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COMPUTATIONS AND CONSERVATIVENESS OF TRACES OF ONE-DIMENSIONAL DIFFUSIONS

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Abstract. We compute explicitly traces of one-dimensional diffusion processes. The obtained trace forms can be regarded as Dirichlet forms on graphs. Then we discuss conditions ensuring the trace forms to be conservative. Finally, the obtained results are applied to the Bessel process of order ν.

1. Introduction

Throughout this paper we are concerned with the computation and conservativeness of traces of one-dimensional diffusions generated by the Feller operator $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$. We recall the known fact [3] that such diffusions on an open interval I are characterized by a scale function s, i.e., a continuous strictly increasing function on I and a measure m. Moreover, they are related in an appropriate way to Dirichlet forms with domains in $L^2(I,m)$ defined by

$$\mathcal{E}^{(s)}[u] := \int_I (\frac{\mathrm{d}u}{\mathrm{d}s}(x))^2\,\mathrm{d}s(x)$$

on their domains $\operatorname{dom}\mathcal{E}^{(s)}$. Further details about the form are given in the next section. Given a diffusion of the above type, a positive measure μ with support $V \subset I$ and a linear operator $J : \operatorname{dom}\mathcal{E}^{(s)} \cap L^2(V,\mu) \to L^2(V,\mu)$ we shall first compute the trace of $\mathcal{E}^{(s)}$ with respect to the measure μ by means of the method elaborated in [1]. We shall demonstrate in particular that the obtained trace form in $L^2(V,\mu)$ is in fact a graph Dirichlet form if the measure μ is discrete.

Once the computation has been performed we shall turn our attention to study the conservativeness property, i.e., conservation of total mass, for the trace Dirichlet form. We shall show that, for fixed $\mathcal{E}^{(s)}$, conservativeness depends strongly on the measure μ and its support V.

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The motivation rests on two facts: first to put the particular case for the Bessel's process analyzed in [2] in a general framework. Second, the significance of the conservativeness property both in analysis and in probability. In fact in analysis conservativeness is equivalent to the existence and uniqueness of a bounded stationnary solution of the heat equation. Whereas in probability conservativeness implies that, almost surely, the related stochastic process starting at any point will have an infinite lifetime.

At this stage we mention that there is a huge literature concerned with conservative Dirichlet forms. Regarding the subject we refer the reader to [6–8, 11, 14, 15].

The paper is organized as follows. In Section 2 we introduce some necessary definitions and notations concerning Dirichlet forms related to one-dimensional diffusions as well as Feller's classification of boundary points. Section 3 is devoted to the computation of the trace of the considered Dirichlet forms on discrete sets as well as on composites of continuous and discrete sets. In Section 4 we study the conservativeness property for the traces on discrete sets. In this respect we shall give necessary and sufficient conditions ensuring the trace form to be conservative. Thereby we extend [2, Theorem 3.7] to this general framework. The obtained theoretical results will be applied to the Bessel process of order ν, in the last section.

2. Framework and basic notations

We start by introducing some notations.

Let $I := (r_1, r_2)$, where $-\infty \le r_1 < r_2 \le \infty$. Let us consider a continuous strictly increasing function $s : I \to \mathbb{R}$. It is well known that s is almost everywhere differentiable.

Furthermore the scale function s can be also considered as a scaling measure on Borel subsets of $\mathbb{R}$ which we still denote by s, setting

$$s([\alpha,\beta]) = s(\beta) - s(\alpha) = \int_\alpha^\beta \mathrm{d}s(x),\ \forall \alpha,\beta \in \mathbb{R},\ \alpha < \beta.$$

Let us designate by $AC_{loc}(I)$ the space of locally absolutely continuous functions on I and by $AC_s(I)$ the space of s-absolutely continuous functions on I, i.e., the set of functions $u : I \to \mathbb{R}$ such that there exists a locally absolutely continuous function ϕ with $u = \phi \circ s$.

Let us consider a Radon measure m on I with full support. We designate by

$$\mathcal{D}^{(s)} := \Big\{u\colon I \to \mathbb{R} :\ u \in AC_s(I),\ \ \int_I \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2\, \mathrm{d}s(x) < \infty\Big\},$$

where $\frac{\mathrm{d}u}{\mathrm{d}s}$ is the Radon Nikodym derivative of $\mathrm{d}u$ with respect to $\mathrm{d}s$.

We define the quadratic form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(I,m)$ by

$$\mathcal{D} := \mathcal{D}^{(s)} \cap L^2(I,m),\ \mathcal{E}[u] := \int_I \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2\, \mathrm{d}s(x) \quad \text{for all } u \in \mathcal{D}. \tag{2.1}$$

It is well known that $\mathcal{E}$ is a regular strongly local Dirichlet form in $L^2(I,m)$ [3]. Moreover, the positive self-adjoint operator associated with the form $\mathcal{E}$ via Kato's representation theorem, which we denote by L, is given by (see [12, Section 2] and [5, Chap. 1])

$$D(L) = \big\{u \in \mathcal{D} : \frac{\mathrm{d}u}{\mathrm{d}s} \text{ is abs. cont. w.r.t. } \mathrm{d}m,\ \mathcal{L}u = -\frac{\mathrm{d}}{\mathrm{d}m}\big(\frac{\mathrm{d}u}{\mathrm{d}s}\big) \in L^2(I,m), \text{ with boundary conditions at } r_1 \text{ and } r_2\big\},$$
$$Lu = \mathcal{L}u \quad \text{for all } u \in D(L).$$

We have the following facts about the boundary conditions at r_1 and r_2 to apply to u in $D(L)$ (we refer to [13, Section 2], [8]):

(i) If r_1 (resp. r_2) is an exit endpoint, then we have the following boundary condition at r_1 (resp. r_2)
$$\lim_{x\to r_1} u(x) = 0 \text{ (resp. } \lim_{x\to r_2} u(x) = 0).$$

(ii) If r_1 (resp. r_2) is an entrance endpoint, then we have the following boundary condition at r_1 (resp. r_2)
$$\lim_{x\to r_1} \frac{\mathrm{d}u}{\mathrm{d}s}(x) = 0 \text{ (resp. } \lim_{x\to r_2} \frac{\mathrm{d}u}{\mathrm{d}s}(x) = 0).$$

Remark 2.1. The second-order ordinary differential operator ([3, pp. 63–64])
$$\mathcal{L}u(x) = a(x)u''(x) + b(x)u'(x)$$
with real-valued functions $a > 0$ and b can be converted into Feller's canonical form $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$ with
$$\mathrm{d}s = \mathrm{e}^{-B(x)}\mathrm{d}x,\ \mathrm{d}m = \frac{\mathrm{e}^{B(x)}}{a(x)}\mathrm{d}x,\quad B(x) = \int_{x_0}^{x} \frac{b(y)}{a(y)}\,\mathrm{d}y.$$
Hence, by formal computation we get
$$-\int \mathcal{L}u \cdot v\ \mathrm{d}m = -\int v \cdot \mathrm{d}\frac{\mathrm{d}u}{\mathrm{d}s} = \int \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}\ \mathrm{d}s,\ \forall u,v \in \mathcal{C}_c^2(I).$$
Owing to Feller's canonical form we can define the differential operator $\mathcal{L}$ on I by
$$\mathcal{L} := -\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}.$$

2.1. Feller's boundary classification

Let us introduce the following quantities

$$\Gamma_1 = \int_{r_1}^{c} \big(m(c) - m(x)\big)\ \mathrm{d}s(x),\quad \Sigma_1 = \int_{r_1}^{c} \big(s(c) - s(x)\big)\ \mathrm{d}m(x) \tag{2.2}$$

and

$$\Gamma_2 = \int_{c}^{r_2} \big(m(x) - m(c)\big)\ \mathrm{d}s(x),\quad \Sigma_2 = \int_{c}^{r_2} \big(s(x) - s(c)\big)\ \mathrm{d}m(x) \tag{2.3}$$

for some $r_1 < c < r_2$.

It is well known (see [10, pp. 271–273]) that the boundaries r_1 and r_2 of I can be classified with respect to the Feller operator $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$ into four classes as follows (we refer the reader to [9, pp. 151–152] or [13, pp. 24–25]):

(a) r_i is a regular boundary if $\Gamma_i < \infty$, $\Sigma_i < \infty$,
(b) r_i is an exit boundary if $\Gamma_i < \infty$, $\Sigma_i = \infty$,
(c) r_i is an entrance boundary if $\Gamma_i = \infty$, $\Sigma_i < \infty$,
(d) r_i is a natural boundary if $\Gamma_i = \infty$, $\Sigma_i = \infty$.

Definition 2.2. We say that the boundary r_1 (resp. r_2) is approachable whenever $s(r_1) > -\infty$ (resp. $s(r_2) < \infty$).

Definition 2.3. The boundary r_1 (resp. r_2) is called regular whenever it is approachable and $m((r_1, c)) < \infty$, (resp. $m((c, r_2)) < \infty$) $\forall\, c \in (r_1, r_2)$.

2.2. Extended Dirichlet spaces

Let us now introduce the extended Dirichlet space of $\mathcal{E}$ ([3, Chap. 1]), which we denote by $\mathcal{D}_e$.

Definition 2.4. Let $(\mathcal{E}, \mathcal{D})$ be a closed symmetric form on $L^2(I, m)$. Denote by $\mathcal{D}_e$ the totality of m-equivalence classes of all m-measurable functions f on I such that $|f| < \infty$, $m - a.e$ and there exists an $\mathcal{E}$-Cauchy sequence $\{f_n, n \geq 1\} \subset \mathcal{D}$ such that $\lim_{n\to\infty} f_n = f$, $m - a.e.$ on I. $\{f_n\} \subset \mathcal{D}$ above is called an approximating sequence of $f \in \mathcal{D}_e$. We call the space $\mathcal{D}_e$ the extended space attached to $(\mathcal{E}, \mathcal{D})$. When the latter is a Dirichlet form on $L^2(I, m)$, the space $\mathcal{D}_e$ will be called its extended Dirichlet space.

Henceforth, to determine the extended Dirichlet space $\mathcal{D}_e$ we shall make use the following proposition. We include it for the convenience of the reader.

Proposition 2.5 (see [3, p. 66])**.** *Assume that for some $i \in 1, 2$, r_i is approachable but non-regular. If we let*

$$\mathcal{D}_0^{(s)} := \{u \in \mathcal{D}^{(s)} : u(r_i) = 0\}, \tag{2.4}$$

then

$$\mathcal{D} \subset \mathcal{D}_e = \mathcal{D}_0^{(s)}, \tag{2.5}$$

and $(\mathcal{E}, \mathcal{D})$ is a regular, strongly local, transient, and irreducible Dirichlet form on $L^2(I, m)$.

It is well known that $\mathcal{E}$ is transient if and only if (see [3, Theorem 2.2.11]) either r_1 or r_2 is approachable and non-regular. Otherwise it is recurrent.

From now on we assume that either r_1 or r_2 is approachable but non-regular. Thus $\mathcal{E}$ is transient. By virtue of Feller's classical test of non-explosion, $\mathcal{E}$ is conservative if and only if (see [3, p. 126] and the discussion made there)

$$\int_{(r_1,c)} m((x, c))\, \mathrm{d}s(x) = \int_{(c,r_2)} m((c, x))\, \mathrm{d}s(x) = \infty,\ \forall\ r_1 < c < r_2. \tag{2.6}$$

In this case, the boundaries r_1 and r_2 are non-exit points.

Remark 2.6. Suppose the following assumptions are fulfilled:

(H1) $s, s^{-1} \in AC_{loc}(I)$.

(H2) $s' = \sigma > 0$ and $\frac{1}{\sigma} \in L^1_{loc}(I)$.

Then

(1) $AC_s(I)$ coincides with $AC_{loc}(I)$. In fact, let $u \in AC_s(I)$. Thus there exists $\phi \in AC_{loc}(I)$ such that $u = \phi \circ s$. As $s \in AC_{loc}(I)$ by assumption (H1), then $u \in AC_{loc}(I)$.

Conversely, let $u \in AC_{loc}(I)$. Then $u = u \circ s^{-1} \circ s$ and by assumption (H1), $u \circ s^{-1} \in AC_{loc}(I)$ and hence $u \in AC_s(I)$.

(2) $\mathcal{C}_c^\infty(I) \subset \mathcal{D}$. Indeed, let $u \in \mathcal{C}_c^\infty(I)$. Obviously $u \in L^2(I, m)$ and is locally absolutely continuous. It remains to show that $\mathcal{E}^{(s)}[u] < \infty$.

As by assumptions s^{-1} and s are locally absolutely continuous, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}x}} = \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{1}{\sigma}.$$

Then the fact $\frac{1}{\sigma} \in L^1_{loc}(I)$ leads to

$$\int_I \Big(\frac{\mathrm{d}u}{\mathrm{d}s}\Big)^2(x)\ \mathrm{d}s(x) = \int_I \left(\frac{\mathrm{d}u}{\mathrm{d}x}.\frac{1}{\sigma}\right)^2(x)\ \sigma(x)\mathrm{d}x = \int_I u'(x)^2 \frac{\mathrm{d}x}{\sigma(x)} < \infty.$$

Accordingly, we obtain

$$\mathcal{D}^{(s)} = \Big\{u \colon I \to \mathbb{R} :\ u \in AC_{loc}(I),\ \ \int_I (u'(x))^2\ \frac{\mathrm{d}x}{\sigma(x)} < \infty\Big\}, \tag{2.7}$$

$$\mathcal{D} := \mathcal{D}^{(s)} \cap L^2(I, m),\ \mathcal{E}[u] := \int_I (u'(x))^2\ \frac{\mathrm{d}x}{\sigma(x)} \ \text{ for all } u \in \mathcal{D}. \tag{2.8}$$

3. Computation of the trace of $\mathcal{E}$

Let $V = \{x_k \in (r_1, r_2),\ k \in \mathbb{N}\} \subset I$ be a countable set, where $(x_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence, i.e., $x_k < x_{k+1}$ for all $k \in \mathbb{N}$. In addition, let $x_\infty = \lim_{k\to\infty} x_k$, where the increasing limit may be finite or infinite.

Next, we will investigate the following two cases for a transient Dirichlet form $\mathcal{E}$:

(1) V has no accumulation point in $\bar{I}$ (the closure of I in $\mathbb{R}$).

(2) V has x_∞ as an accumulation point in $\bar{I}$.

We turn our attention to discuss some properties of capacity which is a set function associated to a Dirichlet form. Also it plays an important role to measure the size of sets adapted to the form.

Definition 3.1. We define the 1-capacity Cap_1 associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ by

$$Cap_1(U) := \inf\ \{\mathcal{E}_1[u] : u \in \mathcal{D},\ u \geq 1,\ m - a.e. \text{ in } U\}$$

for an open set $U \subset I$, and

$$Cap_1(A) := \inf \{Cap_1(U) : U \subset I \text{ open}, \ A \subset U\}$$

for a Borel set $A \subset I$, where

$$\mathcal{E}_1[u] := \mathcal{E}[u] + \int_I u^2(x) \, \mathrm{d}m(x) \text{ for } u, v \in \mathcal{D}.$$

The following lemma shows that a diffusion process associated with a Dirichlet form $\mathcal{E}$ enjoys a strong irreducibility property which means that any two points of I are connected for a diffusion process.

Lemma 3.2. *For all $x \in V$, we have $Cap_1(\{x\}) > 0$.*

Proof. An elementary identity

$$u(\xi) - u(y) = \int_y^\xi \frac{\mathrm{d}u}{\mathrm{d}s}(x) \ \ \mathrm{d}s(x), \ \forall \ r_1 < \xi < y < r_2$$

and Hölder's inequality lead to

$$\big(u(\xi) - u(y)\big)^2 \leq s([y, \xi])\mathcal{E}[u], \ \forall \ r_1 < \xi < y < r_2. \tag{3.1}$$

We get from (3.1)

$$\sup_{\xi \leq z \leq y} u(z)^2 \leq 2 \ s([y, \xi])\mathcal{E}[u] + 2u(x)^2, \ \xi \leq x \leq y. \tag{3.2}$$

Then for each compact set $K \subset I$, there is a positive constant C_K such that

$$\sup_{x \in K} u(x)^2 \leq C_K \ \mathcal{E}_1[u], \ \forall u \in \mathcal{D}. \tag{3.3}$$

Hence $Cap_1(\{x\}) \geq \frac{1}{C_K} > 0$ for every $x \in I$. □

Remark 3.3. According to the inequality (3.1), if r_1 (resp. r_2) is approachable, then for any element from $\mathcal{D}^{(s)}$ we have $u(r_1) = \lim_{x \downarrow r_1, x \in I} u(x) < \infty$, (resp. $u(r_2) = \lim_{x \uparrow r_2, x \in I} u(x) < \infty$) and $u \in \mathcal{C}([r_1, r_2))$ (resp. $u \in \mathcal{C}((r_1, r_2]))$. In particular, $\mathcal{D}^{(s)}$ is a uniformly dense sub-algebra of $\mathcal{C}([r_1, r_2])$ if both r_1 and r_2 are approachable (for more details we refer to [3, Chap. II]).

We shall start by the first case (a) which says that V has no accumulation point in $\bar{I}$, i.e., $x_\infty = \infty$. Assume that $(r_1, r_2) = (r_1, \infty)$.

3.1. V has no accumulation point in $\bar{I}$

Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that $a_k > 0$ for all $k \in \mathbb{N}$. Let us consider the atomic measure defined as follows:

$$\mu = \sum_{k \in \mathbb{N}} a_k \delta_{x_k}.$$

We define now the Hilbert space $\ell^2(V, \mu)$ equipped with the scalar product given by

$$(u, v) := \sum_{k \in \mathbb{N}} u(x_k) v(x_k) \ a_k \ \text{ and } \ \|u\| := \sqrt{(u, u)}.$$

Let $\check{\mathcal{E}}$ be the trace of $\mathcal{E}$ on the discrete set V (see [1, 2, 5]). As by assumptions $\mathcal{E}$ is transient we adopt the method elaborated in [1] to compute explicitly $\check{\mathcal{E}}$.

Let us now explain our computation strategies.

Owing to Proposition 2.5 the extended domain $\mathcal{D}_e$ is given by

$$\mathcal{D}_e = \big\{u\colon I \to \mathbb{R}:\ u \in AC_s(I),\ \int_I \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2\,\mathrm{d}s(x) < \infty \text{ such that } u(r_i) = 0 \\ \text{if } r_i \text{ is approachable and non-regular}\big\}.$$

Henceforth, we define the operator L by

$$D(L) = \big\{u \in \mathcal{D} : \frac{\mathrm{d}u}{\mathrm{d}s} \text{ is abs. cont. w.r.t. } \mathrm{d}m,\quad \lim_{x\to r_1} \frac{\mathrm{d}u}{\mathrm{d}s}(x) = 0, \\ \mathcal{L}u = -\frac{\mathrm{d}}{\mathrm{d}m}\big(\frac{\mathrm{d}u}{\mathrm{d}s}\big) \in L^2(I, m)\big\}, \\ Lu = \mathcal{L}u \quad \text{for all } u \in D(L).$$

Let J be the restriction operator defined from $\operatorname{dom} J \subseteq (\mathcal{E}, \mathcal{D}_e)$ to $\ell^2(V, \mu)$ by

$$\operatorname{dom} J := \big\{u \in \mathcal{D}_e :\ \sum_{k\in\mathbb{N}} a_k u(x_k)^2 < \infty\big\}, \\ Ju := u|_V \quad \text{for all } u \in \operatorname{dom} J.$$

Since $\mathcal{E}$ is regular and the functions with finite support are dense in $\ell^2(V, \mu)$, the operator J has dense range. Obviously

$$\ker J = \big\{u \in \operatorname{dom} J : u(x_k) = 0 \quad \text{for all } k \in \mathbb{N}\big\}.$$

According to [1, Prop. 5.5], the construction of the trace form coincides with Fukushima's construction. Accordingly we can compute $\check{\mathcal{E}}$ following Fukushima's method. To that end we designate by P the orthogonal projection in the Dirichlet space $(\mathcal{E}, \mathcal{D}_e)$ onto the $\mathcal{E}$-orthogonal complement of $\ker J$. Then

$$\operatorname{dom} \check{\mathcal{E}} = \operatorname{ran} J \text{ and } \check{\mathcal{E}}[Ju] = \mathcal{E}[Pu] \quad \text{for all } u \in \operatorname{dom} J.$$

Clearly

$$(\ker J)^{\perp_{\mathcal{E}}} = \{u \in \mathcal{D}_e : \mathcal{E}(u, v) = 0 \quad \text{for all } v \in \ker J\}.$$

Lemma 3.4. *J is closed in $(\mathcal{E}, \mathcal{D}_e)$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} J$ and $u \in \mathcal{D}_e$ such that u_n $\mathcal{E}$-converges to u in $\mathcal{D}_e$ and $(Ju_n)_{n\in\mathbb{N}}$ converges to v in $\ell^2(V, \mu)$. Then according to [5, Theorem 2.1.4] there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ q.e. and hence also μ-a.e. It holds $u = v$ μ-a.e., yielding $u \in \operatorname{dom} J$ and $Ju = v$. □

We are in position now to compute explicitly the $\mathcal{E}$-orthogonal projection Pu.

From now on we set $r_1 = x_0$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 3.5. *Let* $u \in \mathcal{D}_e$. *Then*

$$Pu(x) = u(x_k) + \frac{s(x) - s(x_k)}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big) \tag{3.4}$$

for all $x \in [x_k, x_{k+1}]$ *and* $k \in \mathbb{N}_0$.

Proof. To avoid technicalities, we give the proof in the case r_1 is approachable but non-regular. The proof of the general case is similar.

Let $v \in KerJ$. Then $\mathcal{E}(Pu, v) = 0$, which is equivalent to

$$\sum_{k\in\mathbb{N}_0} \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\frac{\mathrm{d}v}{\mathrm{d}s}(x)\ \mathrm{d}s(x) = 0. \tag{3.5}$$

If $\frac{\mathrm{d}Pu}{\mathrm{d}s} = C_k$ is constant on each interval $[x_k, x_{k+1}]$, we have

$$\int_{x_k}^{x_{k+1}} \frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\frac{\mathrm{d}v}{\mathrm{d}s}(x)\ \mathrm{d}s(x) = C_k\big(v(x_{k+1}) - v(x_k)\big) = 0,$$

and then

$$\sum_{k\in\mathbb{N}_0} \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\frac{\mathrm{d}v}{\mathrm{d}s}(x)\ \mathrm{d}s(x) = 0.$$

Let us show that the function

$$v(x) = u(x_k) + \frac{s(x) - s(x_k)}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big) \text{ for all } x \in [x_k, x_{k+1}] \text{ and } k \in \mathbb{N}_0$$

is in fact Pu.

Obviously $v(x_k) = u(x_k)$ for all integers k, the function v is locally absolutely continuous and

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\big(u(x_{k+1}) - u(x_k)\big)}{s([x_k, x_{k+1}])}. \tag{3.6}$$

Let us show that

$$\int_I \big(\frac{\mathrm{d}v}{\mathrm{d}s}(x)\big)^2\ \mathrm{d}s(x) < \infty.$$

We have

$$\begin{aligned}\int_I \big(\frac{\mathrm{d}v}{\mathrm{d}s}(x)\big)^2\ \mathrm{d}s(x) &= \sum_{k\in\mathbb{N}_0} \int_{x_k}^{x_{k+1}} \frac{\big(u(x_{k+1}) - u(x_k)\big)^2}{s([x_k, x_{k+1}])^2}\mathrm{d}s(x) \\ &= \sum_{k\in\mathbb{N}_0} \big(u(x_{k+1}) - u(x_k)\big)^2\ \frac{1}{s([x_k, x_{k+1}])}.\end{aligned}$$

On the other hand, it holds:

$$u(x_{k+1}) - u(x_k) = \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}u}{\mathrm{d}s}(x)\ \mathrm{d}s(x).$$

Then

$$\big(u(x_{k+1}) - u(x_k)\big)^2 \le s([x_k, x_{k+1}])\ \int_{x_k}^{x_{k+1}} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2\mathrm{d}s(x).$$

Accordingly, using the fact $u \in \mathcal{D}_e$, we achieve

$$\int_I \big(\frac{\mathrm{d}v}{\mathrm{d}s}(x)\big)^2 \,\mathrm{d}s(x) \le \sum_{k\in\mathbb{N}_0} \int_{x_k}^{x_{k+1}} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) = \int_I \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \,\mathrm{d}s(x) < \infty.$$

Thereby $v \in D_e$.

Let us now prove that $\mathcal{E}(v,\varphi) = 0$ for all $\varphi \in kerJ$.

Let $\varphi \in KerJ$. Then

$$\begin{aligned}
\mathcal{E}(v,\varphi) &= \sum_{k\in\mathbb{N}_0} \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}v}{\mathrm{d}s}(x)\frac{\mathrm{d}\varphi}{\mathrm{d}s}(x)\,\mathrm{d}s(x)\\
&= \sum_{k\in\mathbb{N}_0} \frac{u(x_{k+1}) - u(x_k)}{s(x_{k+1}) - s(x_k)} \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}\varphi}{\mathrm{d}s}(x)\,\mathrm{d}s(x)\\
&= \sum_{k\in\mathbb{N}_0} \frac{u(x_{k+1}) - u(x_k)}{s(x_{k+1}) - s(x_k)} \big(\varphi(x_{k+1}) - \varphi(x_k)\big)\\
&= 0.
\end{aligned}$$

□

Theorem 3.6. *It holds* $\mathrm{dom}(\check{\mathcal{E}}) = ranJ$ *and*

$$\check{\mathcal{E}}[Ju] = \sum_{k=0}^{\infty} \frac{1}{s([x_k,x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 \quad \textit{for all } u \in domJ. \tag{3.7}$$

Proof. Let $u \in \mathrm{dom}\,J$. Having formula (3.6), a straightforward computation leads to

$$\begin{aligned}
\check{\mathcal{E}}[Ju] = \mathcal{E}[Pu] &= \int_{r_1}^{r_2} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x)\\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \left(\frac{u(x_{k+1}) - u(x_k)}{s([x_k,x_{k+1}])}\right)^2 \mathrm{d}s(x)\\
&= \sum_{k=0}^{\infty} \frac{1}{s([x_k,x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2, \qquad (3.8)
\end{aligned}$$

which completes the proof. □

Definition 3.7. We say that (V,b) is a weighted graph if V is a discrete and countably infinite space and $b : V \times V \to [0,\infty)$ is a symmetric function which is zero on the diagonal of $V \times V$ such that $\sum_{y\in V} b(x,y) < \infty$, $x \in V$. We say that two vertices x and y in V are neighbors if $b(x,y) > 0$ and we write $x \sim y$ in this case.

Remark 3.8. Let us rewrite $\check{\mathcal{E}}$ as follows

$$\begin{aligned}
\check{\mathcal{E}}[Ju] &= \sum_{x_k\in V} \sum_{x_j\sim x_k} b(x_k,x_j)\big(u(x_k) - u(x_j)\big)^2\\
&= \sum_{k\in\mathbb{N}} b(x_k,x_{k+1})\big(u(x_{k+1}) - u(x_k)\big)^2 \quad \text{for all } u \in \mathrm{dom}\,J,
\end{aligned}$$

where

$$b(x_k,x_{k+1}) = \frac{1}{2\,(s(x_{k+1})-s(x_k))} > 0 \ \text{ if } x_k \sim x_{k+1},\ k\in\mathbb{N},$$

and $b(x_k,x_{k+1})=0$ otherwise.

Assume that $\mu(V)=\infty$. Then the condition (A) of [11] is fulfilled.

Set

$$\tilde{L}u(x_k) = \frac{1}{a_k}\sum_j b(x_k,x_j)\big(u(x_k)-u(x_j)\big) \ \text{ for each } \ k\in\mathbb{N}. \tag{3.9}$$

Then, $\tilde{L}\big(\mathcal{C}_c(V)\big)\subseteq\mathcal{C}_c(V)$.

Hence according to [11, Theorem 6] $\check{L}$ is given by

$$D(\check{L}) := \big\{u\in\ell^2(V,\mu):\tilde{L}u\in\ell^2(V,\mu)\big\},$$
$$\check{L}u=\tilde{L}u.$$

Hence, for all $k\in\mathbb{N}$ and $u\in\mathrm{dom}(\check{L})$ it holds

$$\begin{aligned}\check{L}u(x_k) &= \frac{1}{a_k}b(x_k,x_{k-1})\big(u(x_k)-u(x_{k-1})\big)\\ &\quad+\frac{1}{a_k}b(x_k,x_{k+1})\big(u(x_k)-u(x_{k+1})\big)\\ &= \frac{1}{a_k\ s([x_{k-1},x_k])}\big(u(x_k)-u(x_{k-1})\big)\\ &\quad+\frac{1}{a_k\ s([x_k,x_{k+1}])}\big(u(x_k)-u(x_{k+1})\big)\\ &= -\frac{u(x_{k+1})}{a_k s([x_k,x_{k+1}])}+\frac{s([x_{k-1},x_{k+1}])u(x_k)}{a_k s([x_k,x_{k+1}])s([x_{k-1},x_k])}\\ &\quad-\frac{u(x_{k-1})}{a_k s([x_{k-1},x_k])}.\end{aligned}\tag{3.10}$$

By [11, Theorem 6], once again, we obtain the following description of $\check{\mathcal{E}}$:

$$\operatorname{ran} J = \Big\{u=(u_k)\in\ell^2(V,\mu),\ \sum_{k=0}^{\infty}\frac{1}{s([x_k,x_{k+1}])}\big(u(x_{k+1})-u(x_k)\big)^2<\infty\Big\},\tag{3.11}$$

$$\check{\mathcal{E}}[u] = \sum_{k=0}^{\infty}\frac{1}{s([x_k,x_{k+1}])}\big(u(x_{k+1})-u(x_k)\big)^2.\tag{3.12}$$

Remark 3.9. Assume that $r_1=-\infty$, $V=\mathbb{Z}$ and $a_k=1$ for all $k\in\mathbb{Z}$. Then the expression (3.10) can be regarded as a Jacobi operator which has the following form:

$$\mathcal{J}u(k) := A(k)u(k+1)+B(k)u(k)+A(k-1)u(k-1),\ \forall k\in\mathbb{Z}.\tag{3.13}$$

3.2. V has an accumulation point in $\bar{I}$

Now we consider the second case (b), where the sequence $(x_k)_{k\in\mathbb{N}}$ of the set V is convergent and it has x_∞ as a limit. We keep the same definitions as above. Arguing as in the former case we get that for any $u \in \mathcal{D}_e$, the following expression of Pu

$$Pu(x) = u(x_k) + \frac{s(x) - s(x_k)}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big), \quad \text{in } [x_k, x_{k+1}],\ k \in \mathbb{N}_0,$$

Pu is a constant in (r_1, x_1) and $Pu(x) = u(x_\infty) - \frac{s(x)-s(x_\infty)}{s([x_\infty,\infty))}u(x_\infty)$ in $[x_\infty, \infty)$ with the convention $\frac{1}{s([x_\infty,\infty))} = 0$ if ∞ is non-approachable.

Next, the trace form $\check{\mathcal{E}}$ will be computed.

Proposition 3.10. *For every $u \in \operatorname{dom} J$, it holds*

$$\check{\mathcal{E}}[Ju] = \sum_{k=0}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))}. \tag{3.14}$$

Proof. Let $u \in \operatorname{dom} J$. We get

$$\begin{aligned}
\check{\mathcal{E}}[Ju] &= \int_{r_1}^{r_2} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) = \int_{x_0}^{\infty} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x)\\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) + \int_{x_\infty}^{\infty} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x)\\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \left(\frac{u(x_{k+1}) - u(x_k)}{s([x_k, x_{k+1}])}\right)^2 \mathrm{d}s(x) + \int_{x_\infty}^{\infty} \frac{u(x_\infty)^2}{s([x_\infty, \infty))^2}\mathrm{d}s(x)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))}. \tag{3.15}$$

□

Remark 3.11. In particular, if the end-point ∞ is a non-approachable boundary, i.e., $s(\infty) = \infty$, then we have $s([x_\infty, \infty)) = \infty$ and

$$\check{\mathcal{E}}[Ju] = \sum_{k=0}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2.$$

3.3. Trace of one-dimensional diffusions with respect to mixed type measure

In this subsection we assume that $r_1 < 0$, $r_2 = \infty$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of negative numbers strictly increasing to 0. We consider a measure on (r_1, ∞) of mixed type, i.e., a measure which has an absolutely continuous part and a discrete part as follows

$$\mu := \mu_{disc} + \mu_{abs},$$

where

$$\mu_{disc} = \sum_{k=1}^{\infty} a_k \delta_{x_k},\ a_k > 0, k \in \mathbb{N} \ \text{ and } \ \operatorname{supp}[\mu_{abs}] = [0,\infty).$$

Hence $F = \{x_k < 0 : k \in \mathbb{N}\} \cup [0,\infty)$ is the support of the measure μ. In order to compute the trace of $\mathcal{E}$ with respect to the measure μ we shall define the trace operator J by

$$J : \mathcal{D}_e \cap L^2(F,\mu) \to L^2(F,\mu),\ Ju = u_{|_F}.$$

Obviously, we have

$$Ker(J) := \{u \in \mathcal{D}_e : u(x_k) = 0, \forall k \in \mathbb{N},\ u_{|_{(0,\infty)}} = 0\}.$$

Then Pu can expressed in the same way as follows:

$$Pu(x) = u(x_k) + \frac{s(x) - s(x_k)}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big), \ \text{ in } [x_k, x_{k+1}] \ \ k \in \mathbb{N}_0 \tag{3.16}$$

and

$$Pu = u \text{ in } \ [0,\infty).$$

Lemma 3.12. *It holds* $\operatorname{dom}\check{\mathcal{E}} = ranJ$ *and*

$$\check{\mathcal{E}}[Ju] = \sum_{k=0}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 + \int_0^{\infty} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) \tag{3.17}$$

for all $u \in \operatorname{dom} J$.

Proof. A straightforward computation leads

$$\begin{aligned}
\check{\mathcal{E}}[Ju] &= \int_{r_1}^{r_2} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) = \int_{x_0}^{\infty} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) \\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) + \int_0^{\infty} \big(\frac{\mathrm{d}Pu}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) \\
&= \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \left(\frac{u(x_{k+1}) - u(x_k)}{s([x_k, x_{k+1}])}\right)^2 \mathrm{d}s(x) + \int_0^{\infty} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) \\
&= \sum_{k=0}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 + \int_0^{\infty} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x). \qquad (3.18)
\end{aligned}$$

□

We introduce $\check{\mathcal{D}}_{\max}$ the space of the trace form $\check{\mathcal{E}}$ by

$$\check{\mathcal{D}}_{\max} = \{u \in L^2(F,\mu) : u \in AC_{loc}([0,\infty)),\ \sum_{k=0}^{\infty} \frac{\big(u(x_{k+1}) - u(x_k)\big)^2}{s([x_k, x_{k+1}])} + \int_0^{\infty} \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x) < \infty\}.$$

We denote by $\check{\mathcal{E}}^{(J)}$ and $\check{\mathcal{E}}^{(c)}$ the quadratic forms such that

$$\operatorname{dom}(\check{\mathcal{E}}^{(J)}) = \operatorname{dom}(\check{\mathcal{E}}^{(c)}) = \check{\mathcal{D}}_{\max}$$

and

$$\check{\mathcal{E}}^{(c)}[u] = \int_0^\infty \big(\frac{\mathrm{d}u}{\mathrm{d}s}(x)\big)^2 \mathrm{d}s(x),$$
$$\check{\mathcal{E}}^{(J)}[u] = \sum_{k=0}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k)\big)^2,$$

where $\check{\mathcal{E}}^{(c)}$ and $\check{\mathcal{E}}^{(J)}$ are the strongly local type, non-local type Dirichlet forms respectively.

Finally let

$$\check{\mathcal{E}} := \check{\mathcal{D}} = \check{\mathcal{D}}_{\max}, \quad \check{\mathcal{E}}[u] = \check{\mathcal{E}}^{(J)}[u] + \check{\mathcal{E}}^{(c)}[u] \ \text{ for all } u \in \check{\mathcal{D}}_{\max}.$$

Let us stress that the latter decomposition was inspired by [2, Section 4] for dimension $n = 3$ and $V = (0,1) \cup \mathbb{N}$.

4. Conservativeness of traces of one-dimensional diffusions on discrete sets

In this section we aim to establish necessary and sufficient conditions ensuring the trace form $\check{\mathcal{E}}$ to enjoy the conservativeness property.

The form $\check{\mathcal{E}}$ is said to be conservative (or stochastically complete) if

$$\check{T}_t 1 = 1 \text{ for some and hence for every } t > 0,$$

where $\check{T}_t$ stands for the L^∞-semi-group induced by the Dirichlet form $\check{\mathcal{E}}$.

Let us now start with the case where the set V has no accumulation point in $\bar{I}$.

Theorem 4.1. *Assume that μ is infinite and V has no accumulation point in $\bar{I}$. Then the Dirichlet form $\check{\mathcal{E}}$ is conservative if and only if*

$$\sum_{k=1}^{\infty} \big(s(x_{k+1}) - s(x_k)\big) \sum_{j=1}^{k} a_j = \infty. \tag{4.1}$$

Proof. Remark 3.8, together with [11, Theorem 1] yield that the conservativeness of the Dirichlet form $\check{\mathcal{E}}$ is equivalent to the fact that the equation

$$\tilde{L}u + \alpha u = 0, \ \alpha > 0, \ u \in \ell^\infty, \tag{4.2}$$

has no nontrivial bounded solution where $\tilde{L}$ is given by (3.9).

We rewrite

$$\tilde{L}u(x_k) + \alpha u(x_k) = \frac{1}{a_k} \sum_j \frac{1}{2\ (s(x_k) - s(x_j))} \big(u(x_k) - u(x_j)\big) + \alpha u(x_k)$$
$$= 0. \tag{4.3}$$

This leads to

$$u(x_2) = \big(1 + 2\alpha a_1 (s(x_2) - s(x_1))\big) u(x_1), \tag{4.4}$$

and

$$\frac{(u(x_k)-u(x_{k+1}))}{2a_k\ \big(s(x_{k+1})-s(x_k)\big)}+\frac{(u(x_k)-u(x_{k-1}))}{2a_k\ \big(s(x_k)-s(x_{k-1})\big)}+\alpha u(x_k)=0 \tag{4.5}$$

for all $k\geq 2$.

Thus by induction and the recursive formula we get

$$\begin{aligned}
&u(x_{k+1})-u(x_k)\\
&=\frac{s(x_{k+1})-s(x_k)}{s(x_k)-s(x_{k-1})}(u(x_k)-u(x_{k-1}))+2a_k\alpha\big(s(x_{k+1})-s(x_k)\big)u(x_k)\\
&\ \vdots\\
&=\frac{s(x_{k+1})-s(x_k)}{s(x_2)-s(x_1)}(u(x_2)-u(x_1))+2\alpha\big(s(x_{k+1})-s(x_k)\big)\sum_{j=2}^{k}a_j\ u(x_j)\\
&=2a_1\alpha\big(s(x_{k+1})-s(x_k)\big)u(x_1)+2\alpha\big(s(x_{k+1})-s(x_k)\big)\sum_{j=2}^{k}a_j\ u(x_j)
\end{aligned}$$

$$=2\alpha\big(s(x_{k+1})-s(x_k)\big)\sum_{j=1}^{k}a_j\ u(x_j),\ \forall\, k\geq 1. \tag{4.6}$$

The latter formula gives rise to two observations (which can be proved by induction):

1. $u(x_k)$ has the sign of $u(x_1)$ for all $k\in\mathbb{N}$. This is if $u(x_1)>0$, then $u(x_k)>0$ for all $k\in\mathbb{N}$ and if $u(x_1)<0$, hence $u(x_k)<0$ for all $k\in\mathbb{N}$.
2. $u(x_k)$ is monotone, depending on the sign of $u(x_1)$.

Hence without loss of generality we may and shall assume that $u(x_1)>0$. In this case $(u(x_k))_{k\in\mathbb{N}}$ is positive and strictly increasing.

Accordingly, making use of formula (4.6) we derive

$$u(x_{k+1})-u(x_k)\leq\left(2\alpha\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j\right)u(x_k),\ \forall\, k\geq 1, \tag{4.7}$$

and

$$\frac{u(x_{k+1})}{u(x_k)}\leq 1+2\alpha\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j,\ \forall\, k\geq 1. \tag{4.8}$$

Finally we achieve

$$u(x_{N+1})\leq u(x_1)\prod_{k=1}^{N+1}\left(1+2\alpha\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j\right) \tag{4.9}$$

for all integer N.

Obviously the latter product is finite provided

$$\sum_{k=1}^{\infty}\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j<\infty$$

and then we get a bounded non-zero solution.

Conversely, assume that $\sum_{k=1}^{\infty}\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j=\infty$. Then summing over k in formula (4.6) and keeping in mind that the sequence $(u(x_k))_{k\in\mathbb{N}}$ is increasing and positive. We obtain

$$u(x_{N+1})-u(x_1)=2\alpha\sum_{k=1}^{N}\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_ju(x_j). \tag{4.10}$$

Hence

$$u(x_{N+1})\geq 2\alpha u(x_1)\sum_{k=1}^{N}\big[s(x_{k+1})-s(x_k)\big]\sum_{j=1}^{k}a_j\to\infty \text{ as } N\to\infty,$$

which finishes the proof. □

Theorem 4.2. *If μ is a finite measure, then $\check{\mathcal{E}}$ is not conservative.*

Proof. If μ is finite $\check{\mathcal{E}}$ is conservative if and only if $1\in ranJ$ and $\check{\mathcal{E}}[1]=0$. This implies that there is $u\in D_e$ such that $u|F=1$ and $\mathcal{E}[Pu]=0$. But this contradicts the fact that $\mathcal{E}$ is transient, since $\mathcal{E}$ is irreducible. Thus $\check{\mathcal{E}}$ is not conservative. □

Proposition 4.3. *Assume that V is a finite set. Then there exists $N\in\mathbb{N}$ such that $\check{\mathcal{D}}=\mathbb{R}^N$ and for each $u\in\operatorname{dom}J$*

$$\check{\mathcal{E}}[Ju]=\sum_{k=1}^{N-1}\frac{1}{s(x_{k+1})-s(x_k)}\big(u(x_{k+1})-u(x_k)\big)^2+\frac{u(x_N)^2}{s([x_N,\infty))}.$$

It follows that $\check{\mathcal{E}}$ is not conservative.

Proof. Non-conservativeness of the trace form $\check{\mathcal{E}}$ follows from the fact that $1\in\check{\mathcal{D}}$ and $\check{\mathcal{E}}[1]=\frac{1}{s([x_N,\infty))}\neq 0$. □

Remark 4.4. For the case where V has x_∞ as an accumulation point in $\bar{I}$ and the measure μ is infinite, we remark that the trace form $\check{\mathcal{E}}$ has a killing part in case ∞ is approachable. Hence owing to theoretical results (see [2]) it can't be conservative. However, if ∞ is non-approachable, arguing as in the proof of Theorem 4.1, we conclude that $\check{\mathcal{E}}$ is conservative if and only if condition (4.1) if fulfilled.

5. Application: traces of the one-dimensional diffusion related to Bessel's process

For each $n \in \mathbb{N}$, $n \geq 2$. We consider the speed measure m defined on $I = (0, \infty)$ by

$$\mathrm{d}m(x) = 2x^{2\nu+1}dx, \quad \text{where} \quad \nu = \frac{n}{2} - 1.$$

We define the scale function s as follows

$$s(x) = -\frac{1}{2\nu\ x^{2\nu}}.$$

Obviously the functions s and s^{-1} fulfill all conditions demanded in Remark 2.6. Namely

- s and s^{-1} are absolutely continuous.
- $\frac{1}{\sigma} = x^{2\nu+1} \in L^1_{loc}(I)$.

We shall be concerned with the Dirichlet form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(I, m)$. In this situation we have $L^2(I, m) = L^2(I, 2x^{2\nu+1}\mathrm{d}x)$,

$$\mathcal{D}^{(s)} := \Big\{u\colon (0,\infty) \to \mathbb{R} : u \text{ is loc. abs. cont.}, \int_0^\infty (u'(x))^2 x^{2\nu+1}\mathrm{d}x < \infty\Big\}.$$

$$\mathcal{D} := \mathcal{D}^{(s)} \cap L^2(I, 2x^{2\nu+1}\mathrm{d}x),\ \mathcal{E}[u] := \int_0^\infty (u'(x))^2 x^{2\nu+1}\mathrm{d}x \ \text{ for all } u \in \mathcal{D}.$$

Since $n \geq 3$ the Dirichlet form $\mathcal{E}$ is transient [3, p. 126]. We can easily check that for $n \geq 3$, (i.e., $\nu \geq \frac{1}{2}$) we obtain $r_1 = 0$ is a non-approachable boundary, i.e., $s(0) = \infty$. Whereas the boundary point $r_2 = \infty$ is an approachable boundary, i.e., $s(\infty) < \infty$.

Hence, the extended domain is given by

$$\mathcal{D}_e = \{u \in \mathcal{D}^{(s)} : \lim_{x\to\infty} u(x) = 0\}.$$

According to the Feller's boundary classification, 0 is an entrance boundary. Indeed

$$\Gamma_1(0) = \infty \text{ and } \Sigma_1(0) = \frac{c^2}{2\nu + 2} < \infty \text{ for all constant } c > 0.$$

In this situation, the selfadjoint adjoint operator related to $\mathcal{E}$, which we denote by L is the generator of the Bessel process of index ν on the half-line. Moreover we have the following description of L. Set

$$\mathcal{L} := -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\nu+1}{2x}\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{for all} \quad \nu > -\frac{1}{2}.$$

Then

$$\begin{aligned} D(L) = \big\{u \in \mathcal{D} : u' \in AC_{loc}(I),\ &\lim_{x\downarrow 0^+} x^{2\nu+1}u'(x) = 0, \\ &\mathcal{L}u = -\frac{1}{2}u'' - \frac{2\nu+1}{2x}u' \in L^2(I, m)\big\}, \\ Lu = \mathcal{L}u \ \text{ for all } u \in D(L).& \end{aligned}$$

We start with the case where the sequence $(x_k)_{k\in\mathbb{N}}$ has no accumulation point in $\bar{I}$. Accordingly we consider the discrete measure defined as the first section by

$$\mu = \sum_{k\in\mathbb{N}} a_k \delta_{x_k},$$

which is supported by an infinite countable set $V = \{x_k > 0 : k \in \mathbb{N}\}$.

Remark 5.1. In our case for $n \geq 2$, $r_1 = \{0\}$ is an entrance boundary and it is well known that $Cap_1(\{0\}) = 0$ (we refer to [10] for more details) and for each element $x_k \in V$ we have $Cap_1(\{x_k\}) > 0$.

To compute the trace of the general Bessel's Dirichlet form $\check{\mathcal{E}}$ with domain $\check{\mathcal{D}} \subset \ell^2(V,\mu)$ we have to apply Theorem 3.6 with scale function $s(x) = -\frac{1}{2\nu\ x^{2\nu}}$ to obtain the following expression.

$$\operatorname{dom}\check{\mathcal{E}} = ranJ,$$

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2$$

$$= \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu}\ x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2 \quad \text{for all } u \in \operatorname{dom} J.$$

For the conservativeness property of the general Bessel's Dirichlet forms we have following result as an application of Theorem 4.1.

If μ is infinite, then $\check{\mathcal{E}}$ is conservative if and only if

$$\sum_{k=1}^{\infty} \frac{(x_{k+1}^{2\nu} - x_k^{2\nu})}{x_{k+1}^{2\nu}\ x_k^{2\nu}} \sum_{j=1}^{k} a_j = \infty. \tag{5.1}$$

Finally we consider the case where the sequence $(x_k)_{k\in\mathbb{N}}$ converges to x_∞. According to Proposition 3.10 we obtain:

(1) If ∞ is an approachable boundary. It holds

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)}\big(u(x_{k+1}) - u(x_k)\big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))}$$

$$= 2\nu \sum_{k=1}^{\infty} \frac{x_{k+1}^{2\nu}\ x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2 + 2\nu\ x_\infty^{2\nu} u(x_\infty)^2 \tag{5.2}$$

for all $u \in \operatorname{dom} J$.

Regarding the conservativeness property, according to Remark 4.4, the trace form $\check{\mathcal{E}}$ is not conservative.

(2) If ∞ is an non-approachable boundary, then we have

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)}\big(u(x_{k+1}) - u(x_k)\big)^2$$

$$= 2\nu \sum_{k=1}^{\infty} \frac{x_{k+1}^{2\nu}\ x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})} \big(u(x_{k+1}) - u(x_k)\big)^2 \quad \text{for all } u \in \operatorname{dom} J. \tag{5.3}$$

By the virtue of Remark 4.4, the trace form $\check{\mathcal{E}}$ is conservative if and only if condition (5.1) is fulfilled.

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