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A NOTE ON UNICITY OF MEROMORPHIC FUNCTIONS IN SEVERAL VARIABLES

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Abstract. Let $f(z)$ be a meromorphic function in several variables satisfying

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{r}=0.$$

We mainly investigate the uniqueness problem on f in $\mathbb{C}^m$ sharing polynomial or periodic small function with its difference polynomials from a new perspective. Our main theorems can be seen as the improvement and extension of previous results.

1. Introduction and main results

Let $f(z)$ and $g(z)$ be two meromorphic functions on $\mathbb{C}^m$, and $\alpha\in\mathbb{P}^1=\mathbb{C}^1\cup\{\infty\}$. If $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities (ignoring multiplicities), we say that f and g share α CM (IM). Certainly, all CM shared values are IM shared values as well. It is well-known that in 1926, Nevanlinna [16] proved that if two non-constant meromorphic functions on complex plane $\mathbb{C}^1$ share five distinct values IM in $\mathbb{P}^1$, then they must be equal identically.

As we know, the numbers of distinct values in Nevanlinna five-value theorem cannot be reduced to four. For instance, entire functions e^z and e^{-z} share $0,1,-1$ and ∞ IM in $\mathbb{P}^1$, but $e^z\neq e^{-z}$. Note here that they have a shared value ∞. In 1997, Li [12] further considered the case of sharing small functions (not including the constant function ∞) for entire functions, and proved that two non-constant entire functions in $\mathbb{C}^m$ must be identically equal if they share four distinct small meromorphic functions.

In this paper, we assume that the reader is familiar with standard notations and terms in the value distribution theory such as $T(r,f)$ and $m(r,f)$, etc.

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(see, e.g., [9, 17, 18]). A meromorphic function $\alpha(z)$ on $\mathbb{C}^m$ is called a small function with respect to f if $T(r,\alpha) = o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. Denote by $S(f)$ the family of all small functions of $f(z)$. The order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are, respectively, defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \ \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In the last decades, uniqueness problems on meromorphic function with its derivative or differential polynomial have been studied deeply (see, e.g., [10, 13, 14]). In 2001, Li and Yang [14] investigated the uniqueness of entire function f sharing a finite value $a(\neq 0)$ CM in $\mathbb{P}^1$ with f' and $f^{(n)}$, and they claimed that f must satisfy

$$f(z) = be^{cz} - \frac{a(1-c)}{c},$$

where b, c are non-zero constants with $c^{n-1} = 1$, $n \in \mathbb{N}^+(\geq 2)$. Meanwhile, Li and Yang also considered the case of general higher order differential polynomial

$$L(f) := a_1 f' + a_2 f'' + \cdots + a_n f^{(n)}, \ a_n \neq 0,$$

with $a_1, a_2, \ldots, a_n$ being constants, and the corresponding uniqueness result has been obtained (see, [9, Theorem 2.104]).

With the establishment of difference analogues of the lemma on the logarithmic derivative, the research on uniqueness of meromorphic function with its difference or difference operators was also widely concerned (see, e.g., [3–5, 7, 11, 19]). Let $c = (c_1, c_2, \ldots, c_m) \in \mathbb{C}^m \backslash \{\mathbf{0}\}$. For a given meromorphic function $f(z) : \mathbb{C}^m \to \mathbb{P}^1$, we define its shift by $f(z+c)$ and its difference operators by

$$\Delta_c f(z) = f(z_1 + c_1, z_2 + c_2, \ldots, z_m + c_m) - f(z_1, z_2, \ldots, z_m),$$
$$\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)), \ (n \in \mathbb{N}, n \geq 2)$$

for any $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$. In particular, we set $\Delta_c^0 f(z) = f(z)$. By the definition of $\Delta_c^n f(z)$, one knows that $\Delta_c^n f(z) = \sum_{k=0}^n (-1)^{n-k}\binom{n}{k} f(z+kc)$ and $\sum_{k=0}^n (-1)^{n-k}\binom{n}{k} = 0$ for $k \in \mathbb{N}$. Hence, the exact difference $\Delta_c^n f(z)(\not\equiv 0)$ can be extended to a general form (see [15])

$$P_0(f) = a_0 f(z) + a_1 f(z+c) + \cdots + a_n f(z+nc), \ n \in \mathbb{N}^+,$$

where $z \in \mathbb{C}^m$, $c \in \mathbb{C}^m \backslash \{\mathbf{0}\}$ and $a_k \in \mathbb{C}$ are not all zero complex numbers satisfying $\sum_{k=0}^n a_k = 0$. Obviously, $P_0(f) = \Delta_c^n f(z)$ provided that $a_k = (-1)^{n-k}\binom{n}{k}$. In 2018, Deng et al. [6] studied the uniqueness problem for meromorphic functions sharing polynomial with their difference operators, and obtained the following result.

Theorem 1.1 ([6]). *Let $f(z)$ be a non-constant meromorphic function of finite order, and $a(z)$ be a non-constant polynomial in one complex variable. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ share $a(z)$, ∞ CM, then $f(z) = \Delta_c f(z)$ for all $z \in \mathbb{C}^1$.*

On the other hand, many authors have also investigated the case of meromorphic functions that share periodic small function with their difference operators or difference polynomial. In [8], Gao et al. considered the meromorphic function $f(z)$ with $\rho_2(f) < 1$ on $\mathbb{C}^1$, and proved that if $f(z)$ and its exact difference $\Delta_1^n f(z)(\not\equiv 0)$ share three distinct periodic small functions with period 1 CM, then $\Delta_1^n f(z) \equiv f(z)$. In [15], Liu and Zhang, using the value distribution theory for meromorphic functions in several complex variables, obtained a difference analogue of [9, Theorem 2.104] on meromorphic function $f(z)$ with its difference polynomial $P_0(f)$. We restated it as follows.

Theorem 1.2 ([15])**.** *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a non-constant meromorphic function of finite order, and let $b(z), d(z)(\not\equiv 0) \in S(f)$ be two periodic meromorphic functions with period c, where $z, c \in \mathbb{C}^m$. If $f(z) - b(z)$, $P_0(f) - d(z)$ and $\Delta_c P_0(f) - d(z)$ share $0, \infty$ CM, then $P_0(f) = \Delta_c P_0(f)$.*

It is known that logarithmic difference lemma plays an important role in the study of uniqueness problem. In 2020, Cao and Xu [2, Theorem 2.1] obtained a new version of the logarithmic difference lemma for meromorphic function $f(z)$ on $\mathbb{C}^m$ such that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0. \tag{1.1}$$

By simple analysis, one can deduce that the set of meromorphic functions satisfying the condition (1.1) consists of all meromorphic functions with hyper-order $\rho_2(f) < 1$ and some of hyper-order $\rho_2(f) = 1$. Motivated by this, one naturally asks whether the growth condition of meromorphic function of "finite order" in Theorem 1.1 and Theorem 1.2 can be relaxed to condition (1.1)?

The aim of this paper is to give a positive answer to the above question using different methods from that in [6, 15]. Here, we further consider a more general polynomial

$$P(f) = \sum_{k=0}^{n} m_k(z) f(z+kc), \ (n \in \mathbb{N}, n \geq 2), \tag{1.2}$$

where $m_k(z) \in S(f)$ are non-zero polynomials satisfying $\sum_{k=0}^{n} m_k(z) = 0$ for $z \in \mathbb{C}^m$ $(0 \leq k \leq n)$. It is obvious that $P(f)$ can be degenerate into $P_0(f)$ provided that $m_k(z) = a_k$ $(0 \leq k \leq n)$. Firstly, we get an extension of Theorem 1.1 for meromorphic functions satisfying (1.1) from one complex variable to several complex variables. We show our result as follows.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function for $z \in \mathbb{C}^m$ satisfying*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0,$$

and $a(z)$ be a non-constant polynomial on $\mathbb{C}^m$. If $f(z)$, $\Delta_c f(z)$ and $P(f)$ share $a(z)$, ∞ CM, then $\Delta_c f(z) = Af(z) + (1-A)a(z)$, where A is a non-zero constant. In particular, if $P(f) = \Delta_c^n f(z)$, then $f(z) \equiv \Delta_c f(z)$.

Let $m = 2$, $c = (1, 1)$ and $a(\neq 0)$ be a finite constant. We set $f(z) = e^{\pi i(z_1+z_2)} + a$. By simple calculation, one can deduce that $\Delta_c f(z) = P(f) = 0$, and f, $\Delta_c f(z)$, $P(f)$ share a, ∞ CM, but $\frac{\Delta_c f(z)-a}{f-a} = \frac{-a}{e^{\pi i(z_1+z_2)}}$ is not a constant. This implies that the condition "$a(z)$ is a non-constant polynomial" in Theorem 1.3 is necessary.

Remark 1.4. According to the result of Theorem 1.3, the type of function $f(z)$ can be roughly given. Consider the meromorphic solution $f(z)$ of the difference equation

$$\Delta_c f(z) = Af(z) + (1 - A)a(z)$$

on $\mathbb{C}^m$. If $m = 1$, it follows from the basic knowledge of difference equation that $f(z) = (1 + A)^{\frac{z}{c}}\kappa_1(z) + Q_1(z)$, where $\kappa_1(z)$ is a c-periodic function and $Q_1(z)$ is a polynomial such that $Q_1(z) = 0$ when $A = 1$. For the case of $m \geq 2$, we can also roughly know the expression of such function $f(z)$ under certain conditions. For example, $m = 2$ and $a(z) = \alpha z_1 + \beta z_2 + \gamma$ (α, β, γ are complex numbers), one can deduce that

$$f(z) = (1+A)^{\frac{b'z_1+b''z_2}{b'c_1+b''c_2}}\kappa_2(z) + \frac{A-1}{A}(\alpha z_1 + \beta z_2) + \frac{A-1}{A^2}(\alpha c_1 + \beta c_2) + \frac{A-1}{A}\gamma,$$

where $z = (z_1, z_2)$, $c = (c_1, c_2) \neq (0, 0)$, $b', b'' \in \mathbb{C}^1\backslash\{0\}$ and $\kappa_2(z)$ is a c-periodic function on $\mathbb{C}^2$.

The following examples show that the condition and conclusions of Theorem 1.3 can be satisfied in the higher dimension space $\mathbb{C}^m$ $(m \geq 2)$.

Example 1.5. Let $m = 2$, $n = 3$, $c = (0, 2)$, $z = (z_1, z_2)$. Suppose that $a(z) = \alpha z_1 + \beta z_2$, where $\alpha, \beta \in \mathbb{C}^1\backslash\{0\}$, and that $P(f) = -7f(z) + 15f(z + c) - 10f(z + 2c) + 2f(z + 3c)$, i.e., $P(f) \not\equiv \Delta_c^3 f(z)$. We can easily know that the meromorphic function

$$f(z) = 3^{\frac{z_1+z_2}{2}}\frac{e^{z_1+\pi i z_2}}{z_1} + \frac{\alpha z_1 + \beta z_2}{2} + \frac{\beta}{2}$$

is a solution of difference equation $f(z + c) - 3f(z) + a(z) = 0$. Obviously, $f(z) - a(z)$, $\Delta_c f(z) - a(z)$ and $P(f) - a(z)$ share 0, ∞ CM, and $\Delta_c f(z) \equiv 2f(z) - a(z)$.

Example 1.6. Let $m = 3$, $n = 2$, $c = (1, 1, 0)$, $z = (z_1, z_2, z_3)$. Set $f(z) = 2^{\frac{z_1+z_2}{2}}\frac{e^{\pi i(z_1+z_2)}}{z_3}$. Then $f(z + kc) = 2^k f(z)$ $(k = 0, 1, 2)$ and $P(f) = (m_0(z) + 2m_1(z) + 4m_2(z))f(z)$.

Case 1: Set $m_0(z) = 2(z_1 + z_2 + z_3) + 3$, $m_1(z) = -3(z_1 + z_2 + z_3) - 5$ and $m_2(z) = z_1 + z_2 + z_3 + 2$, i.e., $P(f) \not\equiv \Delta_c^2 f(z)$. Obviously, $f(z)$, $\Delta_c f(z)$ and $P(f)$ share non-constant polynomial $a(z)$, ∞ CM, and $\Delta_c f(z) \equiv f(z)$.

Case 2: Set $m_0(z) = 1$, $m_1(z) = -2$ and $m_2(z) = 1$, i.e., $P(f) \equiv \Delta_c^2 f(z)$. Then $f(z)$, $\Delta_c f(z)$, $\Delta_c^2 f(z)$ share polynomial $a(z)$, ∞ CM, and $f(z) \equiv \Delta_c f(z)$.

Next, we consider the uniqueness problem of meromorphic function $f(z)$ sharing periodic small functions with $P(f)$ and $\Delta_c P(f)$ under the condition (1.1), which is an accurate extension of Theorem 1.2.

Theorem 1.7. *Let $f(z)$ be a transcendental meromorphic function for $z \in \mathbb{C}^m$ satisfying*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0,$$

and let $b(z), d(z)(\not\equiv 0) \in S(f)$ be two periodic meromorphic functions with period c. If $f(z)-b(z)$, $P(f)-d(z)$ and $\Delta_c P(f)-d(z)$ share 0, ∞ CM*, then $P(f) \equiv \Delta_c P(f)$.*

Similarly, the structure of the function $f(z)$ can also be derived from the equation $P(f) = \Delta_c P(f)$ under some certain cases. The following examples show that the conclusion of Theorem 1.7 is reasonable.

Example 1.8. Set $m = 2$, $c = (\pi i, \pi i)$, $z = (z_1, z_2)$, $d(z) \equiv 3$, and $b(z)$ is a small periodic function with period c. Let $f(z) = e^{z_1+z_2} + b(z)$ and $n(\geq 2) \in \mathbb{N}^+$. Then $f(z+kc) \equiv f(z)$ for all $0 \leq k \leq n$. Obviously, $f(z)-b(z) = e^{z_1+z_2}$, $P(f)-d(z) = -3$, $\Delta_c P(f) - d(z) = -3$ share 0 CM, and $P(f) \equiv \Delta_c P(f)$.

Example 1.9. Set $m = 2$, $c = (0, \ln 2)$, $z = (z_1, z_2)$ and $f(z) = \frac{1}{z_1} e^{z_1+z_2}$. Let $n = 2$. Then $P(f) = (m_0(z) + 2m_1(z) + 4m_2(z)) \frac{1}{z_1} e^{z_1+z_2}$.

Case 1: Set $m_0(z) = 2z_1 - 5$, $m_1(z) = -3z_1 + 6$ and $m_2(z) = z_1 - 1$, i.e., $P(f) \not\equiv \Delta_c^2 f(z)$. If $b(z) \equiv 1$, $d(z) \equiv 3$, then $f(z) - b(z) = \frac{1}{z_1} e^{z_1+z_2} - 1$, $P(f) - d(z) = 3(\frac{1}{z_1} e^{z_1+z_2} - 1)$, $\Delta_c P(f) - d(z) = 3(\frac{1}{z_1} e^{z_1+z_2} - 1)$ share 0, ∞ CM. Thus, $P(f) \equiv \Delta_c P(f)$. However, $f(z) \not\equiv P(f)$ and $f(z) \not\equiv \Delta_c P(f)$.

Case 2: Assume that $b(z) \equiv d(z)(\not\equiv 0)$. Set $m_0(z) = 1$, $m_1(z) = -2$ and $m_2(z) = 1$, i.e., $P(f) \equiv \Delta_c^2 f(z)$. Then $f(z)$, $P(f)$, $\Delta_c P(f)$ share $b(z)$, ∞ CM, and $P(f) \equiv \Delta_c P(f) \equiv f(z)$.

If $b(z) \equiv d(z)(\not\equiv 0)$ and $m_k(z) = (-1)^{n-k}\binom{n}{k}$, we obtain following result.

Corollary 1.10. *Let $f(z)$ be a non-constant meromorphic function for $z \in \mathbb{C}^m$ satisfying* (1.1)*, and let $b(z)(\not\equiv 0) \in S(f)$ be a periodic meromorphic function with period c. If $f(z)$, $\Delta_c^n f(z)$, $\Delta_c^{n+1} f(z)$ share $b(z)$,* ∞ CM*, then $\Delta_c^n f(z) = \Delta_c^{n+1} f(z)$.*

The remainder of this paper is organized as follows. In Section 2, some basic notations and auxiliary lemmas on the value distribution theory on $\mathbb{C}^m$ are introduced, which are used frequency in the later proofs. The details of the proofs of our main results are showed in Sections 3 and 4, respectively.

2. Preliminary lemmas

We firstly recall some basis notions in several complex variables (see also [17, 18]). Set $\|z\|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2$ for $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$.

Let

$$S_m(r)=\{z\in\mathbb{C}^m:\|z\|=r\},\ B_m(r)=\{z\in\mathbb{C}^m:\|z\|\le r\}$$

for $r>0$. Introducing the differential operators $d=\partial+\overline{\partial}$, $d^c=\frac{\sqrt{-1}}{4\pi}(\overline{\partial}-\partial)$. This implies $dd^c=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. For $z\in\mathbb{C}^m\backslash\{\mathbf{0}\}$, write

$$\eta_m(z):=dd^c\|z\|^2,\ \sigma_m(z):=d^c\log\|z\|^2\wedge\left(dd^c\log\|z\|^2\right)^{m-1}.$$

Let $f(z)$ be a meromorphic function on $\mathbb{C}^m$ and $a\in\mathbb{P}^1$. If $f^{-1}(a)\neq\mathbb{C}^m$, we denote by ν^0_{f-a} the a-divisor of f, and set

$$n_f(r,a)=r^{2-2m}\int_{B_m(r)\cap\nu^0_{f-a}}\eta_m^{m-1}(z).$$

Then the counting function of ν^0_{f-a} is defined by

$$N_f(r,a)=\int_0^r[n_f(t,a)-n_f(0,a)]\frac{dt}{t}+n_f(0,a)\log r,$$

where $n_f(0,a)$ is the Lelong number of ν^0_{f-a} at the origin. If

$$\overline{\nu}^0_{f-a}=\min\{1,\nu^0_{f-a}\},$$

then we can also define the reduced counting function $\overline{N}_f(r,a)$. Usually, we denote by $N(r,\frac{1}{f-a})=N_f(r,a)$ for $a\in\mathbb{C}$ and $N(r,f)=N_f(r,\infty)$ for $a=\infty$, respectively. The proximity function of f is defined by

$$m_f(r,a)=\begin{cases}\displaystyle\int_{S_m(r)}\log^+\mid f(z)\mid\sigma_m(z), & \text{if } a=\infty,\\ \displaystyle\int_{S_m(r)}\log^+\frac{1}{\mid f(z)-a\mid}\sigma_m(z), & \text{if } a\neq\infty,\end{cases}$$

where $\log^+x=\max\{\log x,0\}$. Similar, we usually replace the notations $m_f(r,a)$ by $m(r,\frac{1}{f-a})$ for $a\in\mathbb{C}$ and $m(r,f)$ for $a=\infty$. Then Nevanlinna characteristic function of f is defined as $T(r,f)=m_f(r,\infty)+N_f(r,\infty)$. A meromorphic function $f(z)$ on $\mathbb{C}^m$ is called transcendental provided that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

When $a\neq\infty$, the first main theorem can be stated as $T(r,f)=m_f(r,a)+N_f(r,a)+O(1)$. To prove our main theorems in this paper, we need the following lemmas.

Lemma 2.1 ([2]). *Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}^m$, and let $c\in\mathbb{C}^m\backslash\{\mathbf{0}\}$. If $\limsup\limits_{r\to\infty}\frac{\log T(r,f)}{r}=0$, then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=o(T(r,f))$$

and

$$T(r, f(z+c)) = T(r, f) + o(T(r, f)),\ N(r, f(z+c)) = N(r, f) + o(N(r, f))$$

holds for all $r \notin E$*, where* E *is a set with zero upper density measure, i.e.,* $\overline{dens}E = \limsup_{r\to\infty} \int_{E\cap[1,r]} \frac{1}{r} dt = 0$.

Lemma 2.2 ([1, Corollary 4.5]). *Let* $f_1(z), f_2(z), \ldots, f_n(z)$ *be* n *non-zero meromorphic functions on* $\mathbb{C}^m$*, and* $g_1(z), \ldots, g_n(z)$ *be* n *entire functions on* $\mathbb{C}^m$ *satisfying*

$$\sum_{i=1}^{n} f_i(z) e^{g_i(z)} \equiv 0.$$

If for all $1 \le i \le n$

$$T(r, f_i) = o(T(r, e^{g_j - g_k})),\ (1 \le j \ne k \le n),$$

then $f_i(z) \equiv 0$ *for* $1 \le i \le n$.

Lemma 2.3 ([15, Lemma 2.7]). *Let* $f(z)$ *be a polynomial in* $z \in \mathbb{C}^m$*. If* $f(z)$ *is of degree* $n(\ge 1)$*, then* $\deg(f(z+c) - f(z)) < n$ *holds for any given* $c = (c_1, c_2, \ldots, c_m) \in \mathbb{C}^m$.

Remark 2.4. Let $f(z)$ be a non-constant polynomial on $\mathbb{C}^m$ and $P(f)$ be defined as in (1.2). Then $\deg P(f(z)) < \deg f(z)$ holds for any given $c = (c_1, c_2, \ldots, c_m)$.

Proof. Based on a similar argument as to that in [15, Lemma 2.7], we assume that $f(z)$ is a polynomial of degree $n(\ge 1)$. Set

$$f(z) = \sum_{|I|=n} a_I (z_1)^{i_1} (z_2)^{i_2} \cdots (z_m)^{i_m} + \sum_{|I|=0}^{n-1} b_I (z_1)^{i_1} (z_2)^{i_2} \cdots (z_m)^{i_m},$$

where $a_I(\not\equiv 0)$, b_I are complex numbers and $I = (i_1, i_2, \ldots, i_m) \in \mathbb{N}^m$ satisfies $|I| = i_1 + i_2 + \cdots + i_m$. For $c = (c_1, c_2, \ldots, c_m) \in \mathbb{C}^m$ and $k \in \mathbb{N}$, one has

$$\begin{aligned} f(z+kc) = & \sum_{|I|=n} a_I (z_1 + kc_1)^{i_1} (z_2 + kc_2)^{i_2} \cdots (z_m + kc_m)^{i_m} \\ & + \sum_{|I|=0}^{n-1} b_I (z_1 + kc_1)^{i_1} (z_2 + kc_2)^{i_2} \cdots (z_m + kc_m)^{i_m} \\ = & \sum_{|I|=n} a_I (z_1)^{i_1} (z_2)^{i_2} \cdots (z_m)^{i_m} + D_{n-1,k}(z), \end{aligned}$$

where $D_{n-1,k}(z)$ is of degree at most $n-1$. Noting that $\sum_{k=0}^{n} m_k(z) = 0$, it follows from (1.2) that

$$P(f) = \sum_{k=0}^{n} m_k(z) \left(\sum_{|I|=n} a_I (z_1)^{i_1} (z_2)^{i_2} \cdots (z_m)^{i_m} \right) + \sum_{k=0}^{n} m_k(z) D_{n-1,k}(z)$$

$$= \sum_{k=0}^{n} m_k(z) D_{n-1,k}(z).$$

Since $m_k(z) \in S(f)$ and $f(z)$ is a non-constant polynomial, then m_k ($0 \le k \le n$) are non-zero constants. It follows that

$$\deg(P(f)) = \max_{0 \le k \le n} \deg\{D_{n-1,k}(z)\} \le n - 1.$$

This completes the proof. □

Lemma 2.5 ([19, Lemma 5]). *Let $f(z)$ be a non-constant meromorphic function for $z \in \mathbb{C}^m$ and $a_j(z)$ ($j = 1, 2, \ldots, q$) be distinct small functions with respect to f. If $q \ge 3$, then*

$$\frac{q}{3} T(r, f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + o(T(r, f))$$

holds for all $r \in [0, +\infty)$ outside a Borel subset F of the interval $[0, +\infty)$ with $\int_F dr < +\infty$.

For brevity, throughout this paper, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ holds for all sufficiently large r at most outside two possible exceptional sets of zero upper density and finite linear measure, which may vary in each appearance.

3. The proof of Theorem 1.3

Since $f(z)$, $\Delta_c f(z)$ and $P(f)$ share $a(z)$, ∞ CM, then there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$\frac{P(f) - a(z)}{f(z) - a(z)} = e^{\alpha(z)}, \quad \frac{\Delta_c f(z) - a(z)}{f(z) - a(z)} = e^{\beta(z)}. \tag{3.1}$$

Set $F(z) = f(z) - a(z)$. Owing to $a(z) \in S(f)$, we have $T(r, f) = T(r, F) + S(r, f)$ and $a(z) \in S(F)$. By the definitions of $P(f)$ and $\Delta_c f(z)$, one has

$$\Delta_c f(z) = \Delta_c F(z) + \Delta_c a(z), \quad P(f(z)) = P(F(z)) + P(a(z)). \tag{3.2}$$

It follows from (3.1) that

$$\frac{P(F) + P(a) - a(z)}{F(z)} = e^{\alpha(z)}, \quad \frac{\Delta_c F(z) + \Delta_c a(z) - a(z)}{F(z)} = e^{\beta(z)}.$$

Noting that $P(a) - a(z) \not\equiv 0$ and $\Delta_c a(z) - a(z) \not\equiv 0$. We set

$$\begin{aligned} \Phi(z) &:= (a(z) - P(a)) e^{\beta(z)} - (a(z) - \Delta_c a(z)) e^{\alpha(z)} \\ &= \frac{(a(z) - P(a)) \Delta_c F(z) - (a(z) - \Delta_c a(z)) P(F)}{F(z)}. \end{aligned} \tag{3.3}$$

Since $a(z) \in S(F)$, together with (3.3), (1.2) and Lemma 2.1, one can obtain that $T(r, \Phi) = S(r, F)$. Next, the fact that $T(r, e^{\alpha}) = S(r, f)$ and $T(r, e^{\beta}) = S(r, f)$ will be proved. We consider two cases.

• $\Phi(z) \equiv 0$. This means $(a(z) - P(a))\Delta_c F(z) = (a(z) - \Delta_c a(z))P(F)$ for any $z \in \mathbb{C}^m$. By (3.2) and some calculations, one can deduce

$$(a(z) - P(a))(\Delta_c f(z) - a(z)) = (a(z) - \Delta_c a(z))(P(f) - a(z)).$$

Further, it follows from (3.1) and the above equality that

$$\frac{P(f) - a(z)}{\Delta_c f(z) - a(z)} = e^{\alpha(z)-\beta(z)} = \frac{a(z) - P(a)}{a(z) - \Delta_c a(z)}.$$

If $e^{\alpha(z)-\beta(z)}$ is not a constant, then it must be transcendental, which is impossible for $a(z)$ is a polynomial. Therefore, we may suppose that there exists a non-zero constant C such that $e^{\alpha(z)-\beta(z)} = C$. It follows that $\frac{a(z)-P(a)}{a(z)-\Delta_c a(z)} = C$. By Lemma 2.3 and Remark 2.4, we know that $\deg \Delta_c a(z) < \deg a(z)$ and $\deg P(a) < \deg a(z)$. Then, one can deduce that $C = 1$ and $a(z)$ is a constant polynomial, which is a contradiction.

• $\Phi(z) \not\equiv 0$. It follows from (3.3) that

$$(a(z) - P(a))\frac{e^{\beta(z)}}{\Phi(z)} = 1 + (a(z) - \Delta_c a(z))\frac{e^{\alpha(z)}}{\Phi(z)}.$$

By Lemma 2.5 and the fact that $T(r, \Phi) = S(r, F)$, $a(z) \in S(F)$, one has

$$\begin{aligned} T\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) &\leq \overline{N}\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) + \overline{N}\left(r, \frac{\Phi}{(a - P(a))e^{\beta}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{(a - P(a))\frac{e^{\beta}}{\Phi} - 1}\right) + S\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) \\ &\leq \overline{N}\left(r, \frac{\Phi}{(a - \Delta_c a)e^{\alpha}}\right) + S\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) + S(r, F) \\ &= S\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) + S(r, F). \end{aligned}$$

This implies that $T(r, (a - P(a))\frac{e^{\beta}}{\Phi}) = S(r, F)$ and

$$\begin{aligned} T(r, e^{\beta}) &\leq T\left(r, (a - P(a))\frac{e^{\beta}}{\Phi}\right) + T\left(r, \frac{\Phi}{a - P(a)}\right) \\ &= S(r, F). \end{aligned}$$

By (3.3), we can also get $T(r, e^{\alpha}) = S(r, F)$. Owing to $T(r, f) = T(r, F) + S(r, f)$, then $T(r, e^{\alpha}) = S(r, f)$, $T(r, e^{\beta}) = S(r, f)$.

To complete the proof of Theorem 1.3, we first consider the case of $\beta(z)$ is a non-constant entire function. Using the second equation in (3.1), we have

$$\begin{aligned} f(z + c) &= \Delta_c f(z) + f(z) \\ &= (e^{\beta(z)} + 1)f(z) + (1 - e^{\beta(z)})a(z). \end{aligned}$$

Define $u_1(z) = e^{\beta(z)} + 1$, $v_1(z) = (1 - e^{\beta(z)})a(z)$. Then

$$f(z + c) = u_1(z)f(z) + v_1(z),$$

$$f(z+2c) = u_1(z+c)u_1(z)f(z) + u_1(z+c)v_1(z) + v_1(z+c),$$
$$\vdots$$

For $k \in \mathbb{N}^+$, using mathematical induction, one can deduce

$$f(z+kc) = u_k(z)f(z) + v_k(z),$$

where

$$u_k(z) = \prod_{i=0}^{k-1}(e^{\beta(z+ic)}+1),\ v_k(z) = \sum_{i=0}^{k-1} v_1(z+ic) \prod_{j=i+1}^{k-1} u_1(z+jc).$$

In particular, if $i > k-2$, then $\prod_{j=i+1}^{k-1} u_1(z+jc) = 1$. For $k=0$, we set $u_0(z) = 1$, $v_0(z) = 0$. By the definition of $P(f)$, we have

$$\begin{aligned} P(f) &= \sum_{k=0}^{n} m_k(z)f(z+kc) \\ &= \sum_{k=0}^{n} m_k(z)u_k(z)f(z) + \sum_{k=0}^{n} m_k(z)v_k(z) \\ &= U_n(z)f(z) + V_n(z), \end{aligned} \tag{3.4}$$

where $U_n(z) = \sum_{k=0}^{n} m_k(z)u_k(z)$ and $V_n(z) = \sum_{k=0}^{n} m_k(z)v_k(z)$.

Set $w(z+ic) = \beta(z+ic) - \beta(z)$ for any $i \in \mathbb{N}$, it can be deduced that

$$\begin{aligned} U_n(z) &= m_0 + m_1(1+e^{\beta(z)}) + m_2(1+e^{\beta(z)})(1+e^{\beta(z+c)}) + \cdots \\ &\quad + m_n(1+e^{\beta(z)})(1+e^{\beta(z+c)})\cdots(1+e^{\beta(z+(n-1)c)}) \\ &= m_0 + m_1(1+e^{\beta(z)}) + m_2(1+e^{\beta(z)})(1+e^{w(z+c)}e^{\beta(z)}) + \cdots \\ &\quad + m_n(1+e^{\beta(z)})(1+e^{w(z+c)}e^{\beta(z)})\cdots(1+e^{w(z+(n-1)c)}e^{\beta(z)}). \end{aligned} \tag{3.5}$$

Noting that $T(r, e^{\beta(z)}) = S(r,f)$, then $\limsup_{r\to\infty} \frac{\log T(r,e^{\beta(z)})}{r} \le \limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$. It follows from Lemma 2.1 that for $0 \le i \le n-1$

$$m(r, e^{w(z+ic)}) = m\left(r, \frac{e^{\beta(z+ic)}}{e^{\beta(z)}}\right) = S(r, e^{\beta(z)}).$$

And since $m_k(z)$ $(0 \le k \le n)$ are polynomials, there are $T(r, m_k) = S(r, e^{\beta(z)})$. Thus, (3.5) can be rewritten as follows:

$$U_n(z) = b_0(z) + b_1(z)e^{\beta(z)} + b_2(z)e^{2\beta(z)} + \cdots + b_n(z)e^{n\beta(z)}, \tag{3.6}$$

where $T(r, b_j(z)) = S(r, e^{\beta(z)})$ for all $0 \le j \le n$. Here, $b_0(z) = m_0 + m_1 + \cdots + m_n = 0$ and $b_n(z) = m_n(z)e^{w(z+c)}\cdots e^{w(z+(n-1)c)} \not\equiv 0$.

Noting that $a(z) \in S(f)$. By the definition of $V_n(z)$ and (3.6), Lemma 2.1, one knows

$$T(r, V_n(z)) + T(r, U_n(z)) \le O(T(r, e^{\beta(z)})) + T(r, a(z)) + S(r, e^{\beta(z)})$$

$$= S(r,f). \tag{3.7}$$

Together with the first equation in (3.1) and (3.4), we have

$$\left(U_n(z) - e^{\alpha(z)}\right) f(z) = a(z)\left(1 - e^{\alpha(z)}\right) - V_n(z).$$

If $U_n(z) - e^{\alpha(z)} \not\equiv 0$, it can be deduced from (3.7) and above equation that

$$\begin{aligned}T(r,f) &\leq T(r,V_n(z)) + T(r,U_n(z)) + 2T(r,e^{\alpha(z)}) + S(r,f)\\ &= 2T(r,e^{\alpha(z)}) + S(r,f) = S(r,f),\end{aligned}$$

which is impossible. Hence, $U_n(z) - e^{\alpha(z)} \equiv 0$, i.e.,

$$b_1(z)e^{\beta(z)} + b_2(z)e^{2\beta(z)} + \cdots + b_n(z)e^{n\beta(z)} \equiv e^{\alpha(z)}. \tag{3.8}$$

Since $\beta(z)$ is a non-constant entire function, then three cases are needed to be discussed.

• $T(r,e^{\alpha}) = S(r,e^{\beta})$. Applying Lemma 2.2 to (3.8), it can be seen that $e^{\alpha(z)} \equiv 0$ and $b_j(z) \equiv 0$ $(1 \leq j \leq n)$, a contradiction.

• $T(r,e^{\alpha}) = O(T(r,e^{\beta}))$ and $T(r,e^{\beta}) = O(T(r,e^{\alpha}))$. It follows from (3.8) that $T(r,e^{\alpha}) = nT(r,e^{\beta}) + S(r,e^{\beta})$. This implies that $S(r,e^{\alpha}) = S(r,e^{\beta})$ and $\alpha(z) - \beta(z)$ is not a constant for $n \geq 2$. From (3.8), we have

$$b_1(z) + b_2(z)e^{\beta(z)} + \cdots + b_n(z)e^{(n-1)\beta(z)} \equiv e^{\alpha(z)-\beta(z)}.$$

By Lemma 2.5, one has

$$\begin{aligned}(n-1)T(r,e^{\beta}) &= T(r,e^{\alpha-\beta}) + S(r,e^{\beta})\\ &\leq \overline{N}(r,e^{\alpha-\beta}) + \overline{N}(r,\frac{1}{e^{\alpha-\beta}}) + \overline{N}(r,\frac{1}{e^{\alpha-\beta}-b_1}) + S(r,e^{\beta})\\ &= \overline{N}\left(r,\frac{1}{e^{\beta}(b_2 + b_3e^{\beta} + \cdots + e^{(n-2)\beta})}\right) + S(r,e^{\beta})\\ &\leq (n-2)T(r,e^{\beta}) + S(r,e^{\beta}),\end{aligned}$$

which is impossible.

• $T(r,e^{\beta}) = S(r,e^{\alpha})$. It follows from $T(r,b_j(z)) = S(r,e^{\beta})$ $(1 \leq j \leq n)$ and (3.8) that

$$\begin{aligned}T(r,e^{\alpha}) &= T(r,b_1e^{\beta} + b_2e^{2\beta} + \cdots + b_ne^{n\beta})\\ &\leq nT(r,e^{\beta}) + S(r,e^{\beta}) = S(r,e^{\alpha}),\end{aligned}$$

a contradiction can also be derived.

Hence, $\beta(z)$ is a constant for $z \in \mathbb{C}^m$. That is, there exists a non-zero constant $A = e^{\beta}$ satisfying $\frac{\Delta_c f(z) - a(z)}{f(z) - a(z)} = A$. The first conclusion of Theorem 1.3 holds.

Furthermore, we consider the case of $P(f) = \Delta_c^n f(z)$ $(n \geq 2)$. By the definition of $\Delta_c^n f(z)$ and the fact that $\Delta_c f(z) = Af(z) + (1-A)a(z)$, one has

$$\Delta_c^2 f = A\Delta_c f + (1-A)\Delta_c a(z) = A^2 f + (1-A)[a(z)A + \Delta_c a(z)].$$

By similar calculation, it is easy get for $n \geq 2$

$$\begin{aligned} \Delta_c^n f(z) &= A^{n-1}\Delta_c f(z) + (1-A)T_n'(z) \\ &= A^n f(z) + (1-A)[A^{n-1}a(z) + T_n'(z)] \\ &= A^n f(z) + (1-A)T_n(z), \end{aligned} \tag{3.9}$$

where $T_n'(z) = \sum_{i=1}^{n-1} A^{n-1-i}\Delta_c^i a(z)$ and $T_n(z) = \sum_{i=0}^{n-1} A^{n-1-i}\Delta_c^i a(z)$.

By (3.9) and the first equality of (3.1), we get

$$(e^{\alpha(z)} - A^n)f(z) = (1-A)T_n(z) - (1-e^{\alpha(z)})a(z).$$

Owing to $a(z) \in S(f)$ and $T(r, e^\alpha) = S(r, f)$, one can deduce that $e^{\alpha(z)} \equiv A^n$. In fact, if $e^{\alpha(z)} - A^n \not\equiv 0$, it follows from Lemma 2.1 and Lemma 2.3 that

$$T(r, f) = T\left(r, \frac{(1-A)T_n(z) - (1-e^{\alpha(z)})a(z)}{e^{\alpha(z)} - A^n}\right) = S(r, f),$$

a contradiction. Then, by the first equation of (3.1) again and $e^{\alpha(z)} \equiv A^n$, one has

$$\begin{aligned} \Delta_c^n f(z) &= f(z)e^\alpha - a(z)e^\alpha + a(z) \\ &= A^n f(z) + (1-A^n)a(z) \\ &= A^{n-1}\Delta_c f(z) + (1-A^{n-1})a(z). \end{aligned} \tag{3.10}$$

Assume that $\Delta_c^n f(z) \not\equiv \Delta_c f(z)$, which means $A^{n-1} \neq 1$. Together with the first equality of (3.9) and (3.10), we have

$$(1-A)\sum_{i=1}^{n-1} A^{n-1-i}\Delta_c^i a(z) = (1-A)T_n'(z) = (1-A^{n-1})a(z). \tag{3.11}$$

Note here that $a(z)$ is a non-constant polynomial on $\mathbb{C}^m$. Using Lemma 2.3, one knows that $\deg \Delta_c^i a(z) < \deg a(z)$ for $i \geq 1$, which is a contradiction.

Hence $\Delta_c^n f(z) \equiv \Delta_c f(z)$ and $A^{n-1} = 1$. Next, we claim that $A = 1$. Otherwise, from (3.11) we obtain

$$T_n'(z) = \sum_{i=1}^{n-1} A^{n-1-i}\Delta_c^i a(z) \equiv 0,$$

which contradicts the fact that $a(z)$ is a non-constant polynomial. That we complete the proof of Theorem 1.3.

4. The proof of Theorem 1.7

Assume to the contrary that $P(f) \neq \Delta_c P(f)$ for some $z \in \mathbb{C}^m$. Obviously, $P(f) \not\equiv 0$. By assumption, there also exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$\frac{P(f) - d(z)}{f(z) - b(z)} = e^{\alpha(z)}, \quad \frac{\Delta_c P(f) - d(z)}{f(z) - b(z)} = e^{\beta(z)}. \tag{4.1}$$

It follows that

$$P(f(z)) = d(z) + e^{\alpha(z)}(f(z) - b(z)) \tag{4.2}$$

and

$$\Delta_c P(f(z)) = d(z) + e^{\beta(z)}(f(z) - b(z))$$

for $z, c \in \mathbb{C}^m$. Note here that $b(z), d(z)(\not\equiv 0) \in S(f)$ are two periodic meromorphic functions with period c. By the definition of $\Delta_c P(f(z))$ and (4.2), one has

$$\begin{aligned} d(z) + e^{\alpha(z+c)}(f(z+c) - b(z)) &= P(f(z+c)) \\ &= \Delta_c P(f(z)) + P(f(z)) \\ &= 2d(z) + (e^{\alpha(z)} + e^{\beta(z)})(f(z) - b(z)). \end{aligned}$$

Therefore,

$$f(z+c) - b(z) = M(z)(f(z) - b(z)) + h(z), \tag{4.3}$$

where $M(z) = (e^{\alpha(z)} + e^{\beta(z)})e^{-\alpha(z+c)}$ and $h(z) = d(z)e^{-\alpha(z+c)}$.

Next we claim that $T(r, e^{\alpha}) = S(r, f)$ and $T(r, e^{\beta}) = S(r, f)$. To this end, we set $\Psi(z) := e^{\alpha(z)} - e^{\beta(z)}(\not\equiv 0)$. Since $b(z) \in S(f)$ with period c and $\sum_{k=0}^{n} m_k(z) = 0$, it follows from (4.1) and Lemma 2.1 that

$$\begin{aligned} T(r, \Psi) &= m(r, e^{\alpha(z)} - e^{\beta(z)}) \\ &= m\left(r, \frac{P(f) - \Delta_c P(f)}{f(z) - b(z)}\right) \\ &\le m\left(r, \frac{P(f)}{f(z) - b(z)}\right) + m\left(r, \frac{\Delta_c P(f)}{f(z) - b(z)}\right) \\ &= m\left(r, \sum_{k=0}^{n} m_k \frac{f(z+kc) - b(z+kc)}{f(z) - b(z)}\right) + m\left(r, \frac{\Delta_c P(f)}{P(f)} \cdot \frac{P(f)}{f(z) - b(z)}\right) \\ &= S(r, f) + S(r, P(f)). \end{aligned}$$

In addition, we know

$$T(r, P(f)) \le \sum_{k=0}^{n} \{T(r, f(z+kc)) + T(r, m_k)\} = O(T(r, f)).$$

This implies that $T(r, \Psi) = S(r, f)$. Further, by Lemma 2.5, one has

$$\begin{aligned} T(r, e^{\alpha}) &\le T\left(r, \frac{e^{\alpha}}{\Psi}\right) + T(r, \Psi) \\ &\le \overline{N}\left(r, \frac{e^{\alpha}}{\Psi}\right) + \overline{N}\left(r, \frac{\Psi}{e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\alpha}}{\Psi} - 1}\right) + S\left(r, \frac{e^{\alpha}}{\Psi}\right) + S(r, f) \\ &\le \overline{N}\left(r, \frac{\Psi}{e^{\beta}}\right) + S(r, f) + S(r, e^{\alpha}) = S(r, f) + S(r, e^{\alpha}). \end{aligned}$$

By the same argument, we can also deduce that $T(r, e^{\beta}) \leq S(r, f) + S(r, e^{\beta})$. It follows that $T(r, e^{\alpha}) = S(r, f)$ and $T(r, e^{\beta}) = S(r, f)$. This means that

$$T(r, M(z)) = S(r, f),\ T(r, h(z)) = S(r, f). \tag{4.4}$$

In the following, we use the short notations for $c \in \mathbb{C}^m \backslash \{\mathbf{0}\}$ and $f : \mathbb{C}^m \to \mathbb{P}^1$ as follows:

$$f(z) := f_0(z), f(z+c) := f_c(z), \ldots, f(z+kc) := f_{kc}(z),\ k \in \mathbb{N}.$$

It follows from (4.3) that

$$\begin{aligned}
f_c(z) - b(z) &= M_0(z)(f_0(z) - b(z)) + h_0(z),\\
f_{2c}(z) - b(z) &= M_c(z)(f_c(z) - b(z)) + h_c(z)\\
&= M_c(z)M_0(z)(f_0(z) - b(z)) + M_c(z)h_0(z) + h_c(z),\\
f_{3c}(z) - b(z) &= M_{2c}(z)(f_{2c}(z) - b(z)) + h_{2c}(z)\\
&= M_{2c}(z)M_c(z)M_0(z)(f_0(z) - b(z)) + M_{2c}(z)M_c(z)h_0(z)\\
&\quad + M_{2c}(z)h_c(z) + h_{2c}(z),\\
&\ \vdots
\end{aligned}$$

By the same method, one can deduce for $k \in \mathbb{N}^+$

$$\begin{aligned}
f_{kc}(z) - b(z) = &\left(\prod_{i=0}^{k-1} M_{ic}(z)\right)(f_0(z) - b(z)) + h_0(z)M_c(z)\cdots M_{(k-1)c}(z)\\
&+ h_c(z)M_{2c}(z)\cdots M_{(k-1)c}(z) + \cdots + h_{(k-1)c}(z).
\end{aligned} \tag{4.5}$$

For brevity, we set

$$\begin{aligned}
\gamma_k &= \prod_{i=0}^{k-1} M_{ic}(z) = M_0(z)M_c(z)\cdots M_{(k-1)c}(z),\\
\zeta_k &= h_0(z)M_c(z)\cdots M_{(k-1)c}(z) + \cdots + h_{(k-1)c}(z).
\end{aligned}$$

Then $f_{kc}(z) - b(z) = \gamma_k(f_0(z) - b(z)) + \zeta_k$. In particular, we define $\gamma_0 = 1$, $\zeta_0 = 0$. Using (4.4) and Lemma 2.1, one knows

$$T(r, \gamma_k) = S(r, f),\ T(r, \zeta_k) = S(r, f). \tag{4.6}$$

On the other hand, it follows from (1.2) and (4.5) that

$$\begin{aligned}
P(f) - d(z) &= \sum_{k=0}^{n} m_k(z)(f_{kc}(z) - b(z)) - d(z)\\
&= (f_0(z) - b(z))\sum_{k=0}^{n} m_k\gamma_k + \sum_{k=0}^{n} m_k\zeta_k - d(z).
\end{aligned} \tag{4.7}$$

Noting that $f(z)-b(z)$ and $P(f)-d(z)$ share 0 CM, which implies that the zeros of $f(z) - b(z)$ must be the zeros of $P(f) - d(z) - (f_0(z) - b(z)) \sum_{k=0}^{n} m_k(z)\gamma_k$.

If $\sum_{k=0}^{n} m_k\zeta_k - d(z) \not\equiv 0$, by (4.6) and (4.7), we get

$$N\left(r, \frac{1}{f-b}\right) \le N\left(r, \frac{1}{\sum_{k=0}^{n} m_k\zeta_k - d}\right) = S(r,f).$$

In addition, owing to $d(z) \not\equiv 0$, by means of (4.1) and Lemma 2.1 we always have

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &\le m\left(r, \frac{d}{f-b}\right) + m\left(r, \frac{1}{d}\right) \\ &\le m\left(r, \frac{P(f)}{f-b}\right) + m\left(r, e^{\alpha}\right) + S(r,f) = S(r,f). \end{aligned}$$

This implies that $T(r,f) = T(r, \frac{1}{f-b}) + O(1) = S(r,f)$, which is impossible. So $\sum_{k=0}^{n} m_k\zeta_k - d(z) \equiv 0$. It follows from (4.7) that

$$\sum_{k=0}^{n} m_k(z)\gamma_k = \frac{P(f) - d(z)}{f(z) - b(z)} = e^{\alpha(z)},\ n \ge 2. \tag{4.8}$$

Let $\omega(z) = \beta(z) - \alpha(z)$. Then $T(r,e^{\omega}) \le T(r,e^{\alpha}) + T(r,e^{\beta}) = S(r,f)$ and $\limsup\limits_{r\to\infty} \frac{\log T(r,e^{\omega})}{r} = 0$. Noting that for all $0 \le i \le k-1$

$$M_{ic}(z) = e^{\alpha_{ic}(z) - \alpha_{(i+1)c}(z)}[1 + e^{\omega(z)} \cdot e^{\omega_{ic}(z) - \omega(z)}] = \eta_i + \theta_i e^{\omega(z)}, \tag{4.9}$$

where $\eta_i = e^{\alpha_{ic}(z) - \alpha_{(i+1)c}(z)}$ and $\theta_i = \eta_i e^{\omega_{ic}(z) - \omega(z)}$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,\eta_i) &= m(r,\eta_i) = S(r,e^{\alpha}), \\ T(r,\theta_i) &= m(r,\theta_i) \le S(r,e^{\alpha}) + S(r,e^{\omega}). \end{aligned} \tag{4.10}$$

Further, by the definition of γ_k and (4.8), we have

$$b_0(z) - e^{\alpha} + b_1(z)e^{\omega} + b_2(z)e^{2\omega} + \cdots + b_n(z)e^{n\omega} = 0, \tag{4.11}$$

where $b_j(z)$ are functions in m_k, η_i, θ_i for all $0 \le j \le n$. In particular, one can know $b_0(z) = m_0 + m_1\eta_0 + m_2\eta_0\eta_1 + \cdots + m_n\eta_0\eta_1\cdots\eta_{n-1}$ and $b_n(z) = m_n\theta_0\theta_1\cdots\theta_{n-1} \not\equiv 0$.

Owing to $\omega(z) = \beta(z) - \alpha(z) \not\equiv 0$, then $e^{\omega(z)}$ and $e^{\alpha(z)}$ must satisfy one of the following: (1) $e^{\omega(z)}$ and $e^{\alpha(z)}$ are constants; (2) $e^{\omega(z)}$ and $e^{\alpha(z)}$ are non-constant entire functions; (3) either $e^{\omega(z)}$ or $e^{\alpha(z)}$ is a non-constant entire function. From the perspective of characteristic function, the above three cases will lead to $T(r,e^{\alpha}) = S(r,e^{\omega})$, $\frac{T(r,e^{\alpha})}{T(r,e^{\omega})} \to O(1)(>0)$ or $T(r,e^{\omega}) = S(r,e^{\alpha})$ as $r \to \infty$. Below we discuss these three cases separately.

- Assume $T(r,e^{\alpha}) = S(r,e^{\omega})$. Since $e^{\omega(z)}$ is a transcendental entire function and $m_k(z)$ $(0 \le k \le n)$ are polynomials, then $T(r,m_k) = S(r,e^{\omega})$. By the definition of $b_j(z)$, one can deduce that $T(r,b_j) = S(r,e^{\omega})$ for all $0 \le j \le n$. Applying Lemma 2.2 and (4.11), we obtain $b_0(z) - e^{\alpha(z)} \equiv 0$ and $b_j(z) \equiv 0$ $(1 \le j \le n)$, which contradict the fact that $b_n(z) \not\equiv 0$.

• $T(r,e^\alpha)=O(T(r,e^\omega))$. Noting that $P(f)\not\equiv\Delta_c P(f)$. If $e^{\alpha(z)}$ and $e^{\omega(z)}$ are constants for $z\in\mathbb{C}^m$, then there exist two distinct constants C_1, C_2 satisfying $e^\alpha=C_1$ and $e^\omega=\frac{e^\beta}{e^\alpha}=\frac{C_2}{C_1}$. In view of (4.3), one can deduce

$$M_{ic}(z)=M=1+\frac{C_2}{C_1},\ h_{ic}(z)=h=\frac{d(z)}{C_1} \tag{4.12}$$

for all $0\le i\le k-1$. By the definitions of γ_k and ζ_k for $0\le k\le n$

$$\gamma_k=M^k=(1+C_2/C_1)^k,\ \zeta_k=h(M^{k-1}+\cdots+M+1)=\frac{h(1-M^k)}{1-M}.$$

Since $d(z)\not\equiv 0$ and $\sum_{k=0}^n m_k(z)\zeta_k\equiv d(z)$, it follows from (4.8) and (4.12) that

$$\begin{aligned}C_1&=\sum_{k=0}^n m_k(z)\gamma_k=\sum_{k=0}^n m_k(z)M^k,\\ C_2&=\frac{C_2}{d}\cdot d=\frac{C_2}{d}\sum_{k=0}^n m_k(z)\frac{h(1-M^k)}{1-M}=\sum_{k=0}^n m_k(z)M^k,\end{aligned}$$

which is impossible.

If $e^{\alpha(z)}$ and $e^{\omega(z)}$ are non-constant entire functions, then $S(r,e^\alpha)=S(r,e^\omega)$. It follows from (4.10) and (4.11) that $T(r,b_j)=S(r,e^\omega)$ for $0\le j\le n$. Next, we consider two conditions for $b_0(z)\not\equiv 0$ and $b_0(z)\equiv 0$. Assume that $b_0(z)\not\equiv 0$. By (4.11) and Lemma 2.5, we have

$$\begin{aligned}nT(r,e^\omega)&=T(r,e^\alpha)+S(r,e^\alpha)\\ &\le\overline{N}(r,e^\alpha)+\overline{N}\left(r,\frac{1}{e^\alpha}\right)+\overline{N}\left(r,\frac{1}{e^\alpha-b_0}\right)+S(r,e^\alpha)\\ &=\overline{N}\left(r,\frac{1}{e^\omega(b_1+b_2e^\omega+\cdots+b_ne^{(n-1)\omega})}\right)+S(r,e^\omega)\\ &\le(n-1)T(r,e^\omega)+S(r,e^\omega),\end{aligned}$$

which is a contradiction. Suppose that $b_0(z)\equiv 0$. Then we can rewritten (4.11) as follows:

$$b_1+b_2e^\omega+\cdots+b_ne^{(n-1)\omega}=e^{\alpha-\omega},\ n\ge 2.$$

If $\alpha(z)-\omega(z)$ is a constant, then applying Lemma 2.2, one can obtain $b_1\equiv e^{\alpha-\omega}$ and $b_j\equiv 0$ for $2\le j\le n$, which yields a contradiction for $b_n(z)\not\equiv 0$. Now let's consider that $\alpha(z)-\omega(z)$ is not a constant. By Lemma 2.5, one has

$$\begin{aligned}(n-1)T(r,e^\omega)&=T(r,e^{\alpha-\omega})+S(r,e^\omega)\\ &\le\overline{N}(r,e^{\alpha-\omega})+\overline{N}\left(r,\frac{1}{e^{\alpha-\omega}}\right)+\overline{N}\left(r,\frac{1}{e^{\alpha-\omega}-b_1}\right)+S(r,e^\omega)\\ &=\overline{N}\left(r,\frac{1}{e^\omega(b_2+b_3e^\omega+\cdots+e^{(n-2)\omega}}\right)+S(r,e^\omega)\\ &\le(n-2)T(r,e^\omega)+S(r,e^\omega),\end{aligned}$$

which is impossible.

- $T(r, e^{\omega}) = S(r, e^{\alpha})$. It follows from (4.9) that for $0 \leq i \leq k-1$

$$T(r, M_{ic}(z)) = T(r, \eta_i + \theta_i e^{\omega}) = S(r, e^{\alpha}).$$

Using (4.8) and the definition of γ_k, we have

$$T(r, e^{\alpha}) \leq \sum_{k=0}^{n} (T(r, m_k) + T(r, \gamma_k)) = S(r, e^{\alpha}),$$

a contradiction can also be derived. This completes the proof of Theorem 1.7.

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