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CUP PRODUCT ON RELATIVE BOUNDED COHOMOLOGY

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Abstract. In this paper, we define cup product on relative bounded cohomology, and study its basic properties. Then, by extending it to a more generalized formula, we prove that all cup products of bounded cohomology classes of an amalgamated free product $G_1 *_A G_2$ are zero for every positive degree, assuming that free factors G_i are amenable and amalgamated subgroup A is normal in both of them. As its consequences, we show that all cup products of bounded cohomology classes of the groups $\mathbb{Z} * \mathbb{Z}$ and $\mathbb{Z}_n *_{\mathbb{Z}_d} \mathbb{Z}_m$, where d is the greatest common divisor of n and m, are zero for every positive degree.

1. Introduction

The theory of bounded cohomology has attracted considerable attention with many applications ([3, 4]). However, computing bounded cohomology groups seems more complicated, and therefore inquiring into various methods of its computation would be interesting. For example, let F_m be a free group with rank $m \geq 2$. As it is well known, its ordinary cohomology $H^n(F_m) = 0$ for every $n > 1$. On the other hand, its bounded cohomology groups $\widehat{H}^n(F_m)$ with real coefficients are computed only up to $n = 3$ and remain unknown for $n \geq 4$. In particular, $\widehat{H}^2(F_m)$ and $\widehat{H}^3(F_m)$ are known as infinite dimensional vector spaces over $\mathbb{R}$ ([3]). In [6], it is proved that $\alpha \cup \beta \in \widehat{H}^4(F_m)$ is zero for $\alpha, \beta \in \widehat{H}^2(F_m)$ if they are represented by quasicharacters. A function $f : G \to \mathbb{R}$ for a group G is called a quasicharacter if there is a constant $C \geq 0$ such that $|f(x)+f(y)-f(xy)| \leq C$ for all $x, y \in G$. Meanwhile, in the ordinary cohomology, the following theorem is proved ([1]):

Theorem 1.1. *Suppose $X = U \cup V$, where U and V are open and acyclic sets. Then $\alpha \cup \beta = 0$ for all cohomology classes $\alpha, \beta \in H^*(X)$.*

The proof of this theorem is based on a generalized cup product

$$H^*(X, U) \times H^*(X, V) \xrightarrow{\cup} H^*(X, U \cup V).$$

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Motivated by this, the main purpose of this paper is to establish a more generalized cup product on the relative bounded cohomology of groups and, as a consequence of it, to compute cup product of bounded cohomology classes of amalgamated free products with amenable free factors.

We first review the definitions of the both absolute and relative bounded cohomology of groups briefly following [3] and [8].

Throughout this paper, all groups are considered discrete and G denotes a group.

The ordinary cohomology group $H^*(G)$ of G with real coefficients $\mathbb{R}$ is given by the cohomology of the following cochain complex $C^*(G)$:

$$0 \to \mathbb{R} \xrightarrow{d_0=0} C^1(G) \xrightarrow{d_1} C^2(G) \xrightarrow{d_2} \cdots \to C^n(G) \xrightarrow{d_n} C^{n+1}(G) \to \cdots,$$

where $C^n(G)$ is a space of all real valued functions $f : G^n = \underbrace{G \times \cdots \times G}_{n} \to \mathbb{R}$ and each boundary operator d_n is given by the formula

$$\begin{aligned} & d_n(f)(g_1, g_2, \ldots, g_{n+1}) \\ &= f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ &\quad + (-1)^{n+1} f(g_1, \ldots, g_n). \end{aligned} \tag{1.1}$$

We consider bounded cochains groups $B^n(G) = \{f \in C^n(G) \mid \|f\| < \infty\}$, where $\|f\| = \sup\{\, |f(x)| \mid x \in G^n\}$. By the same formula for each d_n as in (1.1), we have a complex $B^*(G)$:

$$0 \to \mathbb{R} \xrightarrow{d_0=0} B^1(G) \xrightarrow{d_1} B^2(G) \xrightarrow{d_2} \cdots \tag{1.2}$$

of bounded cochain groups.

Definition 1.2. The nth cohomology of the complex (1.2) is called the *nth bounded cohomology of G* and is denoted by $\widehat{H}^n(G)$.

The inclusion homomorphism $B^*(G) \hookrightarrow C^*(G)$ induces a homomorphism $\widehat{H}^*(G) \to H^*(G)$ which is in general neither injective nor surjective.

We refer the abstract theory of bounded cohomology to [3], [4], and [7].

Remark 1.3. In [7], it is proved that the bounded cohomology groups of an amenable group are zero for every positive degree. Recall that a group G is amenale if there exists a right invariant mean on $B(G)$. For example, abelian groups, finite groups, and also subgroups of an amenable group are amenable.

Similar to $\widehat{H}^*(G)$, the bounded cohomology $\widehat{H}^*(X)$ of a topological space X is defined as the cohomology of the bounded cochain complex $B^*(X)$:

$$0 \to B^0(X) \xrightarrow{d_0} B^1(X) \xrightarrow{d_1} B^2(X) \xrightarrow{d_2} \cdots,$$

where $B^n(X)$ consists of all functions $f\colon S_n(X) \to \mathbb{R}$ which is bounded with respect to the sup norm and $S_n(X)$ is the set of n-dimensional singular simplices in X. For details, we refer to [7].

As one of the important properties in the theory of bounded cohomology, the following theorem is proved in [7]:

Theorem 1.4. *Let X be a connected countable cellular space. Then $\widehat{H}^*(X)$ is canonically isomorphic with $\widehat{H}^*(\pi_1 X)$.*

Now we introduce the basic definition of relative bounded cohomology and refer to [8] for more details. Based on Theorem 1.4, we define relative bounded cohomology in more general aspect: by considering a continuous map $\varphi' : Y \to X$ of spaces and a homomorphism $\varphi : K \to G$ of groups instead of subspaces and subgroups.

Let $\varphi : K \to G$ be a homomorphism of groups. Then there is a cochain map of homomorphisms $\varphi_n : B^n(G) \to B^n(K)$ for $n \geq 1$ defined by the formula: For $\alpha \in B^n(G)$ and $(a_1, a_2, \ldots, a_n) \in G^n$

$$\varphi_n \alpha(a_1, a_2, \ldots, a_n) = \alpha\left(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\right), \; n \geq 1.$$

The relative bounded cochain groups $B^*(K \xrightarrow{\varphi} G)$ of φ is defined by

$$B^n(K \xrightarrow{\varphi} G) = B^n(G) \oplus B^{n-1}(K) \tag{1.3}$$

and a norm $\|\cdot\|$ on $B^n(K \xrightarrow{\varphi} G)$ by $\|(\alpha_n, \gamma_{n-1})\| = \max\{\|\alpha_n\|, \|\gamma_{n-1}\|\}$. It is clear that this norm is bounded.

Also, we define a boundary operator for $n \geq 1$

$$d_n : B^n(K \xrightarrow{\varphi} G) \to B^{n+1}(K \xrightarrow{\varphi} G)$$

by the formula: For $(\alpha_n, \gamma_{n-1}) \in B^n(K \xrightarrow{\varphi} G) = B^n(G) \oplus B^{n-1}(K)$

$$d_n(\alpha_n, \gamma_{n-1}) = \left(\partial_n \alpha_n, \, -\varphi_n \alpha_n - \delta_{n-1}\gamma_{n-1}\right), \tag{1.4}$$

where δ_* and ∂_* are boundary operators of $B^*(K)$ and $B^*(G)$, respectively. Then we have the relative bounded cochain complex $B^*(K \xrightarrow{\varphi} G)$:

$$0 \to \mathbb{R} \xrightarrow{d_0} B^1(K \xrightarrow{\varphi} G) \xrightarrow{d_1} B^2(K \xrightarrow{\varphi} G) \xrightarrow{d_2} \cdots, \tag{1.5}$$

where $d_0(r) = (0, -r)$ for $r \in \mathbb{R}$.

Definition 1.5. The nth cohomology of the complex (1.5) is called *the nth relative bounded cohomology of φ*. We denote it by $\widehat{H}^n(K \xrightarrow{\varphi} G)$.

It is easy to compute that

$$\widehat{H}^0(K \xrightarrow{\varphi} G) = 0 \quad \text{and} \quad \widehat{H}^1(K \xrightarrow{\varphi} G) = 0. \tag{1.6}$$

Remark 1.6. Similar to the case of groups, for a continuous map $\varphi' : Y \to X$ of topological spaces, *the relative bounded cohomology $\widehat{H}^*(Y \xrightarrow{\varphi'} X)$ of φ'* is defined by the cohomology of the complex

$$0 \to B^0(Y \xrightarrow{\varphi'} X) \xrightarrow{d_0} B^1(Y \xrightarrow{\varphi'} X) \xrightarrow{d_1} B^2(Y \xrightarrow{\varphi'} X) \xrightarrow{d_2} \cdots,$$

where $B^n(Y \xrightarrow{\varphi'} X) = B^n(X) \oplus B^{n-1}(Y)$ and the each boundary operator d_n is defined by the same formula as (1.4).

In [8], the following theorem is proved:

Theorem 1.7. *Let $\varphi' : Y \to X$ be a continuous map of topological spaces. Then $\widehat{H}^*(Y \xrightarrow{\varphi'} X)$ is (isometrically) isomorphic to $\widehat{H}^*(\pi_1 Y \xrightarrow{\varphi'_\sharp} \pi_1 X)$.*

By Theorem 1.7, we can study relative bounded cohomology of spaces and groups simultaneously as the absolute case shown in Theorem 1.4.

Recall that the following basic homological algebraic properties hold.

Remark 1.8. Let $\varphi : K \to G$ be a homomorphism of groups. By the natural injection ι_n and projection π_n, there is an exact sequence

$$0 \to B^{n-1}(K) \xrightarrow{\iota_n} B^n(K \xrightarrow{\varphi} G) = B^n(G) \oplus B^{n-1}(K) \xrightarrow{\pi_n} B^n(G) \to 0$$

and it induces a long exact sequence

$$\to \widehat{H}^{n-1}(K) \xrightarrow{\iota^*} \widehat{H}^n(K \xrightarrow{\varphi} G) \xrightarrow{\pi^*} \widehat{H}^n(G) \xrightarrow{\tilde{\delta}^*} \widehat{H}^n(K) \xrightarrow{\iota^*} \widehat{H}^{n+1}(K \xrightarrow{\varphi} G) \to .$$

Notice that, for $[\zeta] \in \widehat{H}^n(K)$ it is clear that

$$\iota^*([\zeta]) = [(0, \zeta)] \in \widehat{H}^{n+1}(K \xrightarrow{\varphi} G).$$

We describe the contents of this paper. In the next section, we define cup product on the relative bounded cohomology $\widehat{H}^*(K \xrightarrow{\varphi} G)$ and study its properties. In Section 3, after establishing a more generalized relative cup product from the point of view of Theorem 1.7, we prove that all cup products of bounded cohomology classes of an amalgamated product $G_1 *_A G_2$ with amenable free factors G_i are zero for all positive degrees, assuming that the amalgamated subgroup A is normal in both G_i. Then, as its applications, we show that all cup products of bounded cohomology classes of $\mathbb{Z} * \mathbb{Z}$ and of $\mathbb{Z}_n *_{\mathbb{Z}_d} \mathbb{Z}_m$, where d is the greatest common divisor of n and m, are zero for all positive degrees.

In the rest of this paper, we consider only the algebraic structure of bounded cohomology as a vector spaces over $\mathbb{R}$.

2. Cup product on relative bounded cohomology

Throughout this section, $\varphi : K \to G$ denotes a homomorphism of groups and $\varphi_n : B^n(G) \to B^n(K)$ its induced homomorphism for every $n \geq 1$. Also, for simplicity, we denote all boundary operators by d_* and omit all subscripts if there is no confusion.

For $\alpha \in C^p(G)$ and $\beta \in C^q(G)$, the cup product $\alpha \cup \beta \in C^{p+q}(G)$ is the cochain whose value on G^{p+q} is given by the formula ([2, 5])

$$(\alpha_p \cup \beta_q)(g_1, \ldots, g_p, g_{p+1}, \ldots, g_{p+q}) = \alpha_p(g_1, \ldots, g_p)\beta_q(g_{p+1}, \ldots, g_{p+q}). \tag{2.1}$$

For $\alpha, \beta \in B^*(G) \subseteq C^*(G)$, notice that $\|\alpha \cup \beta\| \leq \|\alpha\| \cdot \|\beta\| < \infty$ and so $\alpha \cup \beta \in B^*(G)$. Hence there is a cup product

$$\cup : B^p(G) \times B^q(G) \to B^{p+q}(G).$$

Also, as it is shown in [2] and [5], the boundary operator d_* on $B^*(G)$ holds the equation

$$d_{p+q}(\alpha_p \cup \beta_q) = d_p\alpha_p \cup \beta_q + (-1)^p \alpha_p \cup d_q\beta_q \quad \text{for every } p, q \geq 0. \tag{2.2}$$

It follows that there is an induced cup product $\widehat{H}^p(G) \times \widehat{H}^q(G) \xrightarrow{\cup} \widehat{H}^{p+q}(G)$. Moreover, for $[\alpha], [\beta] \in \widehat{H}^*(G)$, we have $[\alpha] \cup [\beta] = [\alpha \cup \beta]$.

By the same method as shown in [2] for the ordinary cohomology, cup products on $\widehat{H}^*(G)$ are bilinear, associative, distributive, anti-commutative, and natural.

Remark 2.1. The external direct sum $\bigoplus_{n\geq 0} \widehat{H}^n(G)$ forms an additive group. Also, there is an identity element $[1] \in \widehat{H}^0(G) = \mathbb{R}$. Thus this additive group forms a graded anti-commutative ring with the identity under the cup product operation.

Now we look into the cup product on the relative bounded cochain groups $B^*(K \xrightarrow{\varphi} G)$. Recall that $B^n(K \xrightarrow{\varphi} G) = B^n(G) \oplus B^{n-1}(K)$ for every $n \geq 0$.

Definition 2.2. We define the relative cup product

$$\cup\colon B^p(K \xrightarrow{\varphi} G) \times B^q(K \xrightarrow{\varphi} G) \to B^{p+q}(K \xrightarrow{\varphi} G)$$

by the following equation

$$(\alpha, \gamma) \cup (\beta, \rho) = \left(\alpha \cup \beta,\, \frac{(-1)^p}{2} \varphi_p\alpha \cup \rho + \frac{1}{2} \gamma \cup \varphi_q\beta \right). \tag{2.2.1}$$

Proposition 2.3. *The cup product on $B^*(K \xrightarrow{\varphi} G)$ is bilinear, distributive, and anti-commutative. Also, for $A \in B^p(K \xrightarrow{\varphi} G)$ and $B \in B^q(K \xrightarrow{\varphi} G)$,*

$$d\,(A \cup B) = dA \cup B + (-1)^p A \cup dB.$$

Proof. Recall that the cup products on bounded cochain groups are bilinear and anti-commutative from Remark 2.1. Using these facts, it is easy to check the bilinear and anti-commutative properties of the cup product on $B^*(K \xrightarrow{\varphi} G)$.

Let $A = (\alpha, \gamma) \in B^p(K \xrightarrow{\varphi} G)$ and $B = (\beta, \rho) \in B^q(K \xrightarrow{\varphi} G)$. By the formulas (1.4) and (2.2.1), we have

$$d(A \cup B) = d\left(\alpha \cup \beta,\, \frac{(-1)^p}{2} \varphi_p\alpha \cup \rho + \frac{1}{2} \gamma \cup \varphi_q\beta \right) = \big(d(\alpha \cup \beta),\, C \big), \tag{2.3.1}$$

where

$$C = -\varphi_{p+q}(\alpha \cup \beta) - \frac{(-1)^p}{2} d\left(\varphi_p\alpha \cup \rho\right) - \frac{1}{2} d\left(\gamma \cup \varphi_q\beta\right).$$

On the other hand,

$$dA \cup B + (-1)^p A \cup dB = d(\alpha, \gamma) \cup (\beta, \rho) + (-1)^p (\alpha, \gamma) \cup d(\beta, \rho)$$

$$= (d\alpha, -\varphi_p\alpha - d\gamma) \cup (\beta, \rho) + (-1)^p(\alpha, \gamma) \cup (d\beta,\ -\varphi_q\beta - d\rho)$$

$$= \left(d\alpha \cup \beta,\ \frac{1}{2}C_1\right) + \left((-1)^p\alpha \cup d\beta,\ \frac{(-1)^p}{2}C_2\right)$$

$$= \left(d(\alpha \cup \beta),\ \frac{1}{2}(C_1 + (-1)^p C_2)\right), \tag{2.3.2}$$

where

$$\begin{cases} C_1 = (-1)^{p+1}\varphi_{p+1}d\alpha \cup \rho + (-\varphi_p\alpha - d\gamma) \cup \varphi_q\beta; \\ C_2 = (-1)^p\varphi_p\alpha \cup (-\varphi_q\beta - d\rho) + \gamma \cup \varphi_{q+1}d\beta. \end{cases}$$

From (2.3.1) and (2.3.2), it is enough to show that $C_1 + (-1)^p C_2 = 2C$. As the cup products on $B^*(G)$ and $B^*(K)$ are bilinear by Remark 2.1,

$$\begin{aligned} &C_1 + (-1)^p C_2 \\ =\ & -(-1)^p\varphi_{p+1}d\alpha \cup \rho - \varphi_p\alpha \cup \varphi_q\beta - d\gamma \cup \varphi_q\beta \\ & -\varphi_p\alpha \cup \varphi_q\beta - \varphi_p\alpha \cup d\rho + (-1)^p\gamma \cup \varphi_{q+1}d\beta \\ =\ & -2\varphi_{p+q}(\alpha \cup \beta) - (-1)^p d\left(\varphi_p\alpha \cup \rho\right) - d\left(\gamma \cup \varphi_q\beta\right) = 2C. \end{aligned}$$

□

Theorem 2.4. *The cup product on $B^*(K \xrightarrow{\varphi} G)$ induces an operation*

$$\cup : \widehat{H}^p(K \xrightarrow{\varphi} G) \times \widehat{H}^q(K \xrightarrow{\varphi} G) \to \widehat{H}^{p+q}(K \xrightarrow{\varphi} G)$$

that is bilinear, distributive, and anti-commutative.

Proof. By the same method shown in [1] and [5], it is easy to check the relative bounded cohomology class of cup product depends only on cohomology classes. So, it follows from Proposition 2.3. □

Notice that the cup product on $B^*(K \xrightarrow{\varphi} G)$ is not associative.

Proposition 2.5. *The cup product on $\widehat{H}^*(K \xrightarrow{\varphi} G)$ is associative.*

Proof. Let $[A], [B], [C] \in \widehat{H}^*(K \xrightarrow{\varphi} G)$ be represented, respectively, by the cocycles $A = (\alpha, \gamma) \in B^p(K \xrightarrow{\varphi} G)$, $B = (\beta, \rho) \in B^q(K \xrightarrow{\varphi} G)$, and $C = (\lambda, \zeta) \in B^r(K \xrightarrow{\varphi} G)$.

From Theorem 2.4, it is enough to show that the cocycles $(A \cup B) \cup C$ and $A \cup (B \cup C)$ differ by a coboundary. Notice that, by Definition 2.2,

$$\begin{cases} (A \cup B) \cup C = \left(\alpha \cup \beta,\ \frac{(-1)^p}{2}\varphi_p\alpha \cup \rho + \frac{1}{2}\gamma \cup \varphi_q\beta\right) \cup (\lambda, \zeta) = ((\alpha \cup \beta) \cup \lambda,\ X); \\ A \cup (B \cup C) = (\alpha, \gamma) \cup \left(\beta \cup \lambda,\ \frac{(-1)^q}{2}\varphi_q\beta \cup \zeta + \frac{1}{2}\rho \cup \varphi_r\lambda\right) = (\alpha \cup (\beta \cup \lambda),\ X'), \end{cases}$$

where

$$\begin{cases} X = \frac{(-1)^{p+q}}{2}(\varphi_p\alpha \cup \varphi_q\beta) \cup \zeta + \frac{(-1)^p}{4}(\varphi_p\alpha \cup \rho) \cup \varphi_r\lambda + \frac{1}{4}(\gamma \cup \varphi_q\beta) \cup \varphi_r\lambda; \\ X' = \frac{(-1)^{p+q}}{4}\varphi_p\alpha \cup (\varphi_q\beta \cup \zeta) + \frac{(-1)^p}{4}\varphi_p\alpha \cup (\rho \cup \varphi_r\lambda) + \frac{1}{2}\gamma \cup (\varphi_q\beta \cup \varphi_r\lambda). \end{cases}$$

As the associative law holds on both $B^*(K)$ and $B^*(G)$ by Remark 2.1,

$$(A \cup B) \cup C - A \cup (B \cup C) = (0,\, X - X').$$

We compute $X - X'$. Since $dA = dB = dC = 0$, by the formula (1.4) we have the following equations:

$$d\gamma = -\varphi_p \alpha; \quad d\rho = -\varphi_q \beta; \quad d\zeta = -\varphi_r \lambda. \tag{2.5.1}$$

Then, by (2.5.1) and the anti-commutative property in Remark 2.1

$$\begin{aligned} X - X' &= \frac{(-1)^{p+q}}{4} \varphi\alpha \cup \varphi\beta \cup \zeta - \frac{1}{4} \gamma \cup \varphi\beta \cup \varphi\lambda \\ &= \frac{(-1)^{p+q}(-1)^{pq}}{4} d\rho \cup d\gamma \cup \zeta - \frac{(-1)^{(p-1)q}}{4} d\rho \cup \gamma \cup d\zeta \\ &= \frac{(-1)^{p+q+pq}}{4} d\left(\rho \cup d(\gamma \cup \zeta)\right). \end{aligned}$$

This shows that

$$(A \cup B) \cup C - A \cup (B \cup C) = \frac{(-1)^{p+q+pq}}{4} d\Big(0,\, -\rho \cup d(\gamma \cup \zeta)\Big)$$

and so by Theorem 2.4

$$([A] \cup [B]) \cup [C] = [(A \cup B) \cup C] = [A \cup (B \cup C)] = [A] \cup ([B] \cup [C]). \qquad \square$$

Remark 2.6. As $\widehat{H}^0(K \xrightarrow{\varphi} G) = 0$ shown in (1.6), the external direct sum $\bigoplus_{n \geq 0} \widehat{H}^n(K \xrightarrow{\varphi} G)$ forms an anti-commutative ring without the identity under relative cup product by Theorem 2.4 and Proposition 2.5.

Definition 2.7. We define the following cup product operations

$$B^p(K \xrightarrow{\varphi} G) \times B^q(G) \xrightarrow{\cup} B^{p+q}(K \xrightarrow{\varphi} G) \text{ and } B^p(K) \times B^q(G) \xrightarrow{\cup} B^{p+q}(K),$$

respectively, by the formulas: For $\beta \in B^q(G)$,

$$(\alpha, \gamma) \cup \beta = (\alpha \cup \beta,\, \gamma \cup \varphi_q \beta) \text{ and } \eta \cup \beta = \eta \cup \varphi_q \beta,$$

where $(\alpha, \gamma) \in B^p(K \xrightarrow{\varphi} G)$, and $\eta \in B^p(K)$.

Now, we show the cup products commute in the sequence in Remark 1.8.

Proposition 2.8. *Let* $[B] \in \widehat{H}^q(G)$. *Then the following diagram commutes*:

$$\begin{array}{ccccccc} \widehat{H}^p(K \xrightarrow{\varphi} G) & \xrightarrow{\pi_*} & \widehat{H}^p(G) & \xrightarrow{\varphi_*} & \widehat{H}^p(K) & \xrightarrow{\iota_*} & \widehat{H}^{p+1}(K \xrightarrow{\varphi} G) \\ \downarrow \cup[B] & & \downarrow \cup[B] & & \downarrow \cup\varphi_*[B] & & \downarrow \cup[B] \\ \widehat{H}^{p+q}(K \xrightarrow{\varphi} G) & \xrightarrow{\pi_*} & \widehat{H}^{p+q}(G) & \xrightarrow{\varphi_*} & \widehat{H}^{p+q}(K) & \xrightarrow{\iota_*} & \widehat{H}^{p+q+1}(K \xrightarrow{\varphi} G). \end{array}$$

Proof. It is clear that the second rectangle is commutative as the cup product is natural. Let $(\alpha,\gamma) \in B^p(K \xrightarrow{\varphi} G)$ be a cocycle and $[B]$ be represented by a cocycle $\beta \in B^q(G)$. By Definition 2.7,

$$\pi_{p+q}((\alpha,\gamma)\cup\beta) = \alpha\cup\beta = \pi_p(\alpha,\gamma)\cup\beta$$

and

$$\iota_{p+q}(\gamma)\cup\beta = (0,\gamma)\cup\beta = (0,\gamma\cup\varphi_q\beta) = \iota_{p+1+q}(\gamma\cup\varphi_q\beta).$$

Hence the first and the third diagrams are also commutative. □

3. Generalized cup product on relative bounded cohomology and its application

As shown in [5], for a CW-complex X and its subcomplexes X_1 and X_2, there is a more generalized relative cup product

$$H^p(X,X_1)\times H^q(X,X_2) \xrightarrow{\cup} H^{p+q}(X, X_1\cup X_2).$$

By imitating this, we define a more generalized relative cup product on bounded cohomology from the point of view of Theorem 1.7. For this, we recollect the amalgamated free product for the fundamental group of $X_1\cup X_2$.

Lemma 3.1. *Any amalgamation diagram $G = G_1 *_A G_2$ with ι_1 and ι_2 injective can be realized by a diagram of $K(\pi_1,1)$ complexes Y_1 and Y_2:*

$$\begin{array}{ccc} A = \pi_1(Y_1\cap Y_2) & \xhookrightarrow{\iota_1} & G_1 = \pi_1 Y_1 \\ \big\downarrow \iota_2 & & \big\downarrow \iota_1' \\ G_2 = \pi_1 Y_2 & \xhookrightarrow{\iota_2'} & G_1 *_A G_2 = \pi_1(Y_1\cup Y_2), \end{array}$$

where $Y_1\cap Y_2$ is connected and nonempty.

Proof. The proof is referred to Theorem 7.2 and Theorem 7.3 in [2]. □

Remark 3.2. We consider an amalgamated free product $G_1 *_A G_2$ with injective homomorphisms $\iota_i : A \to G_i$ for both $i = 1, 2$. Then we can regard A as a subgroup of both factors G_1 and G_2. Let A be normal in both G_1 and G_2. Recall that there is the natural surjective homomorphism

$$G_1 *_A G_2 \xrightarrow{\theta'} G_1/A * G_2/A.$$

So the composition of homomorphisms $G_1 *_A G_2 \xrightarrow{\theta'} G_1/A * G_2/A \to G_i/A$ induces a homomorphism

$$\theta_{i,m} : B^m(G_i/A) \to B^m(G_1 *_A G_2),\ m \geq 0 \text{ and } i = 1, 2.$$

In the rest of this section, when we say $G = G_1 *_A G_2$, we assume A is a subgroup of G_i via injective homomorphisms $\iota_i : A \hookrightarrow G_i$ for $i = 1, 2$.

Lemma 3.3. *Let $G = G_1 *_A G_2$ and A be normal in both G_i. Suppose A is amenable. Then, for every $m \geq 0$ there is a homomorphism*

$$h_{i,m} : B^m(G_i) \to B^m(G_1 *_A G_2). \tag{3.3.1}$$

Proof. Since A is amenable, as shown in [7] there is a homomorphism

$$\tilde{\pi}_{i,m} : B^m(G_i) \to B^m(G_i/A)$$

such that the composition $B^m(G_i/A) \to B^m(G_i) \xrightarrow{\tilde{\pi}_{i,m}} B^m(G_i/A)$ is an identity. Then we set $h_{i,m} = \theta_{i,m} \circ \tilde{\pi}_{i,m}$, where $\theta_{i,m}$ is in Remark 3.2 □

Theorem 3.4. *Let $G = G_1 *_A G_2$ and Γ be a group equipped with homomorphisms $f_i : G_i \to \Gamma$ such that $f_1\iota_1 = f_2\iota_2$, where $\iota_i : A \hookrightarrow G_i$ are injective homomorphisms for $i = 1, 2$. Let A be amenable and normal in both G_i. Then there is a cup product*

$$\widehat{H}^p(G_1 \xrightarrow{f_1} \Gamma) \times \widehat{H}^q(G_2 \xrightarrow{f_2} \Gamma) \xrightarrow{\cup} \widehat{H}^{p+q}(G_1 *_A G_2 \xrightarrow{\widetilde{\theta}} \Gamma) \tag{3.4.1}$$

that is bilinear and associative.

Proof. First, we define a cup product

$$B^p(G_1 \xrightarrow{f_1} \Gamma) \times B^q(G_2 \xrightarrow{f_2} \Gamma) \xrightarrow{\cup} B^{p+q}(G_1 *_A G_2 \xrightarrow{\widetilde{\theta}} \Gamma). \tag{3.4.2}$$

Let $(\alpha, \gamma) \in B^p(\Gamma) \oplus B^{p-1}(G_1)$ and $(\beta, \rho) \in B^q(\Gamma) \oplus B^{q-1}(G_2)$. It is clear that $\alpha \cup \beta \in B^n(\Gamma)$, where $n = p + q$. Notice that there is a unique homomorphism $\tilde{\theta} : G \to \Gamma$ such that $\tilde{\theta} \circ \iota_i' = f_i$, where ι_i' is a natural embedding $G_i \to G$. Then there is an induced homomorphism $\widetilde{\theta}_m : B^m(\Gamma) \to B^m(G_1 *_A G_2)$. Then

$$\widetilde{\theta}_p\alpha \cup h_{2,q-1}\rho \in B^{n-1}(G_1 *_A G_2) \text{ and } h_{1,p-1}\gamma \cup \widetilde{\theta}_q\beta \in B^{n-1}(G_1 *_A G_2),$$

where $h_{i,m} : B^m(G_i) \to B^m(G_1 *_A G_2)$ is defined in Lemma 3.3. Thus, from Definition 2.2, there is a cup product in (3.4.2) defined by

$$(\alpha, \gamma) \cup (\beta, \rho) = \left(\alpha \cup \beta,\ \frac{(-1)^p}{2}\widetilde{\theta}_p\alpha \cup h_{2,q-1}\rho + \frac{1}{2}h_{1,p-1}\gamma \cup \widetilde{\theta}_q\beta\right).$$

Hence, by Proposition 2.3, it induces a cup product in (3.4.1). The bilinear and associative properties follow from Theorem 2.4 and Proposition 2.5. □

There is a topological space version of Theorem 3.4.

Corollary 3.5. *Let Y be a CW-complex which is the union of two connected subcomplexes Y_1 and Y_2 whose intersection $Z = Y_1 \cap Y_2$ is connected and nonempty. Suppose $\pi_1 Z$ is amenable and normal in both $\pi_1 Y_1$ and $\pi_1 Y_2$. Then there is a cup product on the relative bounded cochains*

$$\widehat{H}^p(Y_1 \hookrightarrow X) \times \widehat{H}^q(Y_2 \hookrightarrow X) \xrightarrow{\cup} \widehat{H}^{p+q}(Y_1 \cup Y_2 \to X).$$

Proof. Recall that $\pi_1(Y_1 \cup Y_2) = \pi_1 Y_1 *_{\pi_1 Z} \pi_1 Y_2$ by Lemma 3.3. Then, it follows from Lemma 3.1 and Theorem 1.7. □

Theorem 3.6. *Let $G = G_1 *_A G_2$ and A be normal in both G_1 and G_2. Suppose G_1 and G_2 are amenable. Then, for $\alpha, \beta \in \widehat{H}^*(G_1 *_A G_2)$, their cup product $\alpha \cup \beta$ is zero for all positive degrees.*

Proof. For $p, q \geq 0$ with $p + q > 0$, let $\alpha \in \widehat{H}^p(G)$ and $\beta \in \widehat{H}^q(G)$. From Remark 1.8, for each $i = 1, 2$ there is an exact sequence

$$\to \widehat{H}^{m-1}(G_i) \to \widehat{H}^m(G_i \xrightarrow{\iota_i'} G) \to \widehat{H}^m(G) \to \widehat{H}^m(G_i) \to .$$

Since bounded cohomology groups of an amenable group are zero for all positive degrees by Remark 1.3, $\widehat{H}^*(G_i \xrightarrow{\iota_i'} G)$ and $\widehat{H}^*(G)$ are isomorphic. Hence there are the relative bounded cohomology classes $\widetilde{\alpha} \in \widehat{H}^p(G_1 \xrightarrow{\iota_1'} G)$ and $\widetilde{\beta} \in \widehat{H}^q(G_2 \xrightarrow{\iota_2'} G)$ such that $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively, map to α and β. Since subgroups of an amenable group are amenable, A is also amenable. So from Corollary 3.4, we have a commutative diagram:

$$\begin{array}{ccc}
\widehat{H}^p(G_1 \xrightarrow{\iota_1'} G) \times \widehat{H}^q(G_2 \xrightarrow{\iota_2'} G) & \xrightarrow{\cup} & \widehat{H}^{p+q}(G_1 *_A G_2 \xrightarrow{\mathrm{id}} G) = 0 \\
\downarrow \cong & & \downarrow \\
\widehat{H}^p(G) \times \widehat{H}^q(G) & \xrightarrow{\cup} & \widehat{H}^{p+q}(G).
\end{array}$$

As $\widetilde{\alpha} \cup \widetilde{\beta} \in \widehat{H}^p(G_1 *_A G_2 \to G) = 0$, we have $\alpha \cup \beta = 0$. □

Corollary 3.7. *Let $G = \mathbb{Z} * \mathbb{Z}$. Then $\alpha \cup \beta = 0$ for all $\alpha, \beta \in \widehat{H}^*(G)$ of positive degree.*

Proof. Notice that a free group $\mathbb{Z} * \mathbb{Z}$ is a free product with the trivial group amalgamated. Also, an abelian group $\mathbb{Z}$ is amenable. Hence it follows from Theorem 3.6. □

Corollary 3.8. *Let $G = \mathbb{Z}_n *_{\mathbb{Z}_d} \mathbb{Z}_m$, where d is the greatest common divisor of n and m. Then $\alpha \cup \beta = 0$ for all $\alpha, \beta \in \widehat{H}^*(G)$ of positive degree.*

Proof. Let $n = ds$ and $m = dt$ for positive integers s, t. Recall that

$$\mathbb{Z}_n *_{\mathbb{Z}_d} \mathbb{Z}_m = \left\langle a, b \,|\, a^n = b^m = 1, \, a^s = b^t \right\rangle .$$

Then $\mathbb{Z}_d \cong \left\langle a^s | \, a^{ds} \right\rangle$ and $\mathbb{Z}_d \cong \left\langle b^t | \, b^{dt} \right\rangle$. As we know it, we can take injective homomorphisms $\iota_1 : \mathbb{Z}_d \to \mathbb{Z}_n$ by $\iota_1(x) = x^s$ and similarly $\iota_2 : \mathbb{Z}_d \to \mathbb{Z}_m$ by $\iota_2(y) = y^t$. Notice that $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are amenable. Also, $\mathbb{Z}_d$ can be regarded as a normal subgroup of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then it follows from Theorem 3.6. □

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