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CONSTRUCTION OF A MATTILA–SJÖLIN TYPE FUNCTION OVER A FINITE FIELD

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Abstract. Let $\mathbb{F}_q$ be a finite field with q elements. A function $f : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$ is called a Mattila–Sjölin type function of index $\gamma \in \mathbb{R}$ if γ is the smallest real number such that whenever $|E| \geq Cq^\gamma$ for a sufficiently large constant C, the set $f(E,E) := \{f(x,y) : x, y \in E\}$ is equal to $\mathbb{F}_q$. In this article, we construct an example of a Mattila–Sjölin type function f and provide its index, generalizing the result of Cheong, Koh, Pham and Shen [1].

1. Introduction

The Falconer distance problem concerns the minimal Hausdorff dimension of compact sets E of $\mathbb{R}^d$ such that the Lebesgue measure of the distance set $\Delta(E)$ is positive, where $\Delta(E) := \{|x-y| : x, y \in E\}$. Falconer in [6] conjectured that for subsets E in $\mathbb{R}^d$, $d \geq 2$, if the Hausdorff dimension $\dim_H(E)$ of E is strictly greater than $d/2$, then $\Delta(E)$ has a positive Lebesgue measure. Ever since he used the Fourier analysis machinery to obtain the lower bound $(d+1)/2$ for $\dim_H(E)$, there has been much progress on this problem by many authors, using new approaches such as the polynomial method and the decoupling inequalities. Interested readers can find details in [3–5, 9].

In 1999, Mattila–Sjölin [17] posed a much stronger version of the Falconer distance problem. Namely, they asked for the minimal Hausdorff dimension $\dim_H(E)$ of the compact sets E in $\mathbb{R}^d$ such that the distance set $\Delta(E)$ contains an interval. We refer to this problem as the Mattila–Sjölin problem. They conjectured that the minimal Hausdorff dimension would be $d/2$, which is the same as that on the Falconer distance problem, and obtained the lower bound $(d+1)/2$ for $\dim_H(E)$. Since then, the threshold $(d+1)/2$ has not been improved. Several extensions of this result to general configurations have been

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only made (see, for instance, [7, 8]). In a very recent paper, Koh, Pham and Shen [14] obtained an improvement for Cartesian product sets, namely, when $E = A^d \subset \mathbb{R}^d$ for some $A \subset \mathbb{R}$.

It is natural to consider the distance problem over a finite field. Over the past decades, there has been much development in finite field distance problems; see [10, 12–15, 18–20]. Among them, let us only touch on the finite field version of Mattila–Sjölin problem which has a relation with our result.

For $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{F}_q^d$, define

$$h(x, y) = ||x - y|| := \sum_{j=1}^{d} (x_j - y_j)^2.$$

We will call $h(x, y)$ a distance between x and y as they do in the literature. For $E \subset \mathbb{F}_q^d$, define the distance set of E as

$$\Delta(E) := \{||x - y|| : x, y \in E\}.$$

Since $\Delta(E)$ is a subset of $\mathbb{F}_q$, we have $|\Delta(E)| \leq q$, where $|\cdot|$ denotes the cardinality of a set.

Problem 1.1 (The finite field Mattila–Sjölin problem)**.** Let E be a subset of $\mathbb{F}_q^d$, $d \geq 2$. What is the smallest exponent $\kappa > 0$ such that $\Delta(E) = \mathbb{F}_q$ whenever $|E| \geq Cq^\kappa$ for a sufficiently large constant $C > 0$?

For this problem, there is only one known theorem due to Iosevich and Rudnev [13], which states that if $|E| \geq 2q^{\frac{d+1}{2}}$, then $\Delta(E) = \mathbb{F}_q$. This result is optimal for the general odd dimensions $d \geq 3$ since the exponent $(d+1)/2$ is the best possible result for the Erdős-Falconer distance problem [13] in odd dimensions, which calls for much weaker conclusion than the finite field Mattila–Sjölin problem. For even dimensions, it is believed that the exponent $(d+1)/2$ can be improved but there does not exist a reasonable conjecture. Murphy and Petridis [18] showed that in dimension two the optimal exponent cannot be lower than $4/3$. It would be interesting to investigate whether the exponent $(d+1)/2$ can be improved. For some specific sets such as product sets, it is known that the exponent $(d+1)/2$ can be improved (see, for example, [2, 19]).

The distance problem in the finite setting can be extended in various directions. For example, the distance $h(x, y) = ||x - y||$ can be replaced by various 'reasonable' functions on $\mathbb{F}_q^d \times \mathbb{F}_q^d$ with values in $\mathbb{F}_q$, for which one can consider a certain distance type problem. Indeed, Cheong, Koh, Pham, and Shen [1] gave the following definition.

Definition 1.2 (Mattila–Sjölin type function of index γ)**.** Let $f : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$ be a function. We say that f is a Mattila–Sjölin type function of index $\gamma \in \mathbb{R}$ if γ is the smallest positive real number such that whenever $|E| \geq Cq^\gamma$ for a sufficiently large constant C, we have $f(E, E) := \{f(x, y) : x, y \in E\} = \mathbb{F}_q$. In this situation, we write $Index(f) = \gamma$.

For instance, by what we described right after Problem 1.1, the function $h(x,y) = ||x-y||$ is a Mattila–Sjölin type function of index $(d+1)/2$ when d is odd. It is not known for the even d. In general, given a function $f : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$, it is not easy to find the index of f. The main purpose of this article is to construct a certain Mattila–Sjölin type function and provide the index of it, generalizing the result of [1]. Now we describe our result.

Notation 1. Fix positive integers d, ℓ, k such that $d \geq 3$ and $d = \ell + k$. Let $I = (i_1, \ldots, i_\ell)$ and $J = (j_1, \ldots, j_k)$ be two subsequences of the sequence $(1, 2, \ldots, d)$ such that as sets

$$I \cup J = \{1, 2, \ldots, d\} \text{ and } I \cap J = \emptyset.$$

For $x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d$, write

$$x_I = (x_{i_1}, \ldots, x_{i_\ell}) \text{ and } x_J = (x_{j_1}, \ldots, x_{j_k}),$$

where $I = (i_1, \ldots, i_\ell)$ and $J = (j_1, \ldots, j_k)$. Especially, we write $I_0 := (1, 2, \ldots, \ell)$ and $J_0 := (\ell+1, \ldots, d)$, and we use x' and x'' for x_{I_0} and x_{J_0}, respectively.

Definition 1.3. Let I, J be as above. Define $\Phi = \Phi(I,J) : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$ as

$$\Phi(x,y) := \begin{cases} \frac{||x_I - y_I||}{||x_J - y_J||} & \text{if } ||x_J - y_J|| \neq 0, \\ 0 & \text{if } ||x_J - y_J|| = 0, \end{cases} \tag{1.1}$$

where $||x_I - y_I|| := (x_{i_1} - y_{i_1})^2 + \cdots + (x_{i_\ell} - y_{i_\ell})^2$. We shall call Φ a quotient distance function and $\Phi(E,E)$ the quotient distance set generated by E.

Let η be the quadratic character of $\mathbb{F}_q^*$; see Example 2.1.

Theorem 1.4. *Let Φ be a function defined in Definition* 1.3. *Let $m := \min\{|I|, |J|\}$. Then Index*(Φ) *can be obtained as follows*:

(1) *If m is odd, then Index*$(\Phi) = d - \frac{m+1}{2}$.

(2) *If m is even and* $(\eta(-1))^{\frac{m}{2}} = 1$, *then Index*$(\Phi) = d - \frac{m}{2}$.

(3) *If m is even and* $(\eta(-1))^{\frac{m}{2}} = -1$, *then Index*$(\Phi) = d - \frac{m+2}{2}$.

We remark that the case where $d = 2\ell$ is even and $I = I_0, J = J_0$ with $|I_0| = |J_0|$ was settled in [1], and so our case can be viewed as a generalization of [1]. Thus, to obtain our result we basically follow the method used in [1], even though we have to work out more technicalities.

One of the advantages in working with more general functions f in the finite field setting is that if the function f is an algebraic function, i.e., a rational function of polynomials, then the distance type problem may have some connections with other areas in mathematics such as algebraic geometry, number theory and algebraic combinatorics. For example, if f is an algebraic function on $\mathbb{F}_q^d \times \mathbb{F}_q^d$ and $t \in \mathbb{F}_q$, then the fiber $f^{-1}(t)$ defines a subvariety of $\mathbb{F}_q^d \times \mathbb{F}_q^d$ if it is not empty. Our mainly concerned quantity $\nu_E(t)$ (in Definition 3.1) is none other than the number of the set $f^{-1}(t) \cap (E \times E)$. In particular, if E is a subvariety of $\mathbb{F}_q^d$, then $f^{-1}(t) \cap (E \times E)$ is a subvariety of $\mathbb{F}_q^d \times \mathbb{F}_q^d$.

On the other hand, counting (or finding a bound on the number of) elements of a (sub)variety is one of the typical problems in algebraic geometry over a finite field. Since our function Φ is algebraic, it would be interesting to try to produce a similar result, using an algebraic method like the cohomological one.

Convention. Throughout, we keep up the following conventions.

- $\mathbb{F}_q$ denotes a finite field with q elements, and of characteristic p, where p is an odd prime.
- For a set A, $|A|$ denotes the cardinality of A.
- d, ℓ, k are positive integers such that $d \geq 3$ and $d = \ell + k$.
- The alphabet E denotes a parameter of subsets of $\mathbb{F}_q^d$.

2. Preliminaries

In this section, we review some basics on the Gauss sum and the discrete Fourier analysis which will play essential roles in later computations. See [13] and [16] for these materials. We begin with the definition of a character of a group. A character ϕ of an abelian group G is a group homomorphism from G to the multiplicative group $\mathbb{C}^*$. Note that if G is a finite group, the image of a character $\phi : G \to \mathbb{C}^*$ is in fact in $S^1 = \{z \in \mathbb{C}^* \,|\, |z|^2=1\}$. In this article, we shall deal with two cases: $G = \mathbb{F}_q$ is an additive group, and $G = \mathbb{F}_q^*$ is a multiplicative group, where $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. We refer to a character of $\mathbb{F}_q$ (resp. $\mathbb{F}_q^*$) as an additive (resp. a multiplicative) character of $\mathbb{F}_q$.

Example 2.1. To give examples of an additive character and a multiplicative character, let $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ be the absolute trace [16, Definition 2.22], where we assume that p is the characteristic of $\mathbb{F}_q$.

(1) For $b \in \mathbb{F}_q$, let $\chi_b : \mathbb{F}_q \to S^1$ be a function defined by

$$\chi_b(c) = e^{2\pi i Tr(bc)/p} \quad \text{for all } c \in \mathbb{F}_q,\ b \in \mathbb{F}_q.$$

Then, it is easy to check that χ_b is an additive character. In particular, the character χ_1 is called the canonical additive character.

(2) To define a multiplicative character of $\mathbb{F}_q$, we recall that $\mathbb{F}_q^*$ is a cyclic group so that we can write $\mathbb{F}_q^* = \{g^k \,|\, k = 0, \ldots, q-2\}$ for a fixed generator g of $\mathbb{F}_q^*$. Then, for each $j = 0, 1, \ldots, q-2$, let $\psi_j : \mathbb{F}_q \to S^1$ be a function defined by

$$\psi_j(g^k) = e^{2\pi i jk/(q-1)}.$$

Then, it is easy to see ψ_j is a multiplicative character. In particular, letting $\eta := \psi_{(q-1)/2}$, we have $\eta(c) = 1$ if c is a square in $\mathbb{F}_q$, and $\eta(c) = -1$ otherwise. The character η is called quadratic.

Proposition 2.2. *All additive* (*resp. multiplicative*) *characters can be obtained as above* χ_b (*resp.* ψ_j *with respect to a fixed generator* $g \in \mathbb{F}_q^*$). *Namely,*

(1) *Given an additive character* χ*, there exists* $b \in \mathbb{F}_q$ *such that* $\chi = \chi_b$.

(2) *Given a multiplicative character* ψ, *there exist* $0 \le j \le q-2$ *such that* $\psi = \psi_j$, *i.e.*, $\psi(g^k) = e^{2\pi ijk/(q-1)}$.

Proof. For the proof of (1) and (2), respectively, we refer to Theorem 5.7 and Theorem 5.8, respectively, in [16]. □

Characters of G enjoy the orthogonality property below which will frequently be used for $G = \mathbb{F}_q$ and $\mathbb{F}_q^*$ in our computation.

Proposition 2.3 (Orthogonality of characters)**.** *Let* G *be a finite abelian group and* ϕ *a character of* G. *Then if* ϕ *is nontrivial, then*

$$\sum_{g\in G} \phi(g) = 0,$$

and $\sum_{g\in G} \phi(g) = |G|$, *otherwise.*

The Gauss sum is an exponential sum related to both an additive character and a multiplicative character. More precisely, it is defined as follows.

Definition 2.4 (Gauss sum)**.** Let ψ (resp. χ) be a multiplicative (resp. an additive) character on $\mathbb{F}_q$. Then, the Gauss sum of ψ and χ is defined by

$$\mathcal{G}(\psi, \chi) = \sum_{c\in\mathbb{F}_q^*} \psi(c)\chi(c). \tag{2.1}$$

For $a \in \mathbb{F}_q$, denote $\mathcal{G}_a = \mathcal{G}(\eta, \chi_a)$. $\mathcal{G}_1$ is called the standard Gauss sum. The standard Gauss sum $\mathcal{G}_1$ has been explicitly computed.

Proposition 2.5 (Theorem 5.15, [16])**.** *The standard Gauss sum* $\mathcal{G}_1$ *can be explicitly computed as follows*:

$$\mathcal{G}_1 = \begin{cases} (-1)^{s-1} q^{\frac{1}{2}} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{s-1} i^s q^{\frac{1}{2}} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

We remark that by Proposition 2.5 and (2) of Example 2.1, we have

$$\mathcal{G}_1^2 = \eta(-1)q, \text{ and } |\mathcal{G}_1| = \sqrt{q}. \tag{2.3}$$

Lemma 2.6. *For* $a, b \in \mathbb{F}_q$ *with* $a \neq 0$, *and* $v \in \mathbb{F}_q^d$, *we have*

(1) $\sum_{s\in\mathbb{F}_q} \chi_1(as^2) = \eta(a)\mathcal{G}_1$.
(2) $\sum_{s\in\mathbb{F}_q} \chi_1(as^2 + bs) = \eta(a)\mathcal{G}_1\chi_1\left(-\frac{b^2}{4a}\right)$.
(3) $\sum_{u\in\mathbb{F}_q^n} \chi_1\left(a\|u\| - u\cdot v\right) = \eta^n(a)\cdot\mathcal{G}_1^n\cdot\chi\left(\frac{||v||}{-4a}\right)$.

Proof. To prove (1), we notice that

$$\sum_{s\in\mathbb{F}_q} \chi_1(as^2) = 1 + \sum_{s\in\mathbb{F}_q^*} \chi_1(as^2).$$

Since $(-s)^2 = s^2$ for any $s \in \mathbb{F}_q^*$, by a change of variables, $s^2 = t$, we observe that

$$1 + \sum_{s\in\mathbb{F}_q^*} \chi_1(as^2) = 1 + 2 \sum_{\substack{t\in\mathbb{F}_q^*:\\ t \text{ is a square}}} \chi_1(at) = 1 + \sum_{t\in\mathbb{F}_q^*} \chi_1(at)(\eta(t)+1)$$

$$= 1 + \sum_{t\in\mathbb{F}_q^*} \chi_1(at) + \sum_{t\in\mathbb{F}_q^*} \chi_1(at)\eta(t) = \sum_{t\in\mathbb{F}_q^*} \chi_1(at)\eta(t).$$

Here the last equality follows from the orthogonality of characters for χ_1. Now, using a change of variables $t = a^{-1}\theta$ and the relation $\eta(a) = \eta(a^{-1})$, we have

$$\sum_{t\in\mathbb{F}_q^*} \chi_1(at)\eta(t) = \sum_{\theta\in\mathbb{F}_q^*} \eta(a)\chi_1(\theta)\eta(\theta) = \eta(a)\mathcal{G}_1.$$

(2) and (3) follow from (1) by completing the square and using a change of variables. □

2.1. Discrete Fourier analysis machinery

Recall that $\chi_1 : \mathbb{F}_q \to S^1$ denotes the canonical additive character. For a function $f : \mathbb{F}_q^n \to \mathbb{C}$, the Fourier transform of f is defined by

$$\widehat{f}(\alpha) = q^{-n} \sum_{u\in\mathbb{F}_q^n} \chi_1(-u\cdot\alpha) f(u).$$

We list the finite field versions of basic theorems for Fourier analysis.

Proposition 2.7. *Let $f : \mathbb{F}^n \to \mathbb{C}$ be a function. Then we have*

(1) (*Inversion*)

$$f(u) = \sum_{\alpha\in\mathbb{F}_q^n} \chi_1(\alpha\cdot u)\widehat{f}(\alpha).$$

(2) (*Plancherel*)

$$\sum_{\alpha\in\mathbb{F}_q^n} |\widehat{f}(\alpha)|^2 = \frac{1}{q^n} \sum_{u\in\mathbb{F}_q^n} |f(u)|^2.$$

Given subset $A \subset \mathbb{F}_q^n$, let I_A be the characteristic function associated with A, so

$$I_A(x) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

For simplicity, we write $\widehat{A}$ for the Fourier transform $\widehat{I_A}$. For instance, if we take $f \equiv I_A$, then we notice

$$\sum_{\alpha\in\mathbb{F}_q^n} |\widehat{A}(\alpha)|^2 = \frac{|A|}{q^n}. \tag{2.4}$$

Moreover, if $\alpha = (0, \ldots, 0) \in \mathbb{F}_q^n$, then

$$\widehat{A}(0, \ldots, 0) = \frac{|A|}{q^n}. \tag{2.5}$$

The discrete Fourier analysis machinery is very useful in computing the Fourier transform of certain algebraic varieties over finite fields.

3. Reduction and key lemmas

In this section, we will adapt the so-called counting function argument, due to Iosevich and Rudnev [13], to make a reduction of the proof of Theorem 1.4.

From here until Subsection 4.4, we work for the case

$$I = I_0 = (1, 2, \ldots, \ell) \text{ and } J = J_0 = (\ell + 1, \ldots, d),$$

and so use the notations x' and x'' for x_{I_0} and x_{J_0}, respectively.

3.1. Reduction by the counting function argument

We begin with a notation.

Notation 2. For $r \in \mathbb{F}_q$, $S_r^{n-1} = \{\alpha \in \mathbb{F}_q^n \mid \|\alpha\| = r\}$, where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_n^2$. The set S_0^{n-1} is referred to as the zero sphere.

Definition 3.1. Let Φ be the function on $\mathbb{F}_q^d \times \mathbb{F}_q^d$, defined in (1.1) (for $I = I_0, J = J_0$). For $t \in \mathbb{F}_q$ and $E \subset \mathbb{F}_q^d$, we define $\nu_E(t)$ as the number of pair $(x, y) \in E \times E$ such that $\Phi(x, y) = t$.

Remark 3.2. It is clear that $t \in \Phi(E, E)$ if and only if $\nu_E(t) > 0$. In addition, notice that $0 \in \Phi(E, E)$. Hence, in order to prove that the quotient distance set $\Phi(E, E)$ is exactly $\mathbb{F}_q$, it is enough to prove that for each fixed $t \in \mathbb{F}_q^*$, we have $\nu_E(t) > 0$. In other words, the quotient distance problem is reduced to finding a minimal size condition on any set $E \subset \mathbb{F}_q^d$ such that $\nu_E(t) > 0$ for all $t \neq 0$. This procedure is referred to as the counting function argument.

For $t \in \mathbb{F}_q^*$, let $R_t = \{x \in \mathbb{F}_q^d : \Phi(x, \mathbf{0}) = t\}$. By the definition of ν_E, we can write

$$\nu_E(t) = \sum_{x,y \in E : \Phi(x,y) = t} 1 = \sum_{x,y \in \mathbb{F}_q^d} E(x)E(y)R_t(x - y).$$

Applying the Fourier inversion theorem to $R_t(x - y)$, we easily obtain

$$\nu_E(t) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \widehat{R_t}(m) |\widehat{E}(m)|^2. \tag{3.1}$$

Let us measure the size of E such that

$$\nu_E(t) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \widehat{R_t}(m) |\widehat{E}(m)|^2 > 0.$$

First, we compute the Fourier transform $\widehat{R_t}(m)$. For $m \in \mathbb{F}_q^d$, we write $m' = (m_1, \ldots, m_\ell) \in \mathbb{F}_q^\ell$ and $m'' = (m_{\ell+1}, \ldots, m_d) \in \mathbb{F}_q^k$.

Lemma 3.3. *For $t \in \mathbb{F}_q^*$, the Fourier transform $\widehat{R_t}$ can be written*

$$\widehat{R_t}(m) = q^{-1}\delta_0(m) + q^{-d-1}\mathcal{G}_1^d \eta^k(-t)\sum_{s\neq 0}\eta^d(s)\chi_1\left(\frac{t||m'|| - ||m''||}{-4st}\right) - \widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m''),$$

where δ_0 denotes the indicator function $I_{\{0\}}$.

Proof. By the definition of R_t, we have

$$\widehat{R_t}(m) = q^{-d}\sum_{x\in\mathbb{F}_q^d:\Phi(x,0)=t}\chi_1(-m\cdot x).$$

Since $t \neq 0$, by the definition of Φ, we can write

$$\begin{aligned}\widehat{R_t}(m) &= q^{-d}\sum_{x\in\mathbb{F}_q^d:||x''||,||x'||\neq 0,\Phi(x,0)=t}\chi_1(-m\cdot x)\\ &= q^{-d}\sum_{x\in\mathbb{F}_q^d:||x'||-t||x''||=0}\chi_1(-m\cdot x) - q^{-d}\sum_{x\in\mathbb{F}_q^d:||x''||=||x'||=0}\chi_1(-m\cdot x).\end{aligned}$$

By the definitions of the Fourier transform and the zero sphere, the second term above can be written

$$q^{-d}\sum_{x\in\mathbb{F}_q^d:||x''||=||x'||=0}\chi_1(-m\cdot x) = \widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'').$$

By the orthogonality of the additive character χ_1, we have

$$\begin{aligned}&q^{-d}\sum_{x\in\mathbb{F}_q^d:||x'||-t||x''||=0}\chi_1(-m\cdot x)\\ &= q^{-d}\sum_{x\in\mathbb{F}_q^d}q^{-1}\sum_{s\in\mathbb{F}_q}\chi_1(s(||x'||-t||x''||))\chi_1(-m\cdot x).\end{aligned}\tag{3.2}$$

Computing the cases of $s = 0$ and $s \neq 0$ separately, the right-hand side of (3.2) is reduced to

$$q^{-1}\delta_0(m) + q^{-d-1}\sum_{s\neq 0}\sum_{x\in\mathbb{F}_q^d}\chi_1(s||x'|| - m'\cdot x')\chi_1(-st||x''|| - m''\cdot x'').$$

By the formula (3) in Lemma 2.6, this is equal to

$$q^{-1}\delta_0(m) + q^{-d-1}\mathcal{G}_1^d\eta^k(-t)\sum_{s\neq 0}\eta^d(s)\chi_1\left(\frac{t||m'|| - ||m''||}{-4st}\right).$$

Therefore, the lemma follows. $\square$

We can simplify $\Omega := \sum_{s\neq 0}\eta^d(s)\chi_1\left(\frac{t||m'||-||m''||}{-4st}\right)$ into

$$\Omega = \begin{cases} q\delta_0(t||m'||-||m''||)-1 & \text{if } d \text{ is even,}\\ \eta(-t(t||m'||-||m''||))\mathcal{G}_1 & \text{if } d \text{ is odd.}\end{cases} \tag{3.3}$$

Now combining the value Ω with Lemma 3.3, we get the following consequence.

Corollary 3.4. *With the assumption of Lemma 3.3, the following statements hold.*

(1) *If d is even, then $\widehat{R_t}(m)$ is equal to*

$$q^{-1}\delta_0(m)+q^{-d}\mathcal{G}_1^d\eta^k(-t)\delta_0(t||m'||-||m''||)-q^{-d-1}\mathcal{G}_1^d\eta^k(-t)$$
$$-\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'').$$

(2) *If d is odd, then $\widehat{R_t}(m)$ is equal to*

$$q^{-1}\delta_0(m)+q^{-d-1}\mathcal{G}_1^{d+1}\eta^k(-t)\eta\left(-t(t||m'||-||m''||)\right)$$
$$-\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'').$$

Using Corollary 3.4, $\nu_E(t)$ can be expressed more explicitly. Recall $|I|=\ell$ and $|J|=k$, and for $m=(m_1,m_2,\ldots,m_d)$, $m'=(m_1,m_2,\ldots,m_\ell)$, and $m''=(m_{\ell+1},m_{\ell+2},\ldots,m_d)$.

Lemma 3.5. *For $t\in\mathbb{F}_q^*$ and $E\subset\mathbb{F}_q^d$ with $d=\ell+k$, $\nu_E(t)$ can be expressed as follows*:

(1) *If d is even, then*

$$\nu_E(t)=\frac{|E|^2}{q}+q^d\mathcal{G}_1^d\eta^k(-t)\sum_{m\in\mathbb{F}_q^d}\delta_0(t||m'||-||m''||)|\widehat{E}(m)|^2$$
$$-q^{-1}\mathcal{G}_1^d\eta^k(-t)|E|-q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

(2) *If d is odd, then*

$$\nu_E(t)=\frac{|E|^2}{q}+q^{d-1}\mathcal{G}_1^{d+1}\eta^k(-t)\sum_{m\in\mathbb{F}_q^d}\eta\left(-t(t||m'||-||m''||)\right)|\widehat{E}(m)|^2$$
$$-q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

Proof. Inserting the value $\widehat{R_t}(m)$ in Corollary 3.4 into the formula (3.1), the lemma follows immediately from the equalities (2.4) and (2.5). □

3.2. Key lemmas on Fourier transforms on zero spheres

We compute explicitly the Fourier transform on the zero sphere S_0^{n-1} in $\mathbb{F}_q^n$, which was already used in the literature, e.g., [1] and [11].

Lemma 3.6. *For $\alpha \in \mathbb{F}_q^n$, $n \geq 2$, the following statements hold.*

(1) *If n is odd, then we have*

$$\widehat{S_0^{n-1}}(\alpha) = \begin{cases} q^{-1} & \text{if } \alpha = \mathbf{0}, \\ 0 & \text{if } \alpha \neq \mathbf{0}, ||\alpha|| = 0, \\ q^{-n-1}\mathcal{G}_1^{n+1}\eta(-||\alpha||) & \text{if } ||\alpha|| \neq 0. \end{cases}$$

(2) *If n is even, then we have*

$$\widehat{S_0^{n-1}}(\alpha) = \begin{cases} q^{-1} + q^{-n-1}\mathcal{G}_1^n(q-1) & \text{if } \alpha = \mathbf{0}, \\ q^{-n-1}\mathcal{G}_1^n(q-1) & \text{if } \alpha \neq \mathbf{0}, ||\alpha|| = 0, \\ -q^{-n-1}\mathcal{G}_1^n & \text{if } ||\alpha|| \neq 0. \end{cases}$$

Proof. By the orthogonality of χ_1, we see that

$$\begin{aligned} \widehat{S_0^{n-1}}(\alpha) &= q^{-n} \sum_{u \in \mathbb{F}_q^n : \|u\| = 0} \chi_1(-\alpha \cdot u) \\ &= q^{-n-1} \sum_{u \in \mathbb{F}_q^n} \sum_{s \in \mathbb{F}_q} \chi_1(s\|u\|)\, \chi_1(-\alpha \cdot u) \\ &= \frac{\delta_0(\alpha)}{q} + q^{-n-1} \sum_{s \in \mathbb{F}_q^*} \sum_{u \in \mathbb{F}_q^n} \chi_1(s\|u\| - \alpha \cdot u). \end{aligned}$$

By (3) in Lemma 2.6, we have

$$\widehat{S_0^{n-1}}(\alpha) = \frac{\delta_0(\alpha)}{q} + q^{-n-1}\mathcal{G}_1^n \sum_{s \in \mathbb{F}_q^*} \eta^n(s)\chi_1\left(\frac{||\alpha||}{-4s}\right). \tag{3.4}$$

We now claim that (3.4) is reduced to

$$\widehat{S_0^{n-1}}(\alpha) = \begin{cases} q^{-1}\delta_0(\alpha) + q^{-n-1}\mathcal{G}_1^n\left(q\delta_0(||\alpha||) - 1\right) & \text{if } n \text{ is even}, \\ q^{-1}\delta_0(\alpha) + q^{-n-1}\mathcal{G}_1^{n+1}\eta(-||\alpha||) & \text{if } n \text{ is odd}. \end{cases} \tag{3.5}$$

Indeed, in (3.5) the case when d is even follows from the orthogonality of χ_1. If n is odd, then $\eta^n = \eta$, from which the odd case is immediate. Now, using the definition of δ_0 and η (recall that $\eta(0) = 0$), we can easily get the formulas in Lemma 3.6. ☐

Since the absolute value of the Gauss sum is $\sqrt{q}$ (2.3), the following corollary is a direct consequence of Lemma 3.6.

Corollary 3.7. *For $n \geq 1$, a bound on the Fourier transform $\widehat{S_0^{n-1}}$ can be obtained as follows*:

(1) *If n is odd, then*

$$\widehat{S_0^{n-1}}(\mathbf{0}) = q^{-1} \quad \text{and} \quad |\widehat{S_0^{n-1}}(m)| \le q^{-\frac{n+1}{2}} \text{ if } m \neq \mathbf{0}.$$

(2) *If n is even, then*

$$|\widehat{S_0^{n-1}}(m)| \le \begin{cases} 2q^{-1} & \text{if} \quad m = \mathbf{0}, \\ q^{-\frac{n}{2}} & \text{if} \quad m \neq \mathbf{0}. \end{cases}$$

Lemma 3.6 together with the formula (2.3) provides more concrete value of the Fourier transform on zero spheres in some specific cases.

Lemma 3.8. *Assume $\eta(-1) = -1$ and $n \ge 2$. Then the Fourier transform $\widehat{S_0^{n-1}}$ can be explicitly given as follows*:

(1) *If $n \equiv 2 \pmod 4$, then we have*

$$\widehat{S_0^{n-1}}(\alpha) = \begin{cases} q^{-1} - q^{-\frac{n}{2}} + q^{-\frac{n}{2}-1} & \text{if } \alpha = \mathbf{0}, \\ -q^{-\frac{n}{2}} + q^{-\frac{n}{2}-1} & \text{if } \alpha \neq \mathbf{0}, ||\alpha|| = 0, \\ q^{-\frac{n}{2}-1} & \text{if } ||\alpha|| \neq 0. \end{cases}$$

(2) *If $n \equiv 0 \pmod 4$, then we have*

$$\widehat{S_0^{n-1}}(\alpha) = \begin{cases} q^{-1} + q^{-\frac{n}{2}} - q^{-\frac{n}{2}-1} & \text{if } \alpha = \mathbf{0}, \\ q^{-\frac{n}{2}} - q^{-\frac{n}{2}-1} & \text{if } \alpha \neq \mathbf{0}, ||\alpha|| = 0, \\ -q^{-\frac{n}{2}-1} & \text{if } ||\alpha|| \neq 0. \end{cases}$$

Proof. By Lemma 3.6(2), it is enough to show that

$$\mathcal{G}_1^n = \begin{cases} -q^{\frac{n}{2}} & \text{if} \quad n \equiv 2 \pmod 4, \\ q^{\frac{n}{2}} & \text{if} \quad n \equiv 0 \pmod 4. \end{cases} \tag{3.6}$$

From the assumption that $\eta(-1) = -1$, using the formula (2.3), we have

$$\mathcal{G}_1^2 = \eta(-1)q = -q.$$

Indeed, if $n \equiv 2 \pmod 4$, then $n/2$ is odd and hence

$$\mathcal{G}_1^n = [(\mathcal{G}_1)^2]^{n/2} = (\eta(-1)q)^{n/2} = -q^{n/2}. \tag{3.7}$$

Therefore, the case $n \equiv 2 \pmod 4$ follows from (3.7). The case $n \equiv 0 \pmod 4$ is immediate from the equality $\mathcal{G}_1^n = q^{n/2}$. □

To use a simple notation in the proof of the main theorem, we need the following.

Definition 3.9. For ℓ, k with $\ell + k = d$, and $E \subset \mathbb{F}_q^d$, we define

$$A_E(\ell, k) = q^{2d} \sum_{m \in \mathbb{F}_q^d} |\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')||\widehat{E}(m)|^2.$$

It is clear that $0 \le A_E(\ell, k)$. As we shall see, a lower bound of $-A_E(\ell, k)$ (or an upper bound of $A_E(\ell, k)$) plays a crucial role in proving Theorem 1.4.

Lemma 3.10. *Assume $\ell \geq k \geq 1$ with $\ell + k \geq 3$. Then an upper bound on $A_E(\ell, k)$ is given as follows*:

(1) *If ℓ, k are odd, then*

$$A_E(\ell,k) \leq q^{-2}|E|^2 + q^{d-1-\frac{k+1}{2}}|E| + q^{d-1-\frac{\ell+1}{2}}|E| + q^{d-\frac{\ell+1}{2}-\frac{k+1}{2}}|E|.$$

(2) *If ℓ is even and k is odd, then*

$$A_E(\ell,k) \leq 2q^{-2}|E|^2 + 2q^{d-1-\frac{k+1}{2}}|E| + q^{d-1-\frac{\ell}{2}}|E| + q^{d-\frac{\ell}{2}+\frac{k+1}{2}}|E|.$$

Proof. To estimate an upper bound of $A_E(\ell, k)$, we decompose the sum over $m \in \mathbb{F}_q^d$ as the four subsummands:

$$\sum_{m\in\mathbb{F}_q^d} = \sum_{m'=\mathbf{0}=m''} + \sum_{m'=\mathbf{0},m''\neq\mathbf{0}} + \sum_{m'\neq\mathbf{0},m''=\mathbf{0}} + \sum_{m',m''\neq\mathbf{0}}.$$

Then $A_E(t)$ is the same as

$$q^{2d}\left(\sum_{m'=\mathbf{0}=m''} + \sum_{m'=\mathbf{0},m''\neq\mathbf{0}} + \sum_{m'\neq\mathbf{0},m''=\mathbf{0}} + \sum_{m',m''\neq\mathbf{0}}\right)|\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')||\widehat{E}(m)|^2.$$

Now we prove (1). Since ℓ, k are odd, we can use Corollary 3.7(1) to estimate $|\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')|$. Then by a direct algebra, we have

$$\begin{aligned}
&A_E(\ell,k)\\
&\leq q^{2d}q^{-2}|\widehat{E}(\mathbf{0},\mathbf{0})|^2 + q^{2d}\sum_{m''\neq\mathbf{0}} q^{-1}q^{-\frac{k+1}{2}}|\widehat{E}(\mathbf{0},m'')|^2\\
&\quad + q^{2d}\sum_{m'\neq\mathbf{0}} q^{-\frac{\ell+1}{2}}q^{-1}|\widehat{E}(m',\mathbf{0})|^2 + q^{2d}\sum_{m',m''\neq\mathbf{0}} q^{-\frac{\ell+1}{2}}q^{-\frac{k+1}{2}}|\widehat{E}(m',m'')|^2.
\end{aligned}$$

From the fact that each of the above sums $\sum_{\cdot}|\widehat{E}(\cdot)|^2$ is dominated by $\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2 = q^{-d}|E|$, and $\widehat{E}(\mathbf{0},\mathbf{0}) = q^{-d}|E|$, the proof follows. Next we prove (2), whose proof is similar to that of the first case. Since ℓ is even and k is odd, we apply Corollary 3.7(2) and Corollary 3.7(1) to estimate $|\widehat{S_0^{\ell-1}}(m')|$ and $|\widehat{S_0^{k-1}}(m'')|$, respectively. Then it is not hard to check that

$$A_E(\ell,k) \leq 2q^{-2}|E|^2 + 2q^{d-1-\frac{k+1}{2}}|E| + q^{d-1-\frac{\ell}{2}}|E| + q^{d-\frac{\ell}{2}+\frac{k+1}{2}}|E|.$$

This completes the proof. □

We also use the following definition for a short notation.

Definition 3.11. For $E \subset \mathbb{F}_q^d$, we define

$$\widetilde{A_E}(\ell,k) = q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

Notice that the Fourier transform on the zero sphere is a real valued function. So $\widetilde{A_E}(\ell,k)$ is a real number. To prove (3) of Theorem 1.4, we will need a more accurate estimate on the value $\widetilde{A_E}(\ell,k)$. More precisely, we need the following.

Lemma 3.12. *Assume ℓ is odd, k is even, and $(\eta(-1))^{\frac{k}{2}}=-1$. Then we have*

$$\widetilde{A_E}(\ell,k)\le \frac{|E|^2}{q^2}+q^{d-2-\frac{k}{2}}|E|+q^{d-1-\frac{\ell+1}{2}}|E|+q^{d-\frac{\ell+1}{2}-\frac{k}{2}}|E|+q^{d-1-\frac{\ell+1}{2}-\frac{k}{2}}|E|.$$

Proof. From the estimate (3.5), we can write

$$\begin{aligned}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')= &\left(q^{-1}\delta_0(m')+q^{-\ell-1}\mathcal{G}_1^{\ell+1}\eta(-||m'||)\right)\\ &\times\left(q^{-1}\delta_0(m'')+q^{-k-1}\mathcal{G}_1^{k}\left(q\delta_0(||m''||)-1\right)\right).\end{aligned}$$

After expanding this, we plug it into

$$\widetilde{A_E}(\ell,k)=q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

Then, by a direct computation, we see that $\widetilde{A_E}(\ell,k)$ can be written as

$$\begin{aligned}&q^{2d-2}\sum_{m\in\mathbb{F}_q^d}\delta_0(m')\delta_0(m'')|\widehat{E}(m)|^2+q^{2d-1-k}\mathcal{G}_1^k\sum_{m\in\mathbb{F}_q^d}\delta_0(m')\delta_0(||m^{''}||)|\widehat{E}(m)|^2\\ &-q^{2d-2-k}\mathcal{G}_1^k\sum_{m\in\mathbb{F}_q^d}\delta_0(m')|\widehat{E}(m)|^2+q^{2d-2-\ell}\mathcal{G}_1^{\ell+1}\sum_{m\in\mathbb{F}_q^d}\eta(-||m^{'}||)\delta_0(m^{''})|\widehat{E}(m)|^2\\ &+q^{2d-1-\ell-k}\mathcal{G}_1^{\ell+1}\mathcal{G}_1^k\sum_{m\in\mathbb{F}_q^d}\eta(-||m^{'}||)\delta_0(m^{''})|\widehat{E}(m)|^2\\ &-q^{2d-2-\ell-k}\mathcal{G}_1^{\ell+1}\mathcal{G}_1^k\sum_{m\in\mathbb{F}_q^d}\eta(-||m^{'}||)|\widehat{E}(m)|^2=:I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}$$

Recall from (2.5) that $|\widehat{E}(\mathbf{0},\mathbf{0})|=q^{-d}|E|$. From this, we obtain $I_1=\frac{|E|^2}{q^2}$. To estimate I_2, notice that $k\equiv 2 \pmod 4$, which follows from our assumptions that k is even and $(\eta(-1))^{\frac{k}{2}}=-1$. Hence, by (3.6), we have $\mathcal{G}_1^k=-q^{\frac{k}{2}}<0$. This implies that I_2 is a non-positive integer so that we can remove I_2 when we estimate an upper bound of $\widetilde{A_E}(\ell,k)$.

Since $\delta_0(m')\le 1$ for all $m\in\mathbb{F}_q^d$, we have

$$I_3\le q^{2d-2-k}|\mathcal{G}_1|^k\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2=q^{2d-2-k}q^{\frac{k}{2}}q^{-d}|E|. \tag{3.8}$$

The equality in (3.8) follows from the facts that

$$|\mathcal{G}_1|=\sqrt{q}\quad\text{and}\quad\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2=q^{-d}|E|.$$

To estimate the terms I_4, I_5, I_6, we use the same argument as that for the term I_3. Then one can easily show that

$$I_4 \le q^{d-1-\frac{\ell+1}{2}}|E|, \quad I_5 \le q^{d-\frac{\ell+1}{2}-\frac{k}{2}}|E|, \quad I_6 \le q^{d-1-\frac{\ell+1}{2}-\frac{k}{2}}|E|.$$

Hence, the theorem follows by putting all estimates together. □

We also need the following estimates, for which we heavily use the explicit value of the Fourier transform on the zero sphere in Lemma 3.8.

Lemma 3.13. *Suppose that* $\eta(-1) = -1$, $\ell \equiv 0 \pmod 4)$, *and* $k \equiv 2 \pmod 4$. *Then we have*

$$\widetilde{A_E}(\ell, k) \le \frac{2|E|^2}{q^2} + 2q^{d-2-\frac{k}{2}}|E| + q^{d-1-\frac{\ell}{2}}|E| + 2q^{\frac{d-2}{2}}|E|.$$

Proof. We want to find a larger real number than

$$\widetilde{A_E}(\ell, k) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

Therefore, we may ignore some negative terms that will appear in computing an upper bound of $\widetilde{A_E}(\ell, k)$. Hence, we break down the above sum $\sum_{m\in\mathbb{F}_q^d}$ into 9 subsummands:

$$\begin{aligned} &\sum_{m',m''=\mathbf{0}} + \sum_{m'=\mathbf{0},||m''||=0,m''\neq\mathbf{0}} + \sum_{m'=\mathbf{0},||m''||\neq 0} \\ &+ \sum_{||m'||=0,m'\neq\mathbf{0},m''=\mathbf{0}} + \sum_{\substack{||m'||=0,m'\neq\mathbf{0},\\ ||m''||=0,m''\neq\mathbf{0}}} + \sum_{\substack{||m'||=0,m'\neq\mathbf{0},\\ ||m''||\neq 0}} \\ &+ \sum_{||m'||\neq 0,m''=\mathbf{0}} + \sum_{\substack{||m'||\neq 0,\\ ||m''||=0,m''\neq\mathbf{0}}} + \sum_{||m'||\neq 0,||m''||\neq\mathbf{0}} \\ =:&\ J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9, \end{aligned}$$

and then we only consider such sums that $\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')$ takes a positive value, which can be easily evaluated by Lemma 3.8. More precisely, Lemma 3.8 with our assumptions on ℓ, k, implies that

$$\widehat{S_0^{\ell-1}}(m') = \begin{cases} q^{-1} + q^{-\frac{\ell}{2}} - q^{-\frac{\ell}{2}-1} & \text{if } m' = \mathbf{0}, \\ q^{-\frac{\ell}{2}} - q^{-\frac{\ell}{2}-1} & \text{if } m' \neq \mathbf{0}, ||m'|| = 0, \\ -q^{-\frac{\ell}{2}-1} & \text{if } ||m'|| \neq 0, \end{cases}$$

and

$$\widehat{S_0^{k-1}}(m'') = \begin{cases} q^{-1} - q^{-\frac{k}{2}} + q^{-\frac{k}{2}-1} & \text{if } m'' = \mathbf{0}, \\ -q^{-\frac{k}{2}} + q^{-\frac{k}{2}-1} & \text{if } m'' \neq \mathbf{0}, ||m''|| = 0, \\ q^{-\frac{k}{2}-1} & \text{if } ||m''|| \neq 0. \end{cases}$$

Notice that $J_2, J_5, J_7, J_9 \leq 0$. Hence we have

$$\widetilde{A_E}(\ell,k) \leq J_1 + J_3 + J_4 + J_6 + J_8.$$

J_1 is bounded above by $2q^{-2}|E|^2$ since

$$J_1 \leq q^{2d}(q^{-1}+q^{-\frac{\ell}{2}})q^{-1}|\widehat{E}(\mathbf{0},\mathbf{0})|^2 \leq 2q^{2d}q^{-2}(q^{-d}|E|)^2.$$

J_3 is bounded above by $2q^{d-2-\frac{k}{2}}$, because

$$J_3 \leq q^{2d}\sum_{m\in\mathbb{F}_q^d} 2q^{-1}q^{-\frac{k}{2}-1}|\widehat{E}(m)|^2 = 2q^{2d}q^{-1}q^{-\frac{k}{2}-1}q^{-d}|E|.$$

Similarly, a direct computation shows that

$$J_4 \leq q^{d-1-\frac{\ell}{2}}|E|.$$

J_6 is bounded above by $q^{\frac{d}{2}-1}|E|$ since

$$J_6 \leq q^{2d}\sum_{m\in\mathbb{F}_q^d} q^{-\frac{\ell}{2}}q^{-\frac{k}{2}-1}|\widehat{E}(m)|^2 = q^{2d}q^{-\frac{\ell}{2}}q^{-\frac{k}{2}-1}q^{-d}|E| = q^{\frac{d}{2}-1}|E|.$$

Similarly, we can show that J_8 is bounded above by $q^{\frac{d}{2}-1}|E|$. Putting together all estimates above, the lemma follows. □

We set

$$\Gamma(E) := q^{\frac{3d}{2}} \sum_{\substack{m\in\mathbb{F}_q^d:||m'||=0\\ ||m''||=0}} |\widehat{E}(m)|^2.$$

The following estimate is one of the key ingredients in the proof of our main result.

Lemma 3.14. *Suppose that $\eta(-1)=-1$, $\ell\equiv 2 \pmod 4$, and $k\equiv 2 \pmod 4$. Then for $E\subset\mathbb{F}_q^d$, with $d=\ell+k$, we have*

$$\widetilde{A_E}(\ell,k) - \Gamma(E) \leq \frac{|E|^2}{q^2} + q^{d-2-\frac{k}{2}}|E| + q^{d-2-\frac{\ell}{2}}|E| + q^{\frac{d-4}{2}}|E|.$$

Proof. The proof will proceed by the same argument as in Lemma 3.13. Let $J_j, 1\leq j\leq 9$, be the term defined in the proof of Lemma 3.13. Namely,

$$\widetilde{A_E}(\ell,k) = q^{2d}\sum_{m\in\mathbb{F}_q^d} \widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2 = \sum_{i=1}^{9} J_i.$$

Unlike in the proof of Lemma 3.13, here we only invoke Lemma 3.8(1) with ℓ, k both odd. Then we see that

$$\widehat{S_0^{\ell-1}}(m') = \begin{cases} q^{-1}-q^{-\frac{\ell}{2}}+q^{-\frac{\ell}{2}-1} & \text{if } m'=\mathbf{0}, \\ -q^{-\frac{\ell}{2}}+q^{-\frac{\ell}{2}-1} & \text{if } m'\neq\mathbf{0}, ||m'||=0, \\ q^{-\frac{\ell}{2}-1} & \text{if } ||m'||\neq 0, \end{cases}$$

and

$$\widehat{S_0^{k-1}}(m'') = \begin{cases} q^{-1} - q^{-\frac{k}{2}} + q^{-\frac{k}{2}-1} & \text{if } m'' = \mathbf{0}, \\ -q^{-\frac{k}{2}} + q^{-\frac{k}{2}-1} & \text{if } m'' \neq \mathbf{0}, ||m''|| = 0, \\ q^{-\frac{k}{2}-1} & \text{if } ||m''|| \neq 0. \end{cases}$$

Notice that $J_2, J_4, J_6, J_8 \leq 0$. Hence we have

$$\widetilde{A_E}(\ell, k) \leq J_1 + J_3 + J_5 + J_7 + J_9.$$

Equivalently we can write

$$\widetilde{A_E}(\ell, k) - \Gamma(E) \leq J_1 + J_3 + (J_5 - \Gamma(E)) + J_7 + J_9.$$

Since $d = \ell + k$, it is not hard to check that $J_5 - \Gamma(E) \leq 0$. Indeed, we have

$$\begin{aligned} J_5 &= q^{2d} \sum_{\substack{m' \neq \mathbf{0}, ||m'||=0 \\ m'' \neq \mathbf{0}, ||m''||=0}} \left(-q^{-\frac{\ell}{2}} + q^{-\frac{\ell}{2}-1}\right)\left(-q^{-\frac{k}{2}} + q^{-\frac{k}{2}-1}\right) |\widehat{E}(m)|^2 \\ &\leq q^{2d} \sum_{\substack{m' \neq \mathbf{0}, ||m'||=0 \\ m'' \neq \mathbf{0}, ||m''||=0}} q^{-\frac{\ell}{2}} q^{-\frac{k}{2}} |\widehat{E}(m)|^2 \leq \Gamma(E). \end{aligned}$$

Thus we get

$$\widetilde{A_E}(\ell, k) - \Gamma(E) \leq J_1 + J_3 + J_7 + J_9.$$

As in the proof of Lemma 3.13, a direct computation shows that

$$\begin{aligned} J_1 &\leq q^{2d} q^{-1} q^{-1} |\widehat{E}(\mathbf{0}, \mathbf{0})|^2 = q^{-2}|E|^2, \\ J_3 &\leq q^{2d} \sum_{m \in \mathbb{F}_q^d} q^{-1} q^{-\frac{k}{2}-1} |\widehat{E}(m)|^2 = q^{d-2-\frac{k}{2}}|E|, \\ J_7 &\leq q^{2d} \sum_{m \in \mathbb{F}_q^d} q^{-\frac{\ell}{2}-1} q^{-1} |\widehat{E}(m)|^2 = q^{d-2-\frac{\ell}{2}}|E|, \end{aligned}$$

and

$$J_9 \leq q^{2d} \sum_{m \in \mathbb{F}_q^d} q^{-\frac{\ell}{2}-1} q^{-\frac{k}{2}-1} |\widehat{E}(m)|^2 = q^{d-2-\frac{\ell+k}{2}}|E| = q^{\frac{d}{2}-2}|E|.$$

Putting together all the estimates above gives the desired result. $\square$

4. Proof of the main result

In this section, we give a proof of Theorem 1.4. Propositions 4.1 and 4.3 together will prove the special case, i.e., $I = I_0, J = J_0$ and $\ell \geq k$, of Theorem 1.4. Recall that $d = \ell + k \geq 3$ and $\ell, k \geq 1$. A proof of the general case will be given in the last part.

Proposition 4.1. *Assume $\ell \geq k$. Then a lower bound on $Index(\Phi)$ is given as follows*:

(1) *If k is odd, then $Index(\Phi) \geq d - \frac{k+1}{2}$.*

(2) *If k is even and* $(\eta(-1))^{\frac{k}{2}} = 1$, *then* $Index(\Phi) \geq d - \frac{k}{2}$.
(3) *If k is even and* $(\eta(-1))^{\frac{k}{2}} = -1$, *then* $Index(\Phi) \geq d - \frac{k+2}{2}$.

To give the proof of Proposition 4.1, we invoke the following result.

Lemma 4.2 (Lemma 2.1, [20])**.** *If H is a subspace of maximal dimension contained in the zero sphere S_0^{n-1}, then we have the following facts*:

(1) *If n is odd, then* $|H| = q^{\frac{n-1}{2}}$.
(2) *If n is even,* $(\eta(-1))^{\frac{n}{2}} = 1$ *then* $|H| = q^{\frac{n}{2}}$.
(3) *If n is even,* $(\eta(-1))^{\frac{n}{2}} = -1$ *then* $|H| = q^{\frac{n-2}{2}}$.

4.1. Proof of Proposition 4.1

Let H be a maximal subspace lying in the zero sphere S_0^{k-1} in $\mathbb{F}_q^k$. Set $E = \mathbb{F}_q^\ell \times H$. Then it is clear that

$$E \subset \mathbb{F}_q^d,\ |E| = q^\ell \times |H|. \tag{4.1}$$

By Lemma 4.2, we have

$$|E| = \begin{cases} q^{\ell+\frac{k-1}{2}} & \text{if} \quad k \text{ is odd,} \\ q^{\ell+\frac{k}{2}} & \text{if} \quad k \text{ is even and } (\eta(-1))^{\frac{k}{2}} = 1, \\ q^{\ell+\frac{k-2}{2}} & \text{if} \quad k \text{ is even and } (\eta(-1))^{\frac{k}{2}} = -1. \end{cases} \tag{4.2}$$

Now we show that

$$\Phi(E,E) = \{0\}.$$

To prove this, by the definition of Φ, it is enough to prove that

$$\Delta(H) := \{||\alpha - \beta|| : \alpha, \beta \in H\} = \{0\}. \tag{4.3}$$

Once this is proved, the formula (4.2) implies Proposition 4.1. Let us justify (4.3). Since H is a subspace contained in the zero sphere S_0^{k-1}, we see that for $\alpha, \beta \in H \subset S_0^{k-1}$, we have $\alpha - \beta$ is also contained in the zero sphere. Hence, $||\alpha - \beta|| = 0$. This proves Proposition 4.1. Now we give an upper bound of the index of Φ.

Proposition 4.3. *Assume $\ell \geq k$. Then an upper bound on $Index(\Phi)$ is given as follows*:

(1) *If k is odd, then* $Index(\Phi) \leq d - \frac{k+1}{2}$.
(2) *If k is even and* $(\eta(-1))^{\frac{k}{2}} = 1$, *then* $Index(\Phi) \leq d - \frac{k}{2}$.
(3) *If k is even and* $(\eta(-1))^{\frac{k}{2}} = -1$, *then* $Index(\Phi) \leq d - \frac{k+2}{2}$.

4.2. Proof of (1) in Proposition 4.3

Assume k is an odd integer. By the definition of the index of Φ, it is enough to prove that there is sufficiently large $C > 0$ such that for each $t \in \mathbb{F}_q^*$, $\nu_E(t) > 0$ whenever E is a subset $\mathbb{F}_q^d$ with $|E| \geq Cq^{d-\frac{k+1}{2}}$. Since the value of $\nu_E(t)$ depends on the dimension, we consider two cases separately.

Case 1: d is even. In this case, ℓ is odd since k is odd. Fix $t \neq 0$. Then it follows from Lemma 3.5(1) that

$$\nu_E(t) = \frac{|E|^2}{q} + q^d \mathcal{G}_1^d \eta^k(-t) \sum_{m\in\mathbb{F}_q^d} \delta_0(t||m'|| - ||m''||)|\widehat{E}(m)|^2$$
$$- q^{-1}\mathcal{G}_1^d \eta^k(-t)|E| - q^{2d} \sum_{m\in\mathbb{F}_q^d} \widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

We will show that if $|E| \geq Cq^{d-\frac{k+1}{2}}$, then a lower bound of $\nu_E(t)$ is positive. By the trivial estimate for a lower bound, it follows that

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^d|\mathcal{G}_1|^d \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2$$
$$- q^{-1}|\mathcal{G}_1|^d|E| - q^{2d} \sum_{m\in\mathbb{F}_q^d} |\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')||\widehat{E}(m)|^2.$$

Since $|\mathcal{G}_1|^d = q^{d/2}$ and $\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-d}|E|$, we obtain

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{d/2}|E| - q^{d/2-1}|E| - A_E(\ell,k).$$

Since ℓ, k are odd, it follows from Lemma 3.10(1) that

$$0 \leq A_E(\ell,k) \leq q^{-2}|E|^2 + q^{d-1-\frac{k+1}{2}}|E| + q^{d-1-\frac{\ell+1}{2}}|E| + q^{d-\frac{\ell+1}{2}-\frac{k+1}{2}}|E|.$$

Combining this estimate with the above lower bound of $\nu_E(t)$, we get

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{d/2}|E| - q^{d/2-1}|E|$$
$$- q^{-2}|E|^2 - q^{d-1-\frac{k+1}{2}}|E| - q^{d-1-\frac{\ell+1}{2}}|E| - q^{d-\frac{\ell+1}{2}-\frac{k+1}{2}}|E|.$$

We note that the first term $\frac{|E|^2}{q}$ of RHS of the above inequality dominates all other terms of it and hence $\nu_E(t) > 0$, provided that

$$|E| \gg \max\left\{q^{\frac{d+2}{2}}, q^{d-\frac{k+1}{2}}, q^{d-\frac{\ell+1}{2}}, q^{d+1-\frac{\ell+1}{2}-\frac{k+1}{2}}\right\}.$$

Since $\ell \geq k$, the maximum value is the same as $q^{d-\frac{k+1}{2}}$. Hence, if $|E| \gg q^{d-\frac{k+1}{2}}$, then $\nu_E(t) > 0$, as required.

Case 2: d is odd.

The proof is almost identical to that of Case 1. In this case, ℓ is even. To find a lower bound of $\nu_E(t)$, we use Lemma 3.5(2). As in Case 1, we have

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d-1}{2}}|E| - A_E(\ell,k). \tag{4.4}$$

Since ℓ is even and k is odd, it follows from Lemma 3.10(2) that

$$A_E(\ell,k) \leq 2q^{-2}|E|^2 + 2q^{d-1-\frac{k+1}{2}}|E| + q^{d-1-\frac{\ell}{2}}|E| + q^{d-\frac{\ell}{2}+\frac{k+1}{2}}|E|.$$

From this estimate and (4.4), we see that

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d-1}{2}}|E|$$
$$- 2q^{-2}|E|^2 - 2q^{d-1-\frac{k+1}{2}}|E| - q^{d-1-\frac{\ell}{2}}|E| - q^{d-\frac{\ell}{2}-\frac{k+1}{2}}|E|.$$

Since $\ell \geq k$, the first term $\frac{|E|^2}{q}$ dominates all other terms in RHS of the above inequality, provided that $|E| \geq Cq^{d-\frac{k+1}{2}}$. Namely, if $|E| \geq Cq^{d-\frac{k+1}{2}}$, then $\nu_E(t) > 0$. This completes the proof of Case 2.

4.3. Proof of (2) in Proposition 4.3

This proof proceeds as in the proof of (1) in Proposition 4.3. Inserting the value $\widehat{R_t}(m)$ of Lemma 3.3 into the formula (3.1), we obtain

$$\nu_E(t) = \frac{|E|^2}{q} + q^{d-1}\mathcal{G}_1^d\eta^k(-t)\sum_{m\in\mathbb{F}_q^d}\sum_{s\in\mathbb{F}_q^*}\eta^d(s)\chi_1\left(\frac{t||m'|| - ||m''||}{-4st}\right)|\widehat{E}(m)|^2$$
$$- q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

It is obvious that

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{d-1}|\mathcal{G}_1|^d\sum_{m\in\mathbb{F}_q^d}(q-1)|\widehat{E}(m)|^2$$
$$- q^{2d}\sum_{m\in\mathbb{F}_q^d}|\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')||\widehat{E}(m)|^2.$$

By the Gauss sum estimate (2.3) and the fact that $\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2 = q^{-d}|E|$,

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d}{2}}|E| - A_E(\ell,k). \tag{4.5}$$

Decompose $A_E(\ell,k)$ as

$$q^{2d}\left(\sum_{m'=\mathbf{0}=m''} + \sum_{m'=\mathbf{0},m''\neq\mathbf{0}} + \sum_{m'\neq\mathbf{0},m''=\mathbf{0}} + \sum_{m',m''\neq 0}\right)|\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')||\widehat{E}(m)|^2.$$

Notice that Corollary 3.7 implies that for any dimension n,

$$|\widehat{S_0^{n-1}}(\mathbf{0})| \leq 2q^{-1} \quad \text{and} \quad |\widehat{S_0^{n-1}}(\mathbf{0})| \leq q^{\frac{n}{2}} \quad \text{for all } m \neq \mathbf{0}.$$

We use these facts to estimate $|\widehat{S_0^{\ell-1}}(m')||\widehat{S_0^{k-1}}(m'')|$. Then we can obtain

$$A_E(\ell,k) \leq 4q^{-2}|E|^2 + 2q^{d-1-\frac{k}{2}}|E| + 2q^{d-1-\frac{\ell}{2}}|E| + q^{d-\frac{\ell}{2}-\frac{k}{2}}|E|.$$

Combining this estimate and (4.5) gives

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d}{2}}|E| - 4q^{-2}|E|^2 - 2q^{d-1-\frac{k}{2}}|E| - 2q^{d-1-\frac{\ell}{2}}|E| - q^{d-\frac{\ell}{2}-\frac{k}{2}}|E|.$$

Since $\ell \geq k$, a direct computation shows that if $|E| \geq Cq^{d-\frac{k}{2}}$, then $\nu_E(t) > 0$. This finishes the proof.

4.4. Proof of (3) in Proposition 4.3

We recall that our assumption of (3) is equivalent to the condition that $k \equiv 2 \pmod 4$ and $\eta(-1) = -1$. To prove this, we treat separately two cases: d is even, and d is odd.

Case 1: d is odd. In this case, ℓ is odd, because k is even. By Lemma 3.5(2), we have

$$\nu_E(t) = \frac{|E|^2}{q} + q^{d-1}\mathcal{G}_1^{d+1}\eta^k(-t)\sum_{m\in\mathbb{F}_q^d}\eta\left(-t(t||m'|| - ||m''||)\right)|\widehat{E}(m)|^2 - q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S_0^{\ell-1}}(m')\widehat{S_0^{k-1}}(m'')|\widehat{E}(m)|^2.$$

By the trivial estimate, this implies that

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{d-1}|\mathcal{G}_1|^{d+1}\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2 - \widetilde{A_E}(\ell,k).$$

Using the formulas (2.3), (2.4), and Lemma 3.12, we obtain

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d-1}{2}}|E| - \frac{|E|^2}{q^2} - q^{d-2-\frac{k}{2}}|E| - q^{d-1-\frac{\ell+1}{2}}|E| - q^{d-\frac{\ell+1}{2}-\frac{k}{2}}|E| - q^{d-1-\frac{\ell+1}{2}-\frac{k}{2}}|E|.$$

By a direct comparison, since $\ell \geq k$, the first term $\frac{|E|^2}{q}$ dominates the other terms of the RHS of the above inequality, provided that $|E| \geq Cq^{d-1-\frac{k}{2}}$. Thus, if $|E| \geq Cq^{d-1-\frac{k}{2}}$, then $\nu_E(t) > 0$, which completes the proof.

Case 2: d is even.

In this case, ℓ is even. By the assumption, we have $k \equiv 2 \pmod 4$. Then by Lemma 3.5(1) we have

$$\nu_E(t) = \frac{|E|^2}{q} + q^d\mathcal{G}_1^d\sum_{m\in\mathbb{F}_q^d}\delta_0(t||m'|| - ||m''||)|\widehat{E}(m)|^2 - q^{-1}\mathcal{G}_1^d|E| - \widetilde{A_E}(\ell,k). \tag{4.6}$$

We will separately deal with two cases:

$$\ell \equiv 0 \pmod 4 \quad \text{and} \quad \ell \equiv 2 \pmod 4.$$

Case 2-(1) Assume that $\ell \equiv 0 \pmod 4$.

Since $k \equiv 2 \pmod 4$, we see that $d \equiv 2 \pmod 4$. Thus, by the formula (3.6), we have

$$\begin{aligned}\nu_E(t) = \frac{|E|^2}{q} - q^{\frac{3d}{2}} \sum_{m \in \mathbb{F}_q^d} \delta_0(t||m'|| - ||m''||)|\widehat{E}(m)|^2 \\ + q^{\frac{d-2}{2}}|E| - \widetilde{A_E}(\ell, k).\end{aligned}$$

Applying a trivial bound and the Plancherel theorem to the above summation, we have

$$\begin{aligned}0 &\leq \sum_{m \in \mathbb{F}_q^d} \delta_0(t||m'|| - ||m''||)|\widehat{E}(m)|^2 \\ &\leq \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-d}|E|.\end{aligned}$$

Hence, we get

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d}{2}}|E| + q^{\frac{d-2}{2}}|E| - \widetilde{A_E}(\ell, k).$$

Using a lower bound of $-\widetilde{A_E}(\ell, k)$ in Lemma 3.13, we have

$$\begin{aligned}\nu_E(t) \geq& \frac{|E|^2}{q} - q^{\frac{d}{2}}|E| + q^{\frac{d-2}{2}}|E| \\ &- \frac{2|E|^2}{q^2} - 2q^{d-2-\frac{k}{2}}|E| - q^{d-1-\frac{\ell}{2}}|E| - 2q^{\frac{d-2}{2}}|E|.\end{aligned}$$

By a direct computation, we notice that the first value $\frac{|E|^2}{q}$ of the RHS dominates all other terms, so $\nu_E(t) > 0$, provided that

$$|E| \geq C \max\left\{q^{\frac{d+2}{2}}, q^{d-1-\frac{k}{2}}, q^{d-\frac{\ell}{2}}\right\}.$$

It remains to show that the maximum value takes the value $q^{d-1-\frac{k}{2}}$. But this is obvious since $d \geq k+4$ and $\ell \geq k+2$ by the assumption. This proves Case 2-(1).

Case 2-(2) Assume that $\ell \equiv 2 \pmod 4$.

In this case, $d \equiv 0 \pmod 4$ and so $\mathcal{G}_1^d = q^{\frac{d}{2}}$. Hence, (4.6) becomes

$$\begin{aligned}\nu_E(t) = \frac{|E|^2}{q} + q^{\frac{3d}{2}} \sum_{m \in \mathbb{F}_q^d} \delta_0(t||m'|| - ||m''||)|\widehat{E}(m)|^2 \\ - q^{\frac{d-2}{2}}|E| - \widetilde{A_E}(\ell, k).\end{aligned}$$

Since the second term above is greater than or equal to

$$\Gamma(E) = q^{\frac{3d}{2}} \sum_{m \in \mathbb{F}_q^d : ||m'||=0=||m''||} |\widehat{E}(m)|^2,$$

we have

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d-2}{2}}|E| - (\widetilde{A_E}(\ell, k) - \Gamma(E)).$$

Using Lemma 3.14, we obtain

$$\nu_E(t) \geq \frac{|E|^2}{q} - q^{\frac{d-2}{2}}|E| - \frac{|E|^2}{q^2} - q^{d-2-\frac{k}{2}}|E| - q^{d-2-\frac{\ell}{2}}|E| - q^{\frac{d-4}{2}}|E|.$$

Since $\ell \geq k$, one can check that if $|E| \geq Cq^{d-1-\frac{k}{2}}$, then $\nu_E(t) > 0$. This proves Case 2-(2).

4.5. Completion of the proof of Theorem 1.4

We complete the proof of the main theorem.

Case (i): $k = \min\{\ell, k\}$, and $I = I_0$, $J = J_0$.

This case is immediate from Propositions 4.1 and 4.3.

Case (ii): $\ell = \min\{\ell, k\}$, and $I = I_0, J = J_0$. For this case, consider the automorphism of the set $\mathbb{F}_q^d$ defined by

$$\Pi(x) = \Pi(x', x'') := (x'', x').$$

Let $\widetilde{E}$ denote the image of E under Π. Then for each $t \in \mathbb{F}_q^*$, we see that $\nu_{\widetilde{E}}(t) = \nu_E(\frac{1}{t})$. Thus $\nu_E(t)$ and $\nu_{\widetilde{E}}(t)$ have the same bound, since the bound was taken independently of t. Thus, if we apply Case (i) to $\nu_{\widetilde{E}}(t)$, Case (ii) follows.

Case (iii): General I, J.

For this case, define $\Lambda : \mathbb{F}_q^d \to \mathbb{F}_q^d$ by

$$\Lambda(x_1, \ldots, x_d) := (x_{i_1}, \ldots, x_{i_\ell}, x_{j_1}, \ldots, x_{j_k}).$$

We may think that the domain (resp. the target) space $\mathbb{F}_q^d$ is a "metric space" equipped with the "metric" $\Phi(I, J)$ (resp. $\Phi(I_0, J_0)$). Now note that Λ is a bijection and Λ preserves the distance. Thus, the domain and target spaces are identified with each other as metric spaces. Therefore, by the definition of the index, the indices of $\Phi(I, J)$ and $\Phi(I_0, J_0)$ are equal. This completes the proof Theorem 1.4.

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