, dashed, "\delta"] & \\
 & {} & {} &
\end{tikzcd}
\tag{4.3}
$$

where all rows and columns are both $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangles in ξ. Since $C \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$, there exists a $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact $\mathbb{E}$-triangle $K_1^C \xrightarrow{p} X_0^C \xrightarrow{q} C \overset{\eta}{\dashrightarrow}$ in ξ with $X_0^C \in \mathcal{X}$ and $K_1^C \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$. Since the second column is $\mathscr{C}(\mathcal{Z}, -)$-exact in (4.3) and $\mathcal{X} \subseteq \mathcal{Z}$, we have the following commutative diagram:

$$
\begin{tikzcd}
K_1^C \arrow[r, "p"] \arrow[d, "x"] & X_0^C \arrow[r, "q"] \arrow[d, "y"] & C \arrow[r, dashed, "\eta"] \arrow[d, equal] & {} \\
M \arrow[r, "s"] & X_0^B \arrow[r, "t"] & C \arrow[r, dashed, "\gamma"] & {}
\end{tikzcd}
$$

with $x_*\eta = \gamma$. By [9, Proposition 1.20], we have an $\mathbb{E}$-triangle

$$
K_1^C \xrightarrow{\left(\begin{smallmatrix} -x \\ p \end{smallmatrix}\right)} M \oplus X_0^C \xrightarrow{(s\ y)} X_0^B \overset{t^*\eta}{\dashrightarrow}
\tag{4.4}
$$

in $\mathscr{C}$. Hence, we can construct the following commutative diagram:

$$
\begin{tikzcd}
K_1^C \arrow[r, "\left(\begin{smallmatrix} -x \\ p \end{smallmatrix}\right)"] \arrow[d, equal] & M \oplus X_0^C \arrow[r, "(s\ y)"] \arrow[d, "(0\ 1)"'] & X_0^B \arrow[r, dashed, "t^*\eta"] \arrow[d, "t"] & {} \\
K_1^C \arrow[r, "p"] & X_0^C \arrow[r, "q"] & C \arrow[r, dashed, "\eta"] & {}
\end{tikzcd}
\tag{4.5}
$$

Since the second row is in ξ and ξ is closed under base change, the $\mathbb{E}$-triangle (4.4) is in ξ. Note that the second row is both $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact. By [4, Lemma 2], the $\mathbb{E}$-triangle (4.4) is $\mathscr{C}(\mathcal{Z}, -)$-exact. Applying $\mathscr{C}(-, \mathcal{Z})$ to (4.5), by the snake lemma, the $\mathbb{E}$-triangle (4.4) is $\mathscr{C}(-, \mathcal{Z})$-exact. Since $K_1^C \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$ and $X_0^B \in \mathcal{X} \subseteq \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$, by Claim 1, $M \oplus X_0^C \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$. Hence, $M \oplus X_0^C$ has a $\mathcal{X}$-resolution which is $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact. Applying Proposition 3.11 to the $\mathbb{E}$-triangle $X_0^C \longrightarrow M \oplus X_0^C \longrightarrow M \dashrightarrow$, we have a $\mathscr{C}(\mathcal{Z}, -)$-exact and $\mathscr{C}(-, \mathcal{Z})$-exact ξ-exact complex

$$
\cdots \longrightarrow X_n^M \longrightarrow \cdots \longrightarrow X_1^M \longrightarrow X_0^M \longrightarrow M
$$

with $X_i^M \in \mathcal{X}$. Since $K_1^B \in \mathcal{GX}_{\mathcal{Z}}(\xi)$, then K_1^B has a $\mathcal{X}$-resolution which is $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact. Applying Proposition 3.11 to the $\mathbb{E}$-triangle $K_1^B \longrightarrow M \longrightarrow A \dashrightarrow$, we have a $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact ξ-exact complex

$$\cdots \longrightarrow X_n^A \longrightarrow \cdots \longrightarrow X_1^A \longrightarrow X_0^A \longrightarrow A \tag{4.6}$$

with $X_i^A \in \mathcal{X}$. Combining (4.2) and (4.6), we get a complete $\mathcal{X}_{\mathcal{Z}}(\xi)$-resolution of A. Hence, $A \in \mathcal{GX}_{\mathcal{Z}}(\xi)$. □

By Remark 4.2 and Theorem 4.3, we have the following corollary.

Corollary 4.4. *Let $\mathcal{X}$ be a subcategory of $\mathscr{C}$. Then $\mathcal{GX}(\xi)$ is a $\mathcal{X}$-resolving and $\mathcal{X}$-coresolving subcategory of $\mathscr{C}$. In particular, $\mathcal{GP}(\xi)$ is a $\mathcal{P}(\xi)$-resolving and $\mathcal{P}(\xi)$-coresolving subcategory of $\mathscr{C}$, $\mathcal{GI}(\xi)$ is a $\mathcal{I}(\xi)$-resolving and $\mathcal{I}(\xi)$-coresolving subcategory of $\mathscr{C}$.*

Corollary 4.5 ([3, Theorem 4.16])**.** *Let $A \longrightarrow B \longrightarrow C \dashrightarrow$ be an $\mathbb{E}$-triangle in ξ with $C \in \mathcal{GP}(\xi)$. Then $A \in \mathcal{GP}(\xi)$ if and only if $B \in \mathcal{GP}(\xi)$.*

Proof. By Corollary 4.4, we know that $\mathcal{GP}(\xi)$ is a $\mathcal{P}(\xi)$-resolving subcategory of $\mathscr{C}$, thus $\mathcal{GP}(\xi)$ are closed under ξ-extensions and Cocones of ξ-deflations. □

Dually, we have the following.

Corollary 4.6. *Let $A \longrightarrow B \longrightarrow C \dashrightarrow$ be an $\mathbb{E}$-triangle in ξ with $A \in \mathcal{GI}(\xi)$. Then $B \in \mathcal{GI}(\xi)$ if and only if $C \in \mathcal{GI}(\xi)$.*

Corollary 4.7. *Let $A \longrightarrow B \longrightarrow C \dashrightarrow$ be a $\mathscr{C}(-,\mathcal{P}(\xi))$-exact $\mathbb{E}$-triangle in ξ with $A \in \mathcal{GP}(\xi)$. Then $B \in \mathcal{GP}(\xi)$ if and only if $C \in \mathcal{GP}(\xi)$.*

Proof. By Corollary 4.4, we know that $\mathcal{GP}(\xi)$ is a $\mathcal{P}(\xi)$-coresolving subcategory of $\mathscr{C}$, thus $\mathcal{P}(\xi)$ are closed under $\mathcal{P}(\xi)$-coproper ξ-extensions and Cones of $\mathcal{P}(\xi)$-coproper ξ-inflations. □

Dually, we have the following.

Corollary 4.8. *Let $A \longrightarrow B \longrightarrow C \dashrightarrow$ be a $\mathscr{C}(\mathcal{I}(\xi),-)$-exact $\mathbb{E}$-triangle in ξ with $C \in \mathcal{GI}(\xi)$. Then $A \in \mathcal{GI}(\xi)$ if and only if $B \in \mathcal{GI}(\xi)$.*

Theorem 4.9. *Let $\mathcal{X} \subseteq \mathcal{Z}$ be subcategories of $\mathscr{C}$ with $\mathcal{X}$ is closed under direct summands, and let C be an object in $\mathscr{C}$ with $\mathcal{X}_{\mathcal{Z}}$-res.dim$(C) < \infty$. Then*

$$(\mathcal{GX}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}\text{-res.dim}(C) = \mathcal{X}_{\mathcal{Z}}\text{-res.dim}(C).$$

Proof. Since $\mathcal{X} \subseteq \mathcal{GX}_{\mathcal{Z}}(\xi)$, we have $(\mathcal{GX}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$-res.dim$(C) \leqslant \mathcal{X}_{\mathcal{Z}}$-res.dim$(C)$. Next, we prove that $\mathcal{X}_{\mathcal{Z}}$-res.dim$(C) \leqslant (\mathcal{GX}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$-res.dim$(C)$. Assume that $\mathcal{X}_{\mathcal{Z}}$-res.dim$(C) = m < \infty$ and $(\mathcal{GX}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$-res.dim$(C) = n < \infty$. Then there exists a $\mathscr{C}(\mathcal{Z},-)$-exact ξ-exact complex

$$X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C$$

with all $X_j \in \mathcal{X}$. If $m > n$, we have the following two ξ-exact complexes

$$K_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C \tag{4.7}$$

and

$$X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_n \longrightarrow K_n. \tag{4.8}$$

Hence, we get the following $\mathbb{E}$-triangle

$$K_{i+1} \longrightarrow X_i \longrightarrow K_i \dashrightarrow$$

in ξ with $K_m = X_m$ and $n \leqslant i \leqslant m-1$. By Theorem 3.16, we know that $K_n \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$ in (4.7). Since $(\mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$ is closed under Cocones of $\mathcal{Z}$-proper ξ-deflations and $X_j \in \mathcal{X} \subseteq \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$, we have $K_i \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$ for all $n \leqslant i \leqslant m-1$. Since $K_{m-1} \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$, there exists a $\mathscr{C}(\mathcal{Z},-)$-exact and $\mathscr{C}(-,\mathcal{Z})$-exact $\mathbb{E}$-triangle

$$G \longrightarrow X \longrightarrow K_{m-1} \dashrightarrow$$

in ξ with $X \in \mathcal{X}$ and $G \in \mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi)$. By [3, Theorem 3.2] and Lemma 2.6, we have the following commutative diagram:

$$\begin{tikzcd}
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
X_m \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & X \arrow[r, dashed] \arrow[d] & {} \\
X_m \arrow[r] & X_{m-1} \arrow[r] \arrow[d, dashed] & K_{m-1} \arrow[r, dashed] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles in ξ. Since the third row is $\mathscr{C}(\mathcal{Z},-)$-exact, by [4, Lemma 2], the second row is $\mathscr{C}(\mathcal{Z},-)$-exact. Note that $X \in \mathcal{X} \subseteq \mathcal{Z}$. Hence, the second row is split and $\mathscr{C}(-,\mathcal{Z})$-exact. Applying $\mathscr{C}(-,\mathcal{Z})$-exact to the above commutative diagram, by snake lemma, we know that the third row is $\mathscr{C}(-,\mathcal{Z})$-exact. Hence, the third row is split and $X_{m-1} \cong X_m \oplus K_{m-1}$. Since $\mathcal{X}$ is closed under direct summands, we have $K_{m-1} \in \mathcal{X}$. Repeating this process, we know that $K_n \in \mathcal{X}$ and $(\mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$-res.dim$(C) \leqslant n$, which is a contradiction. Hence, $m \leqslant n$ and $\mathcal{X}_{\mathcal{Z}}$-res.dim$(C) \leqslant (\mathcal{G}\mathcal{X}_{\mathcal{Z}}(\xi))_{\mathcal{Z}}$-res.dim$(C)$. □

For any object C in $\mathscr{C}$, the definitions of the ξ-projective dimension ξ-pd(C) and ξ-$\mathcal{G}$ projective dimension ξ-$\mathcal{G}$pd(C) and their duality can be found in [3].

Corollary 4.10. *Let C be an object in $\mathscr{C}$ with ξ-pd$(C) < \infty$ (resp., ξ-id$(C) < \infty$). Then ξ-$\mathcal{G}$pd$(C) = \xi$-pd(C) (resp., ξ-$\mathcal{G}$id$(C) = \xi$-id(C)).*

Proof. Note that $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are closed under direct summands. Applying [3, Proposition 5.2] and its duality, the next proof is similar to Theorem 4.9. □

Next, we give some applications of the results obtained in Section 3. The arguments here are similar to that in Section 3, so we omit them.

Proposition 4.11. *For any object M in $\mathscr{C}$ and any positive integer n, the following are equivalent*:

(1) ξ-$\mathcal{G}\mathrm{pd}(M) \leqslant n$.

(2) *For any integer k with $1 \leqslant k \leqslant n$, there exists a ξ-exact complex*

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that $T_i \in \mathcal{P}(\xi)$ if $0 \leqslant i < k$ and $P_j \in \mathcal{GP}(\xi)$ if $j \geqslant k$.

(3) *For any integer k with $0 \leqslant k \leqslant n-1$, there exists a ξ-exact complex*

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that $T_k \in \mathcal{P}(\xi)$ and other $T_i \in \mathcal{GP}(\xi)$.

(4) *For any integer k with $0 \leqslant k \leqslant n-1$, there exists a ξ-exact complex*

$$Y_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

with $Y_n \in \mathcal{GP}(\xi)$, such that $T_k \in \mathcal{GP}(\xi)$ and other $T_i \in \mathcal{P}(\xi)$.

Dually, we have the following.

Proposition 4.12. *For any object M in $\mathscr{C}$ and any positive integer n, the following are equivalent*:

(1) $\mathcal{GP}(\xi)$-cores.dim$(M) \leqslant n$.

(2) *For any integer k with $1 \leqslant k \leqslant n$, there exists a ξ-exact complex*

$$M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

such that $T_i \in \mathcal{GP}(\xi)$ if $0 \leqslant i < k$ and $P_j \in \mathcal{P}(\xi)$ if $j \geqslant k$.

(3) *For any integer k with $1 \leqslant k \leqslant n$, there exists a ξ-exact complex*

$$M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

such that $T_k \in \mathcal{P}(\xi)$ and other $T_i \in \mathcal{GP}(\xi)$.

(4) *For any integer k with $1 \leqslant k \leqslant n$, there exists a ξ-exact complex*

$$M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

with $T_0 \in \mathcal{GP}(\xi)$, such that $T_k \in \mathcal{GP}(\xi)$ and other $T_i \in \mathcal{P}(\xi)$.

Proposition 4.13. *For any $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in ξ with $C \in \mathcal{GP}(\xi)$, we have ξ-$\mathcal{G}\mathrm{pd}(A) = \xi$-$\mathcal{G}\mathrm{pd}(B)$.*

Corollary 4.14. *For any object C in $\mathscr{C}$ and any non-negative integer n, if ξ-$\mathcal{G}\mathrm{pd}(C) = n < \infty$, then ξ-$\mathcal{G}\mathrm{pd}(C \oplus Y) = n$ for any object Y in $\mathcal{GP}(\xi)$.*

Proposition 4.15. *For any $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in ξ with $A \in \mathcal{GP}(\xi)$ and neither B nor C in $\mathcal{GP}(\xi)$, we have ξ-$\mathcal{G}\mathrm{pd}(B) = \xi$-$\mathcal{G}\mathrm{pd}(C)$.*

Proposition 4.16. *For any object C in $\mathscr{C}$ and any positive integer n, if ξ-$\mathcal{G}\mathrm{pd}(C) \leqslant n < \infty$, we have the following.*

(1) *For any ξ-exact complex*

$$K_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C$$

in $\mathscr{C}$ with $X_i \in \mathcal{P}(\xi)$, we have that $K_n \in \mathcal{GP}(\xi)$.

(2) *For any $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in ξ with $B \in \mathcal{P}(\xi)$, we have ξ-$\mathcal{G}\mathrm{pd}(A) \leqslant n-1$.*

Proposition 4.17. *res $\mathcal{P}(\xi) =$ res $\mathcal{GP}(\xi)$.*

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Songsong Liu
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023, P. R. China
Email address: liuss0308@163.com

Jiaqun Wei
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023, P. R. China
Email address: weijiaqun@njnu.edu.cn