J. Korean Math. Soc. **60** (2023), No. 4, pp. 745–777
https://doi.org/10.4134/JKMS.j220200
pISSN: 0304-9914 / eISSN: 2234-3008

SOME ALGEBRAS HAVING RELATIONS LIKE THOSE FOR THE 4-DIMENSIONAL SKLYANIN ALGEBRAS

Alexandru Chirvasitu and S. Paul Smith

Abstract. The 4-dimensional Sklyanin algebras are a well-studied 2-parameter family of non-commutative graded algebras, often denoted $A(E,\tau)$, that depend on a quartic elliptic curve $E \subseteq \mathbb{P}^3$ and a translation automorphism τ of E. They are graded algebras generated by four degree-one elements subject to six quadratic relations and in many important ways they behave like the polynomial ring on four indeterminates except that they are not commutative. They can be seen as "elliptic analogues" of the enveloping algebra of $\mathfrak{gl}(2,\mathbb{C})$ and the quantized enveloping algebras $U_q(\mathfrak{gl}_2)$.

Recently, Cho, Hong, and Lau conjectured that a certain 2-parameter family of algebras arising in their work on homological mirror symmetry consists of 4-dimensional Sklyanin algebras. This paper shows their conjecture is false in the generality they make it. On the positive side, we show their algebras exhibit features that are similar to, and differ from, analogous features of the 4-dimensional Sklyanin algebras in interesting ways. We show that most of the Cho-Hong-Lau algebras determine, and are determined by, the graph of a bijection between two 20-point subsets of the projective space $\mathbb{P}^3$.

The paper also examines a 3-parameter family of 4-generator 6-relator algebras admitting presentations analogous to those of the 4-dimensional Sklyanin algebras. This class includes the 4-dimensional Sklyanin algebras and most of the Cho-Hong-Lau algebras.

1. Introduction

1.1. This paper examines three families of graded algebras with four generators and six quadratic relations. The only commutative algebra in these families is the polynomial ring on 4 variables. All algebras in these families

Received April 24, 2022; Revised March 10, 2023; Accepted April 26, 2023.

2020 *Mathematics Subject Classification.* Primary 14A22, 16E65, 16S38, 16W50; Secondary 17B37, 14H52.

Key words and phrases. Sklyanin algebras, graded algebras, 4 generators and 6 relations, elliptic curve.

The work of the first author was partially supported by NSF grants DMS-1565226, DMS-1801011 and DMS-2001128.

are, like the polynomial ring on 4 variables, generated by 4 elements subject to 6 homogeneous quadratic relations.

The members of the first of these families are denoted by $A(\alpha,\beta,\gamma)$, depending on a parameter $(\alpha,\beta,\gamma)\in\Bbbk^3$, where $\Bbbk$ is a field that will be fixed throughout the paper. They are generated by x_0,x_1,x_2,x_3 subject to the relations:

$$\begin{cases} x_0x_1 - x_1x_0 = \alpha(x_2x_3 + x_3x_2), & x_0x_1 + x_1x_0 = x_2x_3 - x_3x_2, \\ x_0x_2 - x_2x_0 = \beta(x_3x_1 + x_1x_3), & x_0x_2 + x_2x_0 = x_3x_1 - x_1x_3, \\ x_0x_3 - x_3x_0 = \gamma(x_1x_2 + x_2x_1), & x_0x_3 + x_3x_0 = x_1x_2 - x_2x_1. \end{cases} \tag{1.1}$$

Among these algebras, those for which

$$\alpha+\beta+\gamma+\alpha\beta\gamma = 0 \qquad \text{and} \qquad \{\alpha,\beta,\gamma\}\cap\{0,\pm1\} = \varnothing, \tag{1.2}$$

are so starkly different from the rest that we consider them as a separate family. These constitute the second of our three families and are called *non-degenerate 4-dimensional Sklyanin algebras.* Algebras in the third family are denoted by $R(a,b,c,d)$, depending on a parameter (a,b,c,d) that is required to lie on the quadric $\{ad+bc=0\}$ in the projective space $\mathbb{P}^3$. They are defined in 1.7.

The algebras $R(a,b,c,d)$ were discovered by Cho, Hong, and Lau in their work on mirror symmetry [8], and the motivation for this paper is their conjecture that these are 4-dimensional Sklyanin algebras. We prove their conjecture is false in the generality in which it is made, but on the positive side

(1) for a Zariski-dense open subset of points on the quadric $\{ad+bc=0\}$, $R(a,b,c,d)$ is isomorphic to $A(\alpha,\beta,\gamma)$ for some (α,β,γ), but (α,β,γ) does not always satisfy the condition $\alpha+\beta+\gamma+\alpha\beta\gamma=0$;
(2) there are two lines $\ell_1,\ell_2\subseteq\{ad+bc=0\}$ such that $R(a,b,c,d)$ is isomorphic to $A(\alpha,1,-1)$ for all $(a,b,c,d)\in\ell_1\cup\ell_2-\{12\text{ points}\}$;
(3) the automorphism group of almost all $R(a,b,c,d)$ has a subgroup isomorphic to the Heisenberg group of order 4^3.

The non-degenerate Sklyanin algebras may be parametrized by pairs (E,τ) consisting of an elliptic curve E and a translation automorphism $\tau: E\to E$. We write $A(E,\tau)$ for the Sklyanin algebra corresponding to this data. It is striking that the translation automorphism for the $R(a,b,c,d)$'s that *are* non-degenerate Sklyanin algebras has order 4; i.e., if $(a,b,c,d)\in\ell_1\cup\ell_2-\{12\text{ points}\}$, then $R(a,b,c,d)\cong A(E,\tau)$ for some elliptic curve E and some τ having order 4 (Propositions 5.2 and 5.3).

1.2. A striking feature of the algebras $R(a,b,c,d)$ is that almost all of them determine, and are determined by, a set of 20 points in the product $\mathbb{P}^3\times\mathbb{P}^3$ of two copies of the three-dimensional projective space.

1.3. Because Sklyanin algebras, which appeared first in [18,19], have played such a large role in the development of non-commutative algebra and algebraic geometry over the past thirty years (see [15,20–22,25] for example), it is sensible to examine the larger class of algebras $A(\alpha,\beta,\gamma)$ defined by the "same" relations minus the constraint $\alpha+\beta+\gamma+\alpha\beta\gamma=0$.

We do not undertake an exhaustive study of the algebras $A(\alpha,\beta,\gamma)$ when $\alpha+\beta+\gamma+\alpha\beta\gamma\neq 0$ but it appears to us that there are interesting questions about them that might be fruitfully pursued. We mention some of these questions in 1.10.

1.4. We use the notation $[x,y]=xy-yx$ and $\{x,y\}=xy+yx$.

1.5. The algebras $A(\alpha,\beta,\gamma)$. Let $\Bbbk$ be an arbitrary field and $\alpha_1,\alpha_2,\alpha_3\in\Bbbk$. Define $A(\alpha_1,\alpha_2,\alpha_3)$, or simply A, to be the free algebra $\Bbbk\langle x_0,x_1,x_2,x_3\rangle$ modulo the six relations:

$$[x_0,x_i]=\alpha_i\{x_j,x_k\},\quad \{x_0,x_i\}=[x_j,x_k], \tag{1.3}$$
$$(i,j,k) \text{ a cyclic permutation of } (1,2,3).$$

We always consider A as an $\mathbb{N}$-graded $\Bbbk$-algebra with $\deg\{x_0,x_1,x_2,x_3\}=1$. Thus, A is the quotient of the free algebra $TV/(R)=\Bbbk\langle x_0,x_1,x_2,x_3\rangle/(R)$, where $V=\operatorname{span}\{x_0,x_1,x_2,x_3\}$ and $R\subseteq V^{\otimes 2}$ is the linear span of the six elements in $V^{\otimes 2}$ corresponding to the relations (1.3).

1.6. Degenerate and non-degenerate 4-dimensional Sklyanin algebras. Suppose $\alpha+\beta+\gamma+\alpha\beta\gamma=0$. We call $A(\alpha,\beta,\gamma)$ a 4-*dimensional Sklyanin algebra* in this case. If, in addition, $\{\alpha,\beta,\gamma\}\cap\{0,\pm1\}=\varnothing$ we call $A(\alpha,\beta,\gamma)$ a *non-degenerate* 4-*dimensional Sklyanin algebra*. If $\alpha+\beta+\gamma+\alpha\beta\gamma=0$ and $\{\alpha,\beta,\gamma\}\cap\{0,\pm1\}\neq\varnothing$, we call $A(\alpha,\beta,\gamma)$ a *degenerate* 4-*dimensional Sklyanin algebra*.

By [21], non-degenerate 4-dimensional Sklyanin algebras are Noetherian domains having the same Hilbert series as the polynomial ring in 4 variables. By [21] and [15], they have excellent homological properties. Their representation theory is intimately related to the geometry of $(E\subseteq\mathbb{P}^3,\tau)$.

Some degenerate 4-dimensional Sklyanin algebras are closely related to better known algebras. For instance, the algebra $A=A(0,0,0)$ has a degree-one central element, z, such that $A/(z-1)\cong A[z^{-1}]_0\cong U(\mathfrak{so}(3,\Bbbk))$, the enveloping algebra of the Lie algebra $\mathfrak{so}(3,\Bbbk)$. Similarly, if $\Bbbk=\mathbb{C}$ and $\beta\neq 0$, then $A=A(0,\beta,-\beta)$ has a degree-two central element Ω such that $A[\Omega^{-1}]_0\cong U_q(\mathfrak{sl}(2,\mathbb{C}))$, a quantized enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$.

If $\alpha+\beta+\gamma+\alpha\beta\gamma=\alpha\beta\gamma=0$, then the structure of $A(\alpha,\beta,\gamma)$ is described in [21, §1.4].

1.7. The algebras of Cho, Hong, and Lau. Let $(a,b,c,d)\in\Bbbk^4$. We write $R(a,b,c,d)$, or simply R, for the free algebra $\Bbbk\langle x_1,x_2,x_3,x_4\rangle$ modulo the relations:

$$ax_4x_3+bx_3x_4+cx_3x_2+dx_4x_1=0, \tag{R1}$$

(R2) $ax_3x_2 + bx_2x_3 + cx_4x_3 + dx_1x_2 = 0,$

(R3) $ax_2x_1 + bx_1x_2 + cx_1x_4 + dx_2x_3 = 0,$

(R4) $ax_1x_4 + bx_4x_1 + cx_2x_1 + dx_3x_4 = 0,$

(R5) $ax_3x_1 - ax_1x_3 + cx_4^2 - cx_2^2 = 0,$

(R6) $bx_4x_2 - bx_2x_4 + dx_3^2 - dx_1^2 = 0.$

Since a, b, c, and d enter into the relations in a homogeneous way, the algebra $R(a,b,c,d)$ depends only on (a,b,c,d) as a point in $\mathbb{P}^3$. If we impose the condition that $ac+bd=0$, we obtain a 2-dimensional family of algebras $R(a,b,c,d)$ parametrized by a quadric (isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) in $\mathbb{P}^3$. Cho, Hong, and Lau conjecture that, when $ac+bd=0$, R is a 4-dimensional Sklyanin algebra [8, Conj. 8.11].

Although only a 1-parameter family of the $R(a,b,c,d)$ are Sklyanin algebras, we find it remarkable that almost all of them (Zariski-densely many, that is) have the "same" relations as the 4-dimensional Sklyanin algebras. We do not understand the deeper reason for this; our proof is just a calculation. We also find it remarkable that the translation automorphism for those that are Sklyanin algebras has order 4—the only translation automorphisms of a degree-four elliptic curve in $\mathbb{P}^3$ that extend to automorphisms of the ambient $\mathbb{P}^3$ are the translations of order 0, 2, and 4. We do not know in what way, in the context of the work of Cho-Hong-Lau, those $R(a,b,c,d)$ that are Sklyanin algebras are special.

1.8. Results about $A(\alpha,\beta,\gamma)$. Suppose $\alpha\beta\gamma \neq 0$. In Section 2 we show that the Heisenberg group of order 4^3 acts as automorphisms of $A(\alpha,\beta,\gamma)$. In Proposition 2.4, we determine exactly when two of these algebras are isomorphic to each other.

In Section 3 we give a geometric interpretation of the relations defining $A(\alpha,\beta,\gamma)$. To do this we first write $A(\alpha,\beta,\gamma)$ as $TV/(R)$, the quotient of the tensor algebra TV on a 4-dimensional vector space V by the ideal generated by a 6-dimensional subspace R of $V^{\otimes 2}$. We then consider elements in $V \otimes V$ as forms of bi-degree $(1,1)$ on the product $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ of two copies of projective 3-space. We now define the closed subscheme $\Gamma \subseteq \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ to be the vanishing locus of the elements in R. Proposition 3.3 shows that Γ is finite if and only if $\alpha\beta\gamma \neq 0$ and $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$. Propositions 3.3 and 3.4 show that in that case

(1) Γ consists of 20 distinct points;
(2) Γ is the graph of a bijection between 20-point subsets of $\mathbb{P}(V^*)$;
(3) $R = \{f \in V^{\otimes 2} \mid f|_\Gamma = 0\}$.

1.9. Centers. Theorem 2 in [18] states that two explicitly given degree-two homogeneous elements, which are denoted there by K_0 and K_1, belong to the center of the 4-dimensional Sklyanin algebra. Although Sklyanin writes that it is "straightforward" to prove these elements are central, the details are left to the reader. We and others have found the calculations less than straightforward.[1] Sklyanin says that an alternative proof can be given by using a lemma in his paper [14] with Kulish. Presumably, the relevant lemma is equation (5.7) in [14]. However, "due to space limitations [they] do not present [t]here the complete proof of (5.7)". We have been unable to find a complete proof of [18, Thm. 2] in the literature, so we give a direct proof that K_0 and K_1 are central in Proposition 6.1 below. We do not use the same notation as Sklyanin, so in this introduction we label the elements in Proposition 6.1 by Ω_0 and Ω_1.

For most 4-dimensional Sklyanin algebras the elements K_0 and K_1, or equivalently Ω_0 and Ω_1, generate the center of the algebra. In sharp contrast, when $\alpha\beta\gamma \neq 0$ and $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$ the elements x_0^2, x_1^2, x_2^2, and x_3^2 belong to the center of $A(\alpha,\beta,\gamma)$ (Proposition 6.2).

Cho, Hong, and Lau write down two degree-two elements in $R(a,b,c,d)$ that they conjecture belong to the center of $R(a,b,c,d)$. We verify their conjecture in Proposition 6.3 and Corollary 6.5.

It is interesting to compare the proof of these results about the centers to the proof that the Casimir elements in the enveloping algebras $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ belong to the center. The latter proofs are absolutely straightforward, whereas the computations involved in describing the centers of $A(\alpha,\beta,\gamma)$ are far less routine because these algebras do not have a PBW basis (or, apparently, any basis that makes computation routine). See however, the notion of an I-algebra in [23].

1.10. Some questions and remarks about $A(\alpha,\beta,\gamma)$. Computer calculations by Frank Moore suggest that the dimensions of the homogeneous components $A(\alpha,\beta,\gamma)_n$ are $1, 4, 10, 16, 19, 20, 20, 20, \ldots$ when $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$. Is this true? If so, then for a generic linear combination Ω of the central elements $x_0^2, x_1^2, x_2^2, x_3^2 \in A(\alpha,\beta,\gamma)_2$ the localization $A[\Omega^{-1}]_0$ is a finite dimensional algebra having dimension 20. What is the structure of this algebra? Is it a product of four copies of $\Bbbk$ and four copies of the 2×2 matrix algebra $M_2(\Bbbk)$ or is it more interesting?

We do not know if $A(\alpha,\beta,\gamma)$ is a Koszul algebra (Sklyanin algebras are) but whether it is or is not its quadratic dual $A(\alpha,\beta,\gamma)^!$ might be interesting.

We show in Section 3, when $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$ and $\alpha\beta\gamma \neq 0$, that the algebra $A(\alpha,\beta,\gamma)$ determines, and is determined by, a configuration of 20

[1] The problem of showing that K_0 and K_1 are central is mentioned in a talk given by Tom Koornwinder at Nijmegen on 12 November 2012—see `https://staff.science.uva.nl/t.h.koornwinder/art/sheets/SklyaninAlgebra1.pdf`, retrieved on 01-20-2017. Koornwinder says that part of the proof is "straightforward" and appeals to the Mathematica package NCAlgebra 4.0.4 at `http://www.math.ucsd.edu/~ncalg/` for the remainder of the proof.

points in $\mathbb{P}^3 \times \mathbb{P}^3$ that is the graph of a bijection between two 20-point elements of $\mathbb{P}^3$. We do not understand this configuration but the representation theory of $A(\alpha, \beta, \gamma)$ and these related algebras is governed by it. The details of this are likely to be interesting and novel.

It would be interesting to understand how the configuration of 20 points relates to the features of $R(a, b, c, d)$ that are relevant to the work of Cho, Hong, and Lau.

The point modules for a non-degenerate Sklyanin algebra $A(E, \tau)$ are parametrized by $E \subseteq \mathbb{P}^3$ together with 4 additional points. However, the other $A(\alpha, \beta, \gamma)$ have only 4 point modules (which are "the same" as the 4 special point modules for the Sklyanin algebra). Does $\mathsf{QGr}\big(A(\alpha, \beta, \gamma)\big)$ have exactly 8 fat point modules of multiplicity 2? (The definitions of point modules and fat point modules can be found in any paper on the 4-dimensional Sklyanin algebras.) Presumably, the fact that the automorphism θ defined in 3.3 has "order" 2 will be relevant.

Over $\mathbb{C}$, the structure constants α, β, γ for the 4-dimensional Sklyanin algebras have a nice description in terms of the theta functions ϑ_{00}, ϑ_{01}, ϑ_{10}, ϑ_{11} [21, §2.10]. Indeed, Sklyanin's original definition involved Jacobi's elliptic functions sn, cn, dn. Furthermore, the condition $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ is a consequence of Riemann's quartic identity $\vartheta_{00}^4(z) + \vartheta_{01}^4(z) = \vartheta_{01}^4(z) + \vartheta_{10}^4(z)$. It is possible that a better understanding of the algebras $R(a, b, c, d)$ might be obtained by realizing a, b, c, d as values of degenerations of elliptic functions. The expressions in Proposition 5.2(2) and the calculations in its proof are reminiscent of certain identities involving ϑ_{00}, ϑ_{01}, ϑ_{10}, ϑ_{11}.

Acknowledgements. We thank Frank Moore whose computer calculations involving the algebras $A(\alpha, \beta, \gamma)$ defined in (1.3) were of great assistance to us at an early stage of this project. His calculations showed that over certain finite fields the elements $x_0^2, \dots, x_3^2$ belong to the center of $A(\alpha, \beta, \gamma)$ when $\alpha\beta\gamma \neq 0$ and $\alpha + \beta + \gamma + \alpha\beta\gamma \neq 0$. Based on those calculations we then proved the centrality of those elements over all fields (Proposition 6.2).

We are also grateful for the anonymous referee's comments and suggestions; they have certainly contributed to the manuscript's improvement.

2. Algebras $A(\alpha, \beta, \gamma)$ with a Sklyanin-like presentation

2.1. Notation. Throughout this paper $\Bbbk$ denotes a field whose characteristic is not 2, and i denotes a fixed square root of -1.

Whenever we use parameters $\alpha, \beta, \gamma \in \Bbbk$ we will assume they have square roots $a, b, c \in \Bbbk$.

We fix a 4-dimensional $\Bbbk$-vector space V. Always, x_0, x_1, x_2, x_3 will denote a basis for V.

2.1.1. We write TV for the tensor algebra on V. Thus TV is the free algebra $\Bbbk\langle x_0, x_1, x_2, x_3\rangle$. We always consider TV as an $\mathbb{Z}$-graded $\Bbbk$-algebra with

$\deg(V) = 1$. All the algebras in this paper are of the form $A = TV/(R)$ for various 6-dimensional subspaces R of $V^{\otimes 2}$.

2.1.2. Let $\alpha, \beta, \gamma \in \Bbbk$. The algebra $A(\alpha, \beta, \gamma)$ is the free algebra TV modulo the relations in (1.1).

2.1.3. We will often write $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, \beta, \gamma)$. In Section 2 and Section 3, a, b, c will denote fixed square roots of α, β, γ. We will often write $(a_1, a_2, a_3) = (a, b, c)$.

2.1.4. Let A be a $\mathbb{Z}$-graded $\Bbbk$-algebra. We write $\operatorname{Aut}_{\mathsf{gr}}(A)$ for the group of graded $\Bbbk$-algebra automorphisms of A.

If $\lambda \in \Bbbk^\times$, ϕ_λ denotes the automorphism of A that is multiplication by λ^n on A_n. The map $\Bbbk^\times \to \operatorname{Aut}_{\mathsf{gr}}(A)$, $\lambda \mapsto \phi_\lambda$, is an injective group homomorphism whose image lies in the center. We will often identify λ with ϕ_λ. If $\psi \in \operatorname{Aut}(A)$, we will write $\psi^m = \lambda$ if $\psi^m = \phi_\lambda$ and $\lambda\psi$ for $\phi_\lambda\psi$.

2.1.5. Suppose $a, b, c \in \Bbbk^\times$. We define $\psi_1, \psi_2, \psi_3 \in \mathrm{GL}(V)$ by declaring that $\psi_i(x_j)$ is the entry in row ψ_i and column x_j in Table 1.

Table 1. Automorphisms ψ_1, ψ_2, ψ_3.

	x_0	x_1	x_2	x_3
ψ_1	bcx_1	$-ix_0$	$-ibx_3$	$-cx_2$
ψ_2	acx_2	$-ax_3$	$-ix_0$	$-icx_1$
ψ_3	abx_3	$-iax_2$	$-bx_1$	$-ix_0$

In the notation of 2.1.3, if (i, j, k) is a cyclic permutation of $(1, 2, 3)$, then

$$\psi_i(x_0) = a_j a_k x_i, \;\; \psi_i(x_i) = -ix_0, \;\; \psi_i(x_j) = -ia_j x_k, \;\; \text{and} \;\; \psi_i(x_k) = -a_k x_j.$$

2.1.6. *The Heisenberg group of order* 4^3. The Heisenberg group of order 4^3 is

$$H_4 \;:=\; \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1,\; \delta\varepsilon_1 = \varepsilon_1\delta,\; \varepsilon_2\delta = \delta\varepsilon_2,\; \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1 \rangle.$$

2.2. By [11] and [22, pp. 64–65], for example, the Heisenberg group H_4 acts as graded $\Bbbk$-algebra automorphisms of the 4-dimensional Sklyanin algebras when $\Bbbk = \mathbb{C}$. The next result records the fact that H_4 acts as graded $\Bbbk$-algebra automorphisms of $A(\alpha, \beta, \gamma)$ whenever $\alpha\beta\gamma \neq 0$ and $\Bbbk$ is a field having square roots of α, β, γ, and -1.

Proposition 2.1. *Suppose* $\alpha\beta\gamma \neq 0$. *Fix* $\nu_1, \nu_2, \nu_3 \in \Bbbk^\times$ *such that* $a\nu_1^2 = b\nu_2^2 = c\nu_3^2 = -iabc$.

(1) *The maps* $\psi_1, \psi_2, \psi_3 : V \to V$ *in Table 1 extend to* $\Bbbk$*-algebra automorphisms of* $A(\alpha, \beta, \gamma)$.

(2) *There is an injective homomorphism* $H_4 \to \mathrm{Aut}_{\mathsf{gr}}(A)$ *given by*

$$\varepsilon_1 \mapsto \nu_1^{-1}\psi_1, \quad \varepsilon_2 \mapsto \nu_2^{-1}\psi_2.$$

Under this map, $\delta \mapsto \phi_i$*, the automorphism that is multiplication by* i^n *on* $A(\alpha,\beta,\gamma)_n$.

(3) *The subgroup of* $\mathrm{Aut}_{\mathsf{gr}}(A)$ *generated by* $\gamma_1 := \varepsilon_1^2$ *and* $\gamma_2 := \varepsilon_2^2$ *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$. *The value of* $\gamma_i(x_j)$ *is the entry in row* γ_i *and column* x_j *of Table 2 below* (*in that table we also define an automorphism* γ_3).

TABLE 2. The action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as automorphisms of A.

	x_0	x_1	x_2	x_3
γ_1	x_0	x_1	$-x_2$	$-x_3$
γ_2	x_0	$-x_1$	x_2	$-x_3$
γ_3	x_0	$-x_1$	$-x_2$	x_3

Proof. Let (i,j,k) be a cyclic permutation of $(1,2,3)$ and let $\lambda_0,\lambda_i,\lambda_j,\lambda_k \in \Bbbk^\times$. In [19, Prop. 4], Sklyanin observed that the linear map $\psi : V \to V$ acting on x_0, x_i, x_j, x_k as

	x_0	x_i	x_j	x_k
ψ	$\lambda_0 x_i$	$\lambda_i x_0$	$\lambda_j x_k$	$\lambda_k x_j$

extends to an automorphism of the Sklyanin algebra if and only if

$$\frac{\lambda_0\lambda_i}{\lambda_j\lambda_k} = -1, \qquad \frac{\lambda_0\lambda_j}{\lambda_k\lambda_i} = -\alpha_j, \quad \text{and} \qquad \frac{\lambda_0\lambda_k}{\lambda_i\lambda_j} = \alpha_k. \tag{2.1}$$

A straightforward calculation shows that ψ extends to an automorphism of $A(\alpha,\beta,\gamma)$ without any restriction on α,β,γ other than $\alpha\beta\gamma \neq 0$ if and only if (2.1) holds. The maps ψ_1, ψ_2, and ψ_3 satisfy these conditions so extend to graded $\Bbbk$-algebra automorphisms of A.

It is easy to check that $\psi_1\psi_2 = \delta\psi_2\psi_1$, $\psi_2\psi_3 = \delta\psi_3\psi_2$, and $\psi_3\psi_1 = \delta\psi_1\psi_3$. It follows that $\varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1$.

It is easy to check that γ_1 and γ_2 act on x_0,x_1,x_2,x_3 as in Table 2. Hence $\varepsilon_1^4 = \varepsilon_2^4 = 1$. Simple calculations show that $\psi_1^2 = -ibc\gamma_1$, $\psi_2^2 = -iac\gamma_2$, and $\psi_3^2 = -iab\gamma_3$, where γ_1, γ_2, and γ_3 are the automorphisms in Table 2. We leave the rest of the proof to the reader. □

2.2.1. The maps $\gamma_i \in \mathrm{GL}(V)$ given by Table 2 extend to graded $\Bbbk$-algebra automorphisms of $A(\alpha,\beta,\gamma)$ for *all* $\alpha,\beta,\gamma \in \Bbbk$.

2.3. In the next result, whose proof we omit, $([A, A])$ denotes the ideal in A generated by all commutators $ab - ba$, $a, b \in A$. Thus, $A/([A, A])$ is the largest commutative quotient of A.

Proposition 2.2. *Suppose $\alpha\beta\gamma \neq 0$. Let $A = A(\alpha, \beta, \gamma)$.*

(1) *As a quotient of the polynomial ring $\Bbbk[x_0, x_1, x_2, x_3]$,*

$$\frac{A}{([A, A])} = \frac{\Bbbk[x_0, x_1, x_2, x_3]}{(x_1, x_2, x_3) \cap (x_0, x_2, x_3) \cap (x_0, x_1, x_3) \cap (x_0, x_1, x_2)}.$$

(2) *As a subscheme of $\mathbb{P}(V^*)$,*

$$\operatorname{Proj}\left(\frac{A}{([A, A])}\right) = \{e_0 = (1, 0, 0, 0), e_1 = (0, 1, 0, 0), e_2 = (0, 0, 1, 0), e_3 = (0, 0, 0, 1)\}.$$

(3) *A has exactly four graded quotients that are polynomial rings in one variable, namely the quotients by the ideals (x_1, x_2, x_3), (x_0, x_2, x_3), (x_0, x_1, x_3), and (x_0, x_1, x_2).*

Lemma 2.3. *There are algebra isomorphisms*

$$\begin{aligned} A(\alpha, \beta, \gamma) \;&\cong\; A(\beta, \gamma, \alpha) \;\cong\; A(\gamma, \alpha, \beta) \\ &\cong\; A(-\alpha, -\gamma, -\beta) \;\cong\; A(-\beta, -\alpha, -\gamma) \;\cong\; A(-\gamma, -\beta, -\alpha). \end{aligned}$$

Proof. There is an isomorphism $A(\alpha, \beta, \gamma) \xrightarrow{\sim} A(\beta, \gamma, \alpha)$ given by $x_0 \mapsto x_0$ and $x_i \mapsto x_{i+1}$ for $i \in \{1, 2, 3\} = \mathbb{Z}/3$. Similarly, $A(\beta, \gamma, \alpha) \cong A(\gamma, \alpha, \beta)$. Since

$$[x_0, -x_1] = -\alpha\{x_3, x_2\}, \quad [x_0, x_3] = -\gamma\{x_2, -x_1\}, \quad [x_0, x_2] = -\beta\{-x_1, x_3\},$$

and

$$\{x_0, -x_1\} = [x_3, x_2], \; \{x_0, x_3\} = [x_2, -x_1], \; \{x_0, x_2\} = [-x_1, x_3],$$

there is an isomorphism $A(\alpha, \beta, \gamma) \xrightarrow{\sim} A(-\alpha, -\gamma, -\beta)$ given by $x_0 \mapsto x_0$, $x_1 \mapsto -x_1$, $x_2 \mapsto x_3$, and $x_3 \mapsto -x_2$. □

Proposition 2.4. *Suppose $\alpha\beta\gamma \neq 0$ and $\alpha'\beta'\gamma' \neq 0$. Then $A(\alpha, \beta, \gamma) \cong A(\alpha', \beta', \gamma')$ as graded $\Bbbk$-algebras if and only if $(\alpha', \beta', \gamma')$ is a cyclic permutation of either (α, β, γ) or $(-\alpha, -\beta, -\gamma)$.*

Proof. ($\Leftarrow$) This is the content of Lemma 2.3.

($\Rightarrow$) Before starting the proof we introduce some notation. If (p, q, r, s) is a permutation of $(0, 1, 2, 3)$, we define

$$\langle p, q, r, s\rangle_A := (\mu_1\nu_1, \mu_2\nu_2, \mu_3\nu_3) \;\in\; \Bbbk^3,$$

where $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ are the unique scalars such that

$$\begin{aligned} [x_p, x_q] &= \mu_1\{x_r, x_s\}, \qquad & \nu_1\{x_p, x_q\} &= [x_r, x_s], \\ [x_p, x_r] &= \mu_2\{x_r, x_s\}, \qquad & \nu_2\{x_p, x_r\} &= [x_s, x_q], \\ [x_p, x_s] &= \mu_3\{x_r, x_s\}, \qquad & \nu_3\{x_p, x_s\} &= [x_q, x_r], \end{aligned}$$

in A. It is easy to see that

$$(2.2)\quad \begin{cases} \langle 0,1,2,3\rangle_A = \langle 1,0,3,2\rangle_A = \langle 2,3,0,1\rangle_A = \langle 3,2,1,0\rangle_A = (\alpha_1,\alpha_2,\alpha_3) \text{ and} \\ \langle 0,1,3,2\rangle_A = \langle 1,2,3,0\rangle_A = \langle 2,1,0,3\rangle_A = \langle 3,1,2,0\rangle_A = (-\alpha_1,-\alpha_3,-\alpha_2). \end{cases}$$

If $\langle p,q,r,s\rangle_A = (\lambda_1,\lambda_2,\lambda_3)$, then $\langle p,r,s,q\rangle_A = (\lambda_2,\, \lambda_3,\lambda_1)$. Using this and the equalities in (2.2), it is easy to compute $\langle p,q,r,s\rangle_A$ for all permutations (p,q,r,s) of $(0,1,2,3)$.

Let's write $A = A(\alpha,\beta,\gamma)$ and $B = A(\alpha',\beta',\gamma')$. To distinguish the presentation of A from that for B we will write x_0,x_1,x_2,x_3 for the generators of A, as in (1.3), and write x_0',x_1',x_2',x_3' for the generators of B. Thus, if $(\beta_1,\beta_2,\beta_3) = (\alpha',\beta',\gamma')$, then $[x_0',x_i'] = \beta_i\{x_j',x_k'\}$ and $\{x_0',x_i'\} = [x_j',x_k']$ for each cyclic permutation (i,j,k) of $(1,2,3)$.

Suppose $\Phi : A \to B$ is an isomorphism of graded $\Bbbk$-algebras. The restriction of Φ to A_1 is a vector space isomorphism $A_1 \to B_1$. It induces an isomorphism $\varphi : \mathbb{P}(B_1^*) \to \mathbb{P}(A_1^*)$. Let's denote the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1) \in \mathbb{P}(B_1^*)$ by e_0', e_1', e_2', e_3', respectively. Since Φ induces an isomorphism $A/([A,A]) \to B/([B,B])$, φ restricts to an isomorphism $\operatorname{Proj}(B/([B,B])) \to \operatorname{Proj}(A/([A,A]))$. Therefore $\varphi(\{e_0',e_1',e_2',e_3'\}) = \{e_0,e_1,e_2,e_3\}$. Since each x_m vanishes at exactly 3 points in $\{e_0,e_1,e_2,e_3\}$, $\Phi(x_m)$ vanishes at exactly 3 points in $\{e_0',e_1',e_2',e_3'\}$. It follows that there are non-zero scalars λ_0, λ_1, λ_2, λ_3 and a permutation (p,q,r,s) of $(0,1,2,3)$ such that $\Phi(x_0) = \lambda_0 x_p'$, $\Phi(x_1) = \lambda_1 x_q'$, $\Phi(x_2) = \lambda_2 x_r'$, and $\Phi(x_3) = \lambda_3 x_s'$.

Since $\{x_0,x_i\} = [x_j,x_k]$ for every cyclic permutation (i,j,k) of $(1,2,3)$,

$$\begin{aligned} \lambda_0\lambda_1\{x_p',x_q'\} &= \lambda_2\lambda_3[x_r',x_s'], \\ \lambda_0\lambda_2\{x_p',x_r'\} &= \lambda_3\lambda_1[x_s',x_q'], \\ \lambda_0\lambda_3\{x_p',x_s'\} &= \lambda_1\lambda_2[x_q',x_r']. \end{aligned}$$

Since $[x_0,x_i] = \alpha_i\{x_j,x_k\}$ for every cyclic permutation (i,j,k) of $(1,2,3)$,

$$\begin{aligned} \lambda_0\lambda_1[x_p',x_q'] &= \alpha_1\lambda_2\lambda_3\{x_r',x_s'\}, \\ \lambda_0\lambda_2[x_p',x_r'] &= \alpha_2\lambda_3\lambda_1\{x_s',x_q'\}, \\ \lambda_0\lambda_3[x_p',x_s'] &= \alpha_3\lambda_1\lambda_2\{x_q',x_r'\}. \end{aligned}$$

It follows that

$$\begin{aligned} [x_p',x_q'] &= \alpha_1\lambda_0^{-1}\lambda_1^{-1}\lambda_2\lambda_3\{x_r',x_s'\}, & [x_r',x_s'] &= \lambda_0\lambda_1\lambda_2^{-1}\lambda_3^{-1}\{x_p',x_q'\}, \\ [x_p',x_r'] &= \alpha_2\lambda_0^{-1}\lambda_2^{-1}\lambda_3\lambda_1\{x_s',x_q'\}, & [x_s',x_q'] &= \lambda_0\lambda_2\lambda_3^{-1}\lambda_1^{-1}\{x_p',x_r'\}, \\ [x_p',x_s'] &= \alpha_3\lambda_0^{-1}\lambda_3^{-1}\lambda_1\lambda_2\{x_q',x_r'\}, & [x_q',x_r'] &= \lambda_0\lambda_3\lambda_1^{-1}\lambda_2^{-1}\{x_p',x_s'\}. \end{aligned}$$

Therefore $\langle p,q,r,s\rangle_B = (\alpha,\beta,\gamma) = \langle 0,1,2,3\rangle_A$. It now follows from (2.2) and the sentence after it that $\langle 0,1,2,3\rangle_B$ is a cyclic permutation of either (α,β,γ) or $(-\alpha,-\beta,-\gamma)$; since $\langle 0,1,2,3\rangle_B = (\alpha',\beta',\gamma')$, the proof is complete. □

3. The zero locus of the relations for $A(\alpha, \beta, \gamma)$

The material in 3.1 applies to all graded algebras defined by 4 generators and 6 quadratic relations, i.e., to all algebras A of the form $TV/(R)$, where V and R are as in the next paragraph.

3.1. Quadratic algebras on 4 generators with 6 relations. Let V be a 4-dimensional vector space over $\Bbbk$, R a 6-dimensional subspace of $V^{\otimes 2}$. Let $\mathbb{P} = \mathbb{P}(V^*) \cong \mathbb{P}^3$. Let $\Gamma \subseteq \mathbb{P} \times \mathbb{P}$ be the scheme-theoretic zero locus of R (viewed as forms of bi-degree $(1,1)$). For example, if A is the polynomial ring, then R consists of the skew-symmetric tensors and Γ is the diagonal.

Since $\dim_{\Bbbk}(R) = 6 = \dim(\mathbb{P}^3 \times \mathbb{P}^3)$, $\Gamma \neq \varnothing$.

Proposition 3.1. *Suppose* $\dim(\Gamma) = 0$. *Then*

(1) Γ *consists of* 20 *points counted with multiplicity, and*
(2) *the subspace of* $V \otimes V$ *that vanishes on* Γ *is* R [17, Thm. 4.1].

Proof. (1) The Chow ring of $\mathbb{P}^3$ is isomorphic to $\mathbb{Z}[t]/(t^4)$ with t the class of a hyperplane. The Chow ring of $\mathbb{P}^3 \times \mathbb{P}^3$ is isomorphic to $\mathbb{Z}[s,t]/(s^4, t^4)$ and the class of the zero locus of a non-zero element in $V \otimes V$ is equal to $s+t$. If $\dim(\Gamma) = 0$, then the class of Γ is $(s+t)^6$ since $\dim(R) = 6$. But $(s+t)^6 = 20s^3t^3$ so the cardinality of Γ is 20 when its points are counted with multiplicity.

(2) This is [17, Thm. 4.1]. □

3.2. We now explain our strategy for computing Γ for $A(\alpha, \beta, \gamma)$.

Let $\mathbf{x}$ denote the row vector (x_0, x_1, x_2, x_3) over $\Bbbk\langle x_0, x_1, x_2, x_3\rangle$ and let $\mathbf{x}^{\mathsf{T}}$ denote its transpose.

The relations defining $A(\alpha, \beta, \gamma)$ can be written as a single matrix equation, $M\mathbf{x}^{\mathsf{T}} = 0$, over $\Bbbk\langle x_0, x_1, x_2, x_3\rangle$, where

$$M := \begin{pmatrix} -x_1 & x_0 & -\alpha x_3 & -\alpha x_2 \\ -x_2 & -\beta x_3 & x_0 & -\beta x_1 \\ -x_3 & -\gamma x_2 & -\gamma x_1 & x_0 \\ -x_3 & -x_2 & x_1 & -x_0 \\ -x_1 & -x_0 & -x_3 & x_2 \\ -x_2 & x_3 & -x_0 & -x_1 \end{pmatrix}. \tag{3.1}$$

The relations can also be written as $\mathbf{x}M' = 0$, where

$$M' := \begin{pmatrix} -x_1 & -x_2 & -x_3 & -x_3 & -x_1 & -x_2 \\ x_0 & \beta x_3 & \gamma x_2 & x_2 & -x_0 & -x_3 \\ \alpha x_3 & x_0 & \gamma x_1 & -x_1 & x_3 & -x_0 \\ \alpha x_2 & \beta x_1 & x_0 & -x_0 & -x_2 & x_1 \end{pmatrix}. \tag{3.2}$$

Consider the entries in M (resp., $\mathbf{x}$) as linear forms on the left-hand (resp., right-hand) factor of $\mathbb{P}^3 \times \mathbb{P}^3 = \mathbb{P}(V^*) \times \mathbb{P}(V^*)$. Then Γ is the scheme-theoretic zero locus of the six entries in $M\mathbf{x}^{\mathsf{T}}$ when those entries are viewed as bi-homogeneous elements in $\Bbbk[x_0, x_1, x_2, x_3] \otimes \Bbbk[x_0, x_1, x_2, x_3]$.

Let $\mathrm{pr}_1 : \Gamma \to \mathbb{P}^3$ and $\mathrm{pr}_2 : \Gamma \to \mathbb{P}^3$ be the projections $\mathrm{pr}_1(p, p') = p$ and $\mathrm{pr}_2(p, p') = p'$.

If $p \in \mathbb{P}^3$, then $p \in \mathrm{pr}_1(\Gamma)$ if and only if there is a point $p' \in \mathbb{P}^3$ such that $M(p)\mathbf{x}^{\mathsf{T}}(p') = 0$; i.e., if and only if $\mathrm{rank}(M(p)) < 4$. Thus, $\mathrm{pr}_1(\Gamma)$ is the scheme-theoretic zero locus of the 4×4 minors of M. Similarly, $\mathrm{pr}_2(\Gamma)$ is the scheme-theoretic zero locus of the 4×4 minors of M'.

Lemma 3.2. *If λ, μ, ν are non-zero scalars, then the intersection of the three quadrics*

$$x_0x_1 - \lambda^2 x_2x_3 = 0, \qquad x_0x_2 - \mu^2 x_1x_3 = 0, \qquad x_0x_3 - \nu^2 x_1x_2 = 0,$$

consists of the eight points

$$\begin{array}{llll} (0,1,0,0), & (0,0,1,0), & (\lambda\mu\nu, \lambda, \mu, \nu), & (\lambda\mu\nu, -\lambda, -\mu, \nu), \\ (1,0,0,0), & (0,0,0,1), & (\lambda\mu\nu, -\lambda, \mu, -\nu), & (\lambda\mu\nu, \lambda, -\mu, -\nu). \end{array}$$

Proof. The line $x_0 - \lambda\mu x_3 = x_1 - \lambda\mu^{-1}x_2 = 0$ lies on the quadric $x_0x_1 - \lambda^2 x_2x_3 = 0$ because

$$x_0x_1 - \lambda^2 x_2x_3 = (x_0 - \lambda\mu x_3)x_1 + (x_1 - \lambda\mu^{-1}x_2)\lambda\mu x_3$$

and on the quadric $x_0x_2 - \mu^2 x_1x_3 = 0$ because

$$x_0x_2 - \mu^2 x_1x_3 = (x_0 - \lambda\mu x_3)x_2 - (x_1 - \lambda\mu^{-1}x_2)\mu^2 x_3.$$

Continuing in this vein, the lines $x_0 = x_3 = 0$, $x_1 = x_2 = 0$, $x_0 - \lambda\mu x_3 = x_1 - \lambda\mu^{-1}x_2 = 0$, and $x_0 + \lambda\mu x_3 = x_1 + \lambda\mu^{-1}x_2 = 0$, lie on the quadrics $x_0x_1 - \lambda^2 x_2x_3 = 0$ and $x_0x_2 - \mu^2 x_1x_3 = 0$. By Bézout's theorem, the intersection of these two quadrics is a curve of degree 4 in $\mathbb{P}^3$ so is the union of these four lines.

The quadric $x_0x_3 - \nu^2 x_1x_2 = 0$ meets the line $x_0 = x_3 = 0$ at $(0,1,0,0)$ and $(0,0,1,0)$; the line $x_1 = x_2 = 0$ at $(1,0,0,0)$ and $(0,0,0,1)$; the line $x_0 - \lambda\mu x_3 = x_1 - \lambda\mu^{-1}x_2 = 0$ at $(\lambda\mu\nu, \lambda, \mu, \nu)$ and $(\lambda\mu\nu, -\lambda, -\mu, \nu)$; and the line $x_0 + \lambda\mu x_3 = x_1 + \lambda\mu^{-1}x_2 = 0$ at $(\lambda\mu\nu, -\lambda, \mu, -\nu)$ and $(\lambda\mu\nu, \lambda, -\mu, -\nu)$. The proof is complete. □

Proposition 3.3. *The scheme Γ associated to the algebra $TV/(R) = A(\alpha, \beta, \gamma)$ is finite if and only if $\alpha\beta\gamma \neq 0$ and $\alpha + \beta + \gamma + \alpha\beta\gamma \neq 0$.*

Proof. Before starting the proof we introduce some notation.

We label the following four polynomials in the symmetric algebra SV:

$$\begin{aligned} q &:= x_0^2 + x_1^2 + x_2^2 + x_3^2, \\ q_1 &:= x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2, \\ q_2 &:= x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2, \\ q_3 &:= x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2. \end{aligned}$$

We write h_{ij} for the 4×4 minor of M obtained by deleting rows i and j. Up to non-zero scalar multiples,

$$\begin{aligned}
h_{23} &= (x_0x_1 - \alpha x_2x_3)q, \quad h_{46} = (x_0x_1 + \alpha x_2x_3)q_1, \quad h_{24} = (x_0x_1 - x_2x_3)q_2,\\
h_{36} &= (x_0x_1 + x_2x_3)q_3, \quad h_{13} = (x_0x_2 - \beta x_1x_3)q, \quad h_{14} = (x_0x_2 + x_1x_3)q_1,\\
h_{45} &= (x_0x_2 + \beta x_1x_3)q_2, \quad h_{35} = (x_0x_2 - x_1x_3)q_3, \quad h_{12} = (x_0x_3 - \gamma x_1x_2)q,\\
h_{16} &= (x_0x_3 - x_1x_2)q_1, \quad h_{25} = (x_0x_3 + x_1x_2)q_2, \quad h_{56} = (x_0x_3 + \gamma x_1x_2)q_3,\\
h_{34} &= (\alpha\beta x_3^2 - x_0^2)(x_1^2 + x_2^2) + (\alpha x_2^2 - \beta x_1^2)(x_0^2 + x_3^2)\\
&= (\alpha\beta x_3^2 - x_0^2)q + (x_0^2 + x_3^2)q_3,\\
h_{26} &= (\alpha\gamma x_2^2 - x_0^2)(x_1^2 + x_3^2) + (\gamma x_1^2 - \alpha x_3^2)(x_0^2 + x_2^2)\\
&= (\alpha\gamma x_2^2 - x_0^2)q + (x_0^2 + x_2^2)q_2,\\
h_{15} &= (\beta\gamma x_1^2 - x_0^2)(x_2^2 + x_3^2) + (\beta x_3^2 - \gamma x_2^2)(x_0^2 + x_1^2)\\
&= (\beta\gamma x_1^2 - x_0^2)q + (x_0^2 + x_1^2)q_1.
\end{aligned}$$

These are the "same" expressions as those in the proof of [21, Prop. 2.4].[2]

We write g_{ij} for the 4×4 minor of M' obtained by deleting columns i and j.

If $p = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^3$ we write $\ominus p$ for the point $(-\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ and $M'(\ominus p)$ for the matrix M' evaluated at $\ominus p$. The matrices $M'(\ominus p)$ and $-M(p)^{\mathsf{T}}$ are almost the same: the only difference is that the top row of $M'(\ominus p)$ is the negative of the top row of $-M(p)^{\mathsf{T}}$. This observation makes it easy to compute the 4×4 minors of M' from the 4×4 minors of M. Doing that, up to non-zero scalar multiples we obtain

$$\begin{aligned}
g_{23} &= (x_0x_1 + \alpha x_2x_3)q, \quad g_{46} = (x_0x_1 - \alpha x_2x_3)q_1, \quad g_{24} = (x_0x_1 + x_2x_3)q_2,\\
g_{36} &= (x_0x_1 - x_2x_3)q_3, \quad g_{13} = (x_0x_2 + \beta x_1x_3)q, \quad g_{14} = (x_0x_2 - x_1x_3)q_1,\\
g_{45} &= (x_0x_2 - \beta x_1x_3)q_2, \quad g_{35} = (x_0x_2 + x_1x_3)q_3, \quad g_{12} = (x_0x_3 + \gamma x_1x_2)q,\\
g_{16} &= (x_0x_3 + x_1x_2)q_1, \quad g_{25} = (x_0x_3 - x_1x_2)q_2, \quad g_{56} = (x_0x_3 - \gamma x_1x_2)q_3,\\
g_{34} &= (\alpha\beta x_3^2 - x_0^2)(x_1^2 + x_2^2) + (\alpha x_2^2 - \beta x_1^2)(x_0^2 + x_3^2)\\
&= (\alpha\beta x_3^2 - x_0^2)q + (x_0^2 + x_3^2)q_3,\\
g_{26} &= (\alpha\gamma x_2^2 - x_0^2)(x_1^2 + x_3^2) + (\gamma x_1^2 - \alpha x_3^2)(x_0^2 + x_2^2)\\
&= (\alpha\gamma x_2^2 - x_0^2)q + (x_0^2 + x_2^2)q_2,\\
g_{15} &= (\beta\gamma x_1^2 - x_0^2)(x_2^2 + x_3^2) + (\beta x_3^2 - \gamma x_2^2)(x_0^2 + x_1^2)\\
&= (\beta\gamma x_1^2 - x_0^2)q + (x_0^2 + x_1^2)q_1.
\end{aligned}$$

In particular, up to non-zero scalar multiples,

$$g_{ij}(x_0, x_1, x_2, x_3) = h_{ij}(-x_0, x_1, x_2, x_3)$$

[2]The phrase "making frequent use of (0.2.1)" in the second sentence in the proof of [21, Prop. 2.4] should be deleted in order to make that sentence true.

for all i and j. Hence $\mathrm{pr}_2(\Gamma) = \ominus \mathrm{pr}_1(\Gamma)$.

It follows that $\mathrm{pr}_1(\Gamma)$ is finite if and only if $\mathrm{pr}_2(\Gamma)$ is finite if and only if Γ is finite.

($\Leftarrow$) Suppose $\alpha\beta\gamma \neq 0$ and $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$.

Let C be an irreducible component of $\mathrm{pr}_1(\Gamma)$. Since h_{12}, h_{13}, and h_{23}, vanish on C, either q vanishes on C or C is in the zero locus of the other factors of h_{12}, h_{13}, and h_{23}; i.e., in the common zero locus of $x_0x_1 - \alpha x_2x_3$, $x_0x_2 - \beta x_1x_3$, and $x_0x_3 - \gamma x_1x_2$; but that common zero locus is finite by Lemma 3.2 so either C is finite or q vanishes on C. Likewise, if $q_j \in \{q_1, q_2, q_3\}$, either C is finite or q_j vanishes on C. Thus, either C is finite or all four of q, q_1, q_2, and q_3, vanish on C. However, the set $\{q, q_1, q_2, q_3\}$ is linearly independent because the determinant

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\beta\gamma & -\gamma & \beta \\ 1 & \gamma & -\alpha\gamma & -\alpha \\ 1 & -\beta & \alpha & -\alpha\beta \end{pmatrix} = -(\alpha+\beta+\gamma+\alpha\beta\gamma)^2$$

is non-zero so the common zero-locus of q, q_1, q_2, and q_3, is empty. We conclude that C is finite. It follows that $\mathrm{pr}_1(\Gamma)$, and hence Γ, is finite.

($\Rightarrow$) Suppose Γ is finite.

If $\alpha+\beta+\gamma+\alpha\beta\gamma = 0$, then $\mathrm{span}\{q, q_1\} = \mathrm{span}\{q, q_2\} = \mathrm{span}\{q, q_3\}$. It follows that all h_{ij} vanish on $\{q = q_1 = 0\}$ whence $\{q = q_1 = 0\} \subseteq \mathrm{pr}_1(\Gamma)$. But this is ridiculous because $\{q = q_1 = 0\}$ is a curve, hence infinite, so we conclude that $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$.

If $\alpha = 0$, then all h_{ij} vanish on the line $x_0 = x_1 = 0$; i.e., $\{x_0 = x_1 = 0\} \subseteq \mathrm{pr}_1(\Gamma)$; this is not the case because Γ is finite so we conclude that $\alpha \neq 0$. If $\beta = 0$, then $\{x_0 = x_2 = 0\} \subseteq \mathrm{pr}_1(\Gamma)$; this is not the case so we conclude that $\beta \neq 0$. If $\gamma = 0$, then $\{x_0 = x_3 = 0\} \subseteq \mathrm{pr}_1(\Gamma)$; this is not the case so we conclude that $\gamma \neq 0$. Thus, $\alpha\beta\gamma \neq 0$. □

3.3. Suppose $\alpha\beta\gamma \neq 0$. Let $\mathfrak{P} \subseteq \mathbb{P}^3$ denote the set 20 of points in the following table.

TABLE 3. The points in $\mathfrak{P}$.

$\mathfrak{P}_\infty$	$\mathfrak{P}_0$	$\mathfrak{P}_1$	$\mathfrak{P}_2$	$\mathfrak{P}_3$	
$(1,0,0,0)$	(abc,a,b,c)	$(-a,ia,i,1)$	$(-b,1,ib,i)$	$(-c,i,1,ic)$	
$(0,1,0,0)$	$(abc,a,-b,-c)$	$(a,-ia,i,1)$	$(b,-1,ib,i)$	$(c,-i,1,ic)$	γ_1
$(0,0,1,0)$	$(abc,-a,b,-c)$	$(a,ia,-i,1)$	$(b,1,-ib,i)$	$(c,i,-1,ic)$	γ_2
$(0,0,0,1)$	$(abc,-a,-b,c)$	$(a,ia,i,-1)$	$(b,1,ib,-i)$	$(c,i,1,-ic)$	γ_3

Let $\ominus : \mathbb{P}(V^*) \to \mathbb{P}(V^*)$ and $\theta : \mathfrak{P} \to \ominus\mathfrak{P}$ be the maps $\ominus(\xi_0, \xi_1, \xi_2, \xi_3) = (-\xi_0, \xi_1, \xi_2, \xi_3)$ and

$$\theta(p) := \begin{cases} p & \text{if } p \in \mathfrak{P}_\infty, \\ \ominus p & \text{if } p \in \mathfrak{P}_0, \\ \ominus\gamma_i(p) & \text{if } p \in \mathfrak{P}_i,\ i = 1, 2, 3, \end{cases}$$

where γ_i is defined in Table 2. As a permutation of $\mathfrak{P} \cup \ominus\mathfrak{P}$, θ has order 2.

Proposition 3.4. *Suppose the subscheme $\Gamma \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ determined by the relations for $A(\alpha, \beta, \gamma)$ is finite. Then Γ is the graph of the bijection $\theta : \mathfrak{P} \to \ominus\mathfrak{P}$ and consists of* 20 *distinct points.*

Proof. Since Γ is finite, both $\alpha\beta\gamma$ and $\alpha + \beta + \gamma + \alpha\beta\gamma$ are non-zero. Since $\alpha\beta\gamma \neq 0$, each column of Table 3 consists of four distinct points. It is easy to see that

$$\mathfrak{P}_\infty \cap (\mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3) = \mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathfrak{P}_2 \cap \mathfrak{P}_3 = \mathfrak{P}_3 \cap \mathfrak{P}_1 = \varnothing.$$

If $(a, \xi_1, \xi_2, \xi_3) \in \mathfrak{P}_0$, then $\xi_1 = a\xi_2\xi_3$; if $(a, \xi_1, \xi_2, \xi_3) \in \mathfrak{P}_1$, then $\xi_1 = -a\xi_2\xi_3$; hence $\mathfrak{P}_0 \cap \mathfrak{P}_1 = \varnothing$. Similarly, if $(b, \xi_1, \xi_2, \xi_3) \in \mathfrak{P}_0$, then $\xi_2 = b\xi_1\xi_3$ whereas if $(b, \xi_1, \xi_2, \xi_3) \in \mathfrak{P}_2$, then $\xi_2 = -b\xi_1\xi_3$ so $\mathfrak{P}_0 \cap \mathfrak{P}_2 = \varnothing$. The same sort of argument shows that $\mathfrak{P}_0 \cap \mathfrak{P}_3 = \varnothing$. Thus, $\mathfrak{P}$ is the disjoint union of five sets each of which consists of four distinct points. Hence $\mathfrak{P}$ consists of 20 distinct points.

Let Γ_θ denote the graph of $\theta : \mathfrak{P} \to \ominus\mathfrak{P}$.

To complete the proof we must show that the vanishing locus in $\mathbb{P} \times \mathbb{P}$ of the polynomials (bilinear forms)

$$(3.3) \qquad \begin{cases} x_0 \otimes x_1 - x_1 \otimes x_0 \ - \ \alpha(x_2 \otimes x_3 + x_3 \otimes x_2), \\ x_0 \otimes x_1 + x_1 \otimes x_0 \ - \ x_2 \otimes x_3 \ + \ x_3 \otimes x_2; \\ x_0 \otimes x_2 - x_2 \otimes x_0 \ - \ \beta(x_3 \otimes x_1 + x_1 \otimes x_3), \\ x_0 \otimes x_2 + x_2 \otimes x_0 \ - \ x_3 \otimes x_1 \ + \ x_1 \otimes x_3; \\ x_0 \otimes x_3 - x_3 \otimes x_0 \ - \ \gamma(x_1 \otimes x_2 + x_2 \otimes x_1), \\ x_0 \otimes x_3 + x_3 \otimes x_0 \ - \ x_1 \otimes x_2 \ + \ x_2 \otimes x_1; \end{cases}$$

is exactly Γ_θ.

Clearly, if $p \in \mathfrak{P}_\infty$, then all six polynomials in (3.3) vanish at $(p, p) = (p, \theta(p))$.

Suppose $p \in \mathfrak{P}_0$. Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. Since $(p, \theta(p)) = (p, \ominus p)$, $x_0 \otimes x_i + x_i \otimes x_0$ and $x_j \otimes x_k - x_k \otimes x_j$ vanish at $(p, \theta(p))$; the three polynomials in the second column of (3.3) therefore vanish at $(p, \theta(p))$. On the other hand, $x_0 \otimes x_i - x_i \otimes x_0 - \alpha_i(x_j \otimes x_k + x_k \otimes x_j)$ vanishes at $(p, \theta(p))$ if and only if $2x_0 \otimes x_i - 2\alpha_i x_j \otimes x_k$ does. This vanishes at $(p, \ominus p)$ because $2x_0x_i - 2\alpha_i x_j x_k$ vanishes at p.

Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. Suppose $p = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathfrak{P}_i$. Then $\theta(p) = (\xi'_0, \xi'_1, \xi'_2, \xi'_3)$, where $\xi'_i = -\xi_i$ and $\xi'_\ell = \xi_\ell$ if $\ell \in \{0, 1, 2, 3\} -$

$\{i\}$. It follows that

$$\begin{aligned}\big(x_0\otimes x_i - x_i\otimes x_0 \;-\; \alpha_i(x_j\otimes x_k + x_k\otimes x_j)\big)\big|_{(p,\theta(p))} &= -2\xi_0\xi_i - 2\alpha_i\xi_j\xi_k,\\ \big(x_0\otimes x_j + x_j\otimes x_0 \;-\; x_k\otimes x_i + x_i\otimes x_k\big)\big|_{(p,\theta(p))} &= 2\xi_0\xi_j + 2\xi_k\xi_i,\\ \big(x_0\otimes x_k + x_k\otimes x_0 \;-\; x_i\otimes x_j + x_j\otimes x_i\big)\big|_{(p,\theta(p))} &= 2\xi_0\xi_k - 2\xi_i\xi_j.\end{aligned}$$

A case-by-case inspection shows that these three expressions are zero; thus, three of the polynomials in (3.3) vanish at $(p,\theta(p))$. The other three polynomials in (3.3) also vanish at $(p,\theta(p))$ because the polynomials

$$\begin{aligned}&x_0\otimes x_i + x_i\otimes x_0, \quad x_0\otimes x_j - x_j\otimes x_0, \quad x_0\otimes x_k - x_k\otimes x_0,\\ &x_i\otimes x_j + x_j\otimes x_i, \quad x_k\otimes x_i + x_i\otimes x_k, \quad x_j\otimes x_k - x_k\otimes x_j,\end{aligned}$$

vanish at $(p,\theta(p))$.

We have shown that the polynomials in (3.3) vanish on Γ_θ. This completes the proof that $\Gamma_\theta \subseteq \Gamma$. In particular, $\mathfrak{P} \subseteq \mathrm{pr}_1(\Gamma)$. To complete the proof of the proposition we must show the polynomials in (3.3) do not vanish outside Γ_θ or, equivalently, that $\mathrm{pr}_1(\Gamma) = \mathfrak{P}$.

With this goal in mind let $p \in \mathrm{pr}_1(\Gamma)$. We observed in the proof of Proposition 3.3, that

$$\{q = q_1 = q_2 = q_3 = 0\} = \varnothing.$$

If q does not vanish at p, then Lemma 3.2 implies that $p \in \mathfrak{P}_\infty \cup \mathfrak{P}_0$. Likewise, if $q_j \in \{q_1,q_2,q_3\}$ and does not vanish at p, then Lemma 3.2 tells us that $p \in \mathfrak{P}_\infty \cup \mathfrak{P}_j$. We conclude that $p \in \mathfrak{P}$. □

Corollary 3.5. *If $\alpha\beta\gamma \neq 0$ and $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$, then $A(\alpha,\beta,\gamma)$ is isomorphic to $TV/(R)$, where $R \subseteq V^{\otimes 2}$ is the subspace consisting of those $(1,1)$ forms that vanish on the graph of the bijection $\theta : \mathfrak{P} \to \ominus\mathfrak{P}$.*

Any deeper meaning of the data $(\mathfrak{P},\theta)$ eludes us.

3.4. Remarks. In 3.4 we assume that $\alpha\beta\gamma \neq 0$ but do not make any assumption on $\alpha+\beta+\gamma+\alpha\beta\gamma$.

3.4.1. Let $\psi_1,\psi_2,\psi_3 \in \mathrm{GL}(V)$ be the maps defined in 2.1.5. Let $\gamma_1,\gamma_2,\gamma_3 \in \mathrm{GL}(V)$ be the maps defined in 2.2.

Let x_0^*,x_1^*,x_2^*,x_3^* be the dual basis to x_0,x_1,x_2,x_3. The contragredient action of the maps ψ_j acting on V^* is given by Table 4. The subgroup of $\mathrm{GL}(V^*)$ generated by ψ_1,ψ_2,ψ_3 is isomorphic to H_4. The center of H_4 acts trivially on $\mathbb{P}(V^*)$ so we obtain an action of $\mathbb{Z}_4\times\mathbb{Z}_4$ on $\mathbb{P}(V^*)$.

It is easy to see that $\psi_j(\mathfrak{P}) = \mathfrak{P}$ for all j. We note that

$$\psi_j(abc,a,b,c) = \begin{cases}(a,-ia,i,1) & \text{if } j=1,\\ (b,1,-ib,i) & \text{if } j=2,\\ (c,i,1,-ic) & \text{if } j=3.\end{cases}$$

It follows rather easily that $\mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$ is a single orbit under the action of H_4 and therefore a single $\mathbb{Z}_4 \times \mathbb{Z}_4$-orbit. The subgroup $\{\mathrm{id}, \gamma_1, \gamma_2, \gamma_3\}$ of $\mathbb{Z}_4 \times \mathbb{Z}_4$ is an essential subgroup and, as is easy to see, none of $\gamma_1, \gamma_2, \gamma_3$ fixes any point in $\mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$ so the homomorphism

$$\mathbb{Z}_4 \times \mathbb{Z}_4 \longrightarrow \{\text{permutations of } \mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3\}$$

is injective. It follows that $\mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$ consists of 16 distinct points. Hence $\mathfrak{P}$ consists of 20 distinct points (even without the hypothesis $\alpha+\beta+\gamma+\alpha\beta\gamma \neq 0$).

Table 4. Contragredient action of H_4 on V^*.

	x_0^*	x_1^*	x_2^*	x_3^*
ψ_1	ix_1^*	$(bc)^{-1}x_0^*$	$-c^{-1}x_3^*$	$ib^{-1}x_2^*$
ψ_2	ix_2^*	$ic^{-1}x_3^*$	$(ac)^{-1}x_0^*$	$-a^{-1}x_1^*$
ψ_3	ix_3^*	$-ib^{-1}x_2^*$	$ia^{-1}x_1^*$	$(ab)^{-1}x_0^*$

3.4.2. The points in $\mathfrak{P}_\infty$ are fixed by the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by Table 2.

3.4.3. If $i \neq \infty$ and p is the topmost point in the column $\mathfrak{P}_i$, then the other points in that column are $\gamma_1(p)$, $\gamma_2(p)$, and $\gamma_3(p)$, in that order from top to bottom. Thus, when $j \neq \infty$, $\mathfrak{P}_j$ is a single $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbit.

3.5. The scheme Γ for the 4-dimensional Sklyanin algebras. We review the Sklyanin algebra case (see [15, 20, 21]). Let $A = A(\alpha, \beta, \gamma)$ be a non-degenerate Sklyanin algebra. Then Γ, which we defined in §3.1, is the graph of an automorphism of $E \cup \{e_0, e_1, e_2, e_3\}$, where $E \subseteq \mathbb{P}^3$ is the quartic elliptic curve cut out by (any two of, or all) the equations:

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \\ x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2 = 0, \\ x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 = 0, \\ x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2 = 0, \end{cases} \tag{3.4}$$

and e_i is the vanishing locus of $\{x_0, x_1, x_2, x_3\} - \{x_i\}$. The points e_i are the vertices of the four singular quadrics that contain E. The automorphism fixes each e_i and its restriction to E is a translation automorphism. Furthermore, $R = \{f \in V \otimes V \mid f|_\Gamma = 0\}$. Thus, R and Γ determine each other.

We fix a point $o \in E \cap \{x_0 = 0\}$ and impose a group law on E such that o is the identity and four points on E are collinear if and only if their sum is o. The 2-torsion subgroup $E[2]$ is $E \cap \{x_0 = 0\}$. We write $\oplus$ for the group law and $\ominus$ for subtraction, i.e., $p \oplus q = r$ if and only if $p = r \ominus q$.

If $p = (\xi_0, \xi_1, \xi_2, \xi_3) \in E$, then $\ominus p = (-\xi_0, \xi_1, \xi_2, \xi_3)$.

See [5, §7] for a longer explanation that uses the same notation as here.

Proposition 3.6. *If A is a non-degenerate 4-dimensional Sklyanin algebra, then*

(1) $\mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3 \subseteq E$, *where E is the elliptic curve given by the equations in* (3.4);
(2) $\mathfrak{P}_0 = \tau' \oplus E[2]$, *where* $\tau' = (abc, a, b, c)$;
(3) $\mathfrak{P}_i = \varepsilon_i \oplus \tau' \oplus E[2]$ *and* $E[2] = \{o, 2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\}$ *for suitable* $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in E[4]$.

4. Point schemes, graphs and flat families

Consider the family of algebras $A(\alpha_1, \alpha_2, \alpha_3)$ as the parameters α_i vary. This section is devoted to studying the behavior of the scheme Γ introduced in 3.1 as the fiber of a family over the parameter space consisting of the points $(\alpha_1, \alpha_2, \alpha_3)$, and, more generally, over the family of six-dimensional relation spaces for four-generator algebras.

4.1. Throughout Section 4, V denotes a fixed four-dimensional space of generators for our quadratic algebras with a fixed basis consisting of the generators x_i, $0 \le i \le 3$, $\mathbb{G} = \mathbb{G}(6, V^{\otimes 2})$ is the Grassmannian of 6-planes in $V^{\otimes 2}$, and we regard the points of $\mathbb{G}$ as spaces of relations for four-generator-six-relator connected graded algebras, so that $\mathbb{G}$ will be the parameter space for the algebras in question. We encode a relation space $R \in \mathbb{G}$ as either a 6×4 matrix M or a 4×6 matrix M' with entries in V analogous to (3.1) and (3.2), respectively; so R is spanned by the entries in either the equation $M\mathbf{x}^{\mathsf{T}} = 0$, or $\mathbf{x}M' = 0$, where $\mathbf{x} = (x_i)$.

In keeping with the notation in §3.1, we denote by Γ the family $\pi : \Gamma \to \mathbb{G}$ whose R-fiber Γ_R for $R \in \mathbb{G}$ is by definition the subscheme of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ consisting of the pairs of points (p, p') as in the discussion from 3.1, whose defining property is $M(p)\mathbf{x}^{\mathsf{T}}(p') = 0$.

We define

$$\mathbb{X} := \mathrm{pr}_1(\Gamma) \quad \text{and} \quad \mathbb{X}' := \mathrm{pr}_2(\Gamma),$$

and

$$U := \{R \in \mathbb{G} \mid \dim(\mathbb{X}_R) = 0\} \quad \text{and} \quad U' := \{R \in \mathbb{G} \mid \dim(\mathbb{X}'_R) = 0\}.$$

Finally, given a family $\pi : \mathbb{S} \to \mathbb{T}$ and an open subset $W \subseteq \mathbb{T}$, we denote the restricted family $\pi^{-1}(W)$ by $\mathbb{S}_W$, slightly abusing notation.

When the algebra A_R corresponding to R is Artin-Schelter regular and has some other good properties that we will not specify here, the scheme $\mathbb{X}_R$ is (one incarnation of) the point scheme of A_R (i.e., the scheme parametrizing the isomorphism classes of point modules in $\mathsf{QGr}(A_R)$). In many cases of interest $\mathbb{X}_R$ equals $\mathbb{X}'_R$, and Γ_R is the graph of an automorphism of this scheme (e.g., for non-degenerate 4-dimensional Sklyanin algebras [21], for 3-dimensional AS-regular algebras [1], et al.). Moreover, we saw above in Proposition 3.4 that even when $\mathbb{X}_R \neq \mathbb{X}'_R$, the scheme Γ_R is often the graph of an isomorphism.

For these reasons, we regard Γ and its projections $\mathbb{X}$ and $\mathbb{X}'$ as stand-ins for the point scheme even when we lack the requisite regularity properties for this to be literally the case.

We first prove a statement analogous to [6, Theorem 2.6]. That result says, essentially, that the line schemes of connected graded algebras with four generators and six quadratic relations form a flat family provided they have minimal dimension. We prove here that the family $\Gamma \to \mathbb{G}$ is similarly well behaved.

First, we have the following analogue of [6, Proposition 2.1].

Proposition 4.1. *The subsets U and U' are dense open subsets of $\mathbb{G}$.*

Proof. This is entirely analogous to the proof of [6, Proposition 2.1]. We focus on the case of U, to fix ideas.

First, Van den Bergh's result [24] that, generically, four-generator-six-relator algebras have twenty point modules ensures that U contains a dense open subset of $\mathbb{G}$. Let $\mathbb{X}_i$ be the irreducible components of $\mathbb{X}$, and $\pi_i : (\mathbb{X}_i)_{\text{red}} \to \pi(\mathbb{X}_i)_{\text{red}}$ the restriction and corestriction of π to $(\mathbb{X}_i)_{\text{red}}$.

Each π_i is projective and hence closed. By [12, Exercise II.3.22(d)] applied to each π_i individually, the complement of U, being the image of the closed subset

$$\{x \in \mathbb{X} \mid x \text{ belongs to some component of } \mathbb{X}_{\pi(x)} \text{ of dimension} \geq 1\}$$

of $\mathbb{X}$, is closed in $\mathbb{G}$. □

Corollary 4.2. *The locus $W \subset \mathbb{G}$ over which the family Γ has zero-dimensional fibers is open and dense.*

Proof. The fiber Γ_R is zero-dimensional if and only if its two projections $\mathbb{X}_R$ and $\mathbb{X}'_R$ are, so W is simply the intersection $U \cap U'$. The conclusion follows from Proposition 4.1. □

We now turn, as hinted above, to proving certain regularity properties for the families $\mathbb{X}$, $\mathbb{X}'$, and Γ over the good open subsets of $\mathbb{G}$ identified in Proposition 4.1 and Corollary 4.2.

Lemma 4.3. *The schemes $\mathbb{X}_U$ and $\mathbb{X}'_{U'}$ are Cohen-Macaulay.*

Proof. Once more, we focus on the case of $\mathbb{X}_U$ without loss of generality.

Locally on U, the equations that define $\mathbb{X}_U$ as a U-subscheme of the relative projective space $\mathbb{P}(V^*)_U = \mathbb{P}(V^*) \times U$ are given by the 4×4 minors of a 6×4 matrix. Moreover, over U, X_U has codimension 3 in $\mathbb{P}(V^*)_U$.

In general, the quotient by the ideal I generated by the $r \times r$ minors of a $p \times q$ matrix in a Cohen-Macaulay ring is again Cohen-Macaulay, provided I has maximal codimension $(p-r+1)(q-r+1)$ (see e.g. the discussion in [10, §18.5] on determinantal rings and [2] for a proof). In our case $p = 6$, $q = 4$, and $r = 4$

so the critical codimension is $(6-4+1)(4-4+1)=3$. This concludes the proof. □

Theorem 4.4. *The families $\mathbb{X}_U \to U$, $\mathbb{X}'_{U'} \to U'$, and $\Gamma_W \to W$ are flat.*

Proof. We divide the argument into two parts.

(1) $\mathbb{X}$ and $\mathbb{X}'$: Symmetry allows us to once again focus on $\mathbb{X}$. Given the Cohen-Macaulay property for $\mathbb{X}_U$, the proof of the theorem mimics that of [6, Theorem 2.6] verbatim.

Let $x \in \mathbb{X}_U$ be a point, and set $B = \mathcal{O}_{\pi(x),U}$ and $A = \mathcal{O}_{x,\mathbb{X}_U}$. In order to show that A is flat as a B-module, denote by $\mathfrak{p}$ the maximal ideal of B. Since the fiber $(\mathbb{X}_U)_{\pi(x)}$ is of minimal dimension 0, we have the equality

$$\dim(A) = \dim(B) + \dim(A/A\mathfrak{p}).$$

This implies the flatness of A over B via [10, Theorem 18.16 (b)], given that A is Cohen-Macaulay by Lemma 4.3 and B is regular.

(2) Γ: The result for $\Gamma_W \to W$ follows from part (1) and the observation that over W the projection $\mathrm{pr}_1 : \Gamma \to \mathbb{X}$ is an isomorphism. □

Finally, as an application of the flatness results just proven, we provide an alternate argument for the fact that the 20 points in Table 3 exhaust the "point scheme" of $A(\alpha_1, \alpha_2, \alpha_3)$ under certain non-degeneracy conditions on the parameters α_i. We begin with:

Corollary 4.5. *For every $R \in U$, the scheme $\mathbb{X}_R$ consists of 20 points counted with multiplicity. Similarly for $\mathbb{X}'_{U'}$ and Γ_W.*

Proof. We can embed the family $\mathbb{X} \to \mathbb{G}$ into the relative projective space $\mathbb{P}(V^*)_{\mathbb{G}} = \mathbb{P}(V^*) \times \mathbb{G}$ in the usual fashion.

The flatness of Theorem 4.4 ensures that all fibers $\mathbb{X}_R$ have the same degree in $\mathbb{P}(V^*)$ so long as $R \in U$, i.e., when $\dim(\mathbb{X}_R) = 0$. But we know there are six-dimensional relation spaces R, where the degree is 20 (e.g. the algebras in [3, 5, 16, 24, 26]).

The case of $\mathbb{X}'_{U'}$ is analogous, while that of Γ_W follows as in the proof of Theorem 4.4, using the fact that $\mathrm{pr}_1 : \Gamma \to \mathbb{X}$ is an isomorphism over W. □

Corollary 4.5 allows us to give a proof of half of Proposition 3.3 that involves less calculation.

Proposition 4.6. *If $\alpha + \beta + \gamma + \alpha\beta\gamma \neq 0$ and $\alpha\beta\gamma \neq 0$ and $A(\alpha, \beta, \gamma) = TV/(R)$, then $\mathbb{X}_R$ consists of the 20 points in Table 3.*

Proof. If we show that every closed point of $\mathbb{X}_R$ is one of the points in Table 3, then $\dim(\mathbb{X}_R) = 0$ so, by Corollary 4.5 above, $\mathbb{X}_R$ will consist of 20 points counted with multiplicity.

Let p be a closed point in $\mathbb{X}_R$. Consider the four quadratic polynomials q, q_1, q_2, and q_3 that are defined in the proof of Proposition 3.3; they are

$$\sum x_i^2, \quad x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2, \quad x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2, \quad x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2.$$

Since

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\beta\gamma & -\gamma & \beta \\ 1 & \gamma & -\alpha\gamma & -\alpha \\ 1 & -\beta & \alpha & -\alpha\beta \end{pmatrix} = -(\alpha+\beta+\gamma+\alpha\beta\gamma)^2$$

is non-zero by hypothesis, at least one of q, q_1, q_2, and q_3 is non-zero at p. Because $\alpha\beta\gamma \neq 0$ we may apply Proposition 2.1. The action of H_4 on V that was defined in Proposition 2.1(2) extends to an action of H_4 as automorphisms of the polynomial ring $\Bbbk[x_0, x_1, x_2, x_3]$. In particular, H_4 permutes the polynomials q, q_1, q_2, and q_3, up to scalar multiples, so we may as well assume that $q = \sum x_i^2$ does not vanish at p. But the minors h_{12}, h_{13}, and h_{23} (defined in Proposition 3.3), all of which are multiples of q, vanish at p so p must be one of the finitely many points in

$$x_0x_3 - \gamma x_1x_2 = x_0x_2 - \beta x_1x_3 = x_0x_1 - \alpha x_2x_3 = 0.$$

But these points belong to $\mathfrak{P}$ so $p \in \mathfrak{P}$. However, it is easy to see that every h_{ij} vanishes on $\mathfrak{P}$, so $\mathbb{X}_R = \mathfrak{P}$. □

5. The algebras $R(a,b,c,d)$ of Cho, Hong, and Lau

In this section a, b, c do not denote square roots of α, β, γ.

5.1. The definition. Let $(a,b,c,d) \in \Bbbk^4 - \{0\}$ and define $R(a,b,c,d)$, or simply R, to be the free algebra $TV = \Bbbk\langle x_1, x_2, x_3, x_4\rangle$ modulo the relations:

$$\begin{aligned}
&\text{(R1)} && ax_4x_3 + bx_3x_4 + cx_3x_2 + dx_4x_1 = 0,\\
&\text{(R2)} && ax_3x_2 + bx_2x_3 + cx_4x_3 + dx_1x_2 = 0,\\
&\text{(R3)} && ax_2x_1 + bx_1x_2 + cx_1x_4 + dx_2x_3 = 0,\\
&\text{(R4)} && ax_1x_4 + bx_4x_1 + cx_2x_1 + dx_3x_4 = 0,\\
&\text{(R5)} && ax_3x_1 - ax_1x_3 + cx_4^2 - cx_2^2 = 0,\\
&\text{(R6)} && bx_4x_2 - bx_2x_4 + dx_3^2 - dx_1^2 = 0.
\end{aligned}$$

For example, $R(1,-1,0,0)$ is the commutative polynomial ring on 4 generators.

Since $R(a,b,c,d)$ depends only on (a,b,c,d) as a point in $\mathbb{P}^3$, the family of algebras $R(a,b,c,d)$ is parametrized by $\mathbb{P}^3$. Proposition 5.2 concerns those algebras $R(a,b,c,d)$ parametrized by the points on the quadric $ac + bd = 0$. That quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The lines

$$\ell_1 := \{a + id = c + ib = 0\} \quad \text{and} \quad \ell_2 := \{a - id = c - ib = 0\}$$

on that quadric and their open subsets

$$\ell_1^\circ := \ell_1 - \{(0,i,1,0), (1,0,0,i), (1,-i,-1,i), (1,i,1,i), (1,-1,i,i), (1,1,-i,i)\}$$

and

$$\ell_2^\circ := \ell_2 - \{(0,-i,1,0),\ (1,0,0,-i),\ (1,i,-1,-i),\ (1,-i,1,-i),\\ (1,-1,-i,-i),\ (1,1,i,-i)\}$$

play a distinguished role.

5.2. At [8, Conj. 8.11], Cho, Hong, and Lau conjecture that when $ac+bd=0$, R is isomorphic to a 4-dimensional Sklyanin algebra, i.e., isomorphic to $A(\alpha,\beta,\gamma)$ for some $\alpha,\beta,\gamma\in\Bbbk$ such that $\alpha+\beta+\gamma+\alpha\beta\gamma=0$. Proposition 5.2 shows that R is isomorphic to a 4-dimensional Sklyanin algebra if and only if $(a,b,c,d)\in\ell_1\cup\ell_2$. Nevertheless, R is always isomorphic to $A(\alpha,\beta,\gamma)$ for *some* α,β,γ.

Proposition 5.1. *Let* $z_0=\frac{1}{2}(x_2+x_4)$, $z_1=\frac{1}{2}(x_1+x_3)$, $z_2=\frac{1}{2}(x_1-x_3)$, *and* $z_3=\frac{1}{2}(x_2-x_4)$. *The algebra* $R(a,b,c,d)$ *is equal to* $\Bbbk\langle z_0,z_1,z_2,z_3\rangle$ *modulo the relations*:

$$\begin{aligned}(a-b-c+d)[z_0,z_1]&=(-a-b+c+d)\{z_2,z_3\},\\ (-a+b+c+d)[z_0,z_2]&=(a+b-c+d)\{z_3,z_1\},\\ (a+b+c+d)\{z_0,z_1\}&=(a-b+c-d)[z_2,z_3],\\ (a+b+c-d)\{z_0,z_2\}&=(-a+b-c-d)[z_3,z_1],\\ b[z_0,z_3]&=d\{z_1,z_2\},\\ c\{z_0,z_3\}&=a[z_1,z_2].\end{aligned}$$

Proof. Since $x_1=z_1+z_2$, $x_2=z_0+z_3$, $x_3=z_1-z_2$, and $x_4=z_0-z_3$, the relations (R1)–(R4) can be replaced by the four relations:

$$\begin{aligned}&\tfrac{1}{2}((\mathrm{R1})+(\mathrm{R3})): (a+d)z_0z_1+(b+c)z_1z_0+(a-d)z_3z_2+(b-c)z_2z_3=0,\\ &\tfrac{1}{2}((\mathrm{R1})-(\mathrm{R3})): (d-a)z_0z_2-(b+c)z_2z_0-(a+d)z_3z_1+(c-b)z_1z_3=0,\\ &\tfrac{1}{2}((\mathrm{R4})+(\mathrm{R2})): (b+c)z_0z_1+(a+d)z_1z_0+(d-a)z_2z_3+(c-b)z_3z_2=0,\\ &\tfrac{1}{2}((\mathrm{R4})-(\mathrm{R2})): (a-d)z_2z_0+(b+c)z_0z_2-(a+d)z_1z_3+(c-b)z_3z_1=0.\end{aligned}$$

$$\begin{aligned}&\tfrac{1}{2}((\mathrm{R1})+(\mathrm{R3})): (a+d)z_{01}+(b+c)z_{10}+(a-d)z_{32}+(b-c)z_{23}=0,\\ &\tfrac{1}{2}((\mathrm{R1})-(\mathrm{R3})): (d-a)z_{02}-(b+c)z_{20}-(a+d)z_{31}+(c-b)z_{13}=0,\\ &\tfrac{1}{2}((\mathrm{R4})+(\mathrm{R2})): (b+c)z_{01}+(a+d)z_{10}+(d-a)z_{23}+(c-b)z_{32}=0,\\ &\tfrac{1}{2}((\mathrm{R4})-(\mathrm{R2})): (a-d)z_{20}+(b+c)z_{02}-(a+d)z_{13}+(c-b)z_{31}=0.\end{aligned}$$

It follows that R is equal to $\Bbbk\langle z_0,z_1,z_2,z_3\rangle$ modulo the relations:

$$\begin{aligned}&\tfrac{1}{2}((\mathrm{R1})+(\mathrm{R3})+(\mathrm{R2})+(\mathrm{R4})): (a+b+c+d)\{z_0,z_1\}+(a-b+c-d)[z_3,z_2]=0,\\ &\tfrac{1}{2}((\mathrm{R1})+(\mathrm{R3})-(\mathrm{R2})-(\mathrm{R4})): (a-b-c+d)[z_0,z_1]+(a+b-c-d)\{z_3,z_2\}=0,\\ &\tfrac{1}{2}((\mathrm{R1})-(\mathrm{R3})-(\mathrm{R2})+(\mathrm{R4})): (-a+b+c+d)[z_0,z_2]+(-a-b+c-d)\{z_1,z_3\}=0,\end{aligned}$$

$$\tfrac{1}{2}((\mathrm{R1})\text{-}(\mathrm{R3})+(\mathrm{R2})\text{-}(\mathrm{R4})) : (-a-b-c+d)\{z_0,z_2\}+(a-b+c+d)[z_1,z_3]=0,$$
$$\tfrac{1}{2}(\mathrm{R5}) : a[z_1,z_2] - c\{z_0,z_3\} = 0,$$
$$\tfrac{1}{2}(\mathrm{R6}) : b[z_0,z_3] - d\{z_1,z_2\} = 0.$$

Rearranging these gives the presentation in the statement of the proposition. □

Proposition 5.2. *Let $a,b,c,d \in \Bbbk$ and define $p := a+b$, $q := a-b$, $r := c+d$, and $s := c-d$. Suppose that $ac+bd = 0$ and*

(5.1) $$abcd(p+r)(p-r)(p+s)(p-s)(q+r)(q-r)(q+s)(q-s) \neq 0.$$

1. *$R(a,b,c,d) \cong A(\alpha,\beta,\gamma)$, where*
$$\alpha = \frac{r^2-p^2}{q^2-s^2}, \qquad \beta = \frac{p^2-s^2}{r^2-q^2}, \quad \text{and} \quad \gamma = \frac{cd}{ab}.$$
2. *$\alpha+\beta+\gamma+\alpha\beta\gamma = 0$ if and only if $(a,b,c,d) \in \ell_1 \cup \ell_2$.*
3. *If $(a,b,c,d) \in \ell_1 \cup \ell_2$, then $R(a,b,c,d)$ is isomorphic to a Sklyanin algebra,*
$$R(a,b,c,d) \;\cong\; A(\alpha,1,-1),$$
where
$$\alpha = \begin{cases} (b+d+ib-id)^2(b+d-ib+id)^{-2} & \text{if } a+id = c+ib = 0, \\ (b+d-ib+id)^2(b+d+ib-id)^{-2} & \text{if } a-id = c-ib = 0, \end{cases}$$
and is generated by homogeneous degree-one elements $Y_\pm, K, K'$ such that
$$KY_\pm = \mp iY_\pm K, \quad K'Y_\pm = \pm iY_\pm K', \quad [Y_+,Y_-] = i(K'^2-K^2), \quad \text{and}$$
$$[K,K'] = i\alpha(Y_+^2 - Y_-^2).$$

Proof. (1) Condition (5.1) ensures that the denominators in the expressions for α,β,γ, are non-zero.

Let z_0,z_1,z_2,z_3 be as in Proposition 5.1. Condition (5.1) ensures that the denominators in the following expressions are non-zero, so R is defined by the following relations:

$$[z_0,z_1] = \frac{r-p}{q-s}\{z_2,z_3\}, \qquad \{z_0,z_1\} = \frac{q+s}{p+r}[z_2,z_3],$$
$$[z_0,z_2] = \frac{p-s}{r-q}\{z_3,z_1\}, \qquad \{z_0,z_2\} = \frac{q+r}{p+s}[z_3,z_1],$$
$$[z_0,z_3] = \frac{d}{b}\{z_1,z_2\}, \qquad \{z_0,z_3\} = \frac{a}{c}[z_1,z_2].$$

For brevity, let's write these relations as

$$[z_0,z_1] = \mu_1\{z_2,z_3\}, \qquad \{z_0,z_1\} = \nu_1[z_2,z_3],$$
$$[z_0,z_2] = \mu_2\{z_3,z_1\}, \qquad \{z_0,z_2\} = \nu_2[z_3,z_1],$$
$$[z_0,z_3] = \mu_3\{z_1,z_2\}, \qquad \{z_0,z_3\} = \nu_3[z_1,z_2].$$

Define $v_0 := z_0$, $v_1 := \sqrt{\nu_2\nu_3}\, z_1$, $v_2 := \sqrt{\nu_3\nu_1}\, z_2$, and $v_3 := \sqrt{\nu_1\nu_2}\, z_3$. Thus, R is the free algebra $\Bbbk\langle v_0, v_1, v_2, v_3\rangle$ modulo the relations:

$$\begin{aligned} [v_0, v_1] &= \alpha\,\{v_2, v_3\}, & \{v_0, v_1\} &= [v_2, v_3],\\ [v_0, v_2] &= \beta\,\{v_3, v_1\}, & \{v_0, v_2\} &= [v_3, v_1],\\ [v_0, v_3] &= \gamma\,\{v_1, v_2\}, & \{v_0, v_3\} &= [v_1, v_2], \end{aligned}$$

where

$$\alpha = \mu_1\nu_1^{-1} = \frac{r^2-p^2}{q^2-s^2}, \quad \beta = \mu_2\nu_2^{-1} = \frac{p^2-s^2}{r^2-q^2}, \quad \text{and} \quad \gamma = \mu_3\nu_3^{-1} = \frac{cd}{ab}.$$

(2) The expression $(q^2-s^2)(r^2-q^2)ab\big(\alpha+\beta+\gamma+\alpha\beta\gamma\big)$ is equal to

$$\begin{aligned} &(r^2-p^2)(r^2-q^2)ab \;+\; (p^2-s^2)(q^2-s^2)ab \;+\; (q^2-s^2)(r^2-q^2)cd\\ &+ (r^2-p^2)(p^2-s^2)cd\\ =\;& ab\big(r^4+s^4+2p^2q^2 \;-\; (p^2+q^2)(r^2+s^2)\big)\\ &- cd\big(p^4+q^4+2r^2s^2 \;-\; (p^2+q^2)(r^2+s^2)\big)\\ =\;& ab\big(2p^2q^2-2r^2s^2+(r^2+s^2-p^2-q^2)(r^2+s^2)\big)\\ &- cd\big(2r^2s^2-2p^2q^2-(r^2+s^2-p^2-q^2)(p^2+q^2)\big)\\ =\;& 2(ab+cd)(p^2q^2-r^2s^2)+(r^2+s^2-p^2-q^2)(abr^2+abs^2+cdp^2+cdq^2). \end{aligned}$$

But $p^2+q^2 = 2(a^2+b^2)$ and $r^2+s^2 = 2(c^2+d^2)$ so

$$abr^2+abs^2+cdp^2+cdq^2 = 2(ac+bd)(ad+bc) = 0.$$

Therefore $(q^2-s^2)(r^2-q^2)ab\big(\alpha+\beta+\gamma+\alpha\beta\gamma\big) = 2(ab+cd)(p^2q^2-r^2s^2)$. Hence $\alpha+\beta+\gamma+\alpha\beta\gamma = 0$ if and only if $(ab+cd)(p^2q^2-r^2s^2) = 0$; i.e., if and only if $(a^2-b^2-c^2+d^2)(a^2-b^2+c^2-d^2)(ab+cd) = 0$.

The locus $ac+bd = (a^2-b^2-c^2+d^2)(a^2-b^2+c^2-d^2)(ab+cd) = 0$ consists of 8 lines: the locus $ac+bd = a^2-b^2-c^2+d^2 = 0$ is the union of the four lines

$$a-b = c+d = 0, \quad a+b = c-d = 0, \quad a+id = c+ib = 0, \quad a-id = c-ib = 0;$$

the locus $ac+bd = a^2-b^2+c^2-d^2 = 0$ is the union of the four lines

$$a-b = c+d = 0, \quad a+b = c-d = 0, \quad a+d = b-c = 0, \quad a-d = b+c = 0;$$

the locus $ac+bd = ab+cd = 0$ is the union of the four lines

$$a = d = 0, \qquad b = c = 0, \qquad a+d = b-c = 0, \qquad a-d = b+c = 0.$$

If (a,b,c,d) is on the line $a-b = c+d = 0$ or on the line $a-d = b+c = 0$, then $0 = a-b-c-d = q-r$; hypothesis (5.1) excludes this possibility so (a,b,c,d) is not on either of those lines. If (a,b,c,d) is on the line $a = d = 0$ or on the line $b = c = 0$, then $abcd = 0$; hypothesis (5.1) excludes this possibility so (a,b,c,d) is not on either of those lines. If (a,b,c,d) is on the line $a+d = b-c = 0$ or on the line $a+b = c-d = 0$, then $0 = a+b-c+d = p-s$; hypothesis (5.1) excludes this possibility so (a,b,c,d) is not on either of those lines. Thus,

$\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ if and only if (a, b, c, d) is on the union of the lines $a + id = c + ib = 0$ and $a - id = c - ib = 0$; i.e., $(a, b, c, d) \in \ell_1 \cup \ell_2$.

Not every $(a, b, c, d) \in \ell_1 \cup \ell_2$ satisfies (5.1). The points (a, b, c, d) on the line $a + id = c + ib = 0$ that do not satisfy (5.1) are

$$(0, i, 1, 0),\ (1, 0, 0, i),\ (1, -i, -1, i),\ (1, i, 1, i),\ (1, -1, i, i),\ (1, 1, -i, i). \tag{5.2}$$

The points (a, b, c, d) on the line $a - id = c - ib = 0$ that do not satisfy (5.1) are

$$\begin{gathered}(0, -i, 1, 0),\ (1, 0, 0, -i),\ (1, i, -1, -i),\\ (1, -i, 1, -i),\ (1, -1, -i, -i),\ (1, 1, i, -i).\end{gathered} \tag{5.3}$$

(3) Suppose $a + id = c + ib = 0$. Then $(\alpha, \beta, \gamma) = (-1, \beta, 1)$, where $\beta = (b + d + ib - id)^2/(b + d - ib + id)^2$. As (a, b, c, d) runs over the points in (5.2), β takes on the values -1, -1, 1, 1, 0, ∞, respectively. As (a, b, c, d) runs over the other points on the line $a + id = c + ib = 0$, β takes on every value in $\Bbbk - \{0, \pm 1\}$.

Suppose $a - id = c - ib = 0$. Then $(\alpha, \beta, \gamma) = (-1, \beta, 1)$, where $\beta = (b + d - ib + id)^2/(b + d + ib - id)^2$. As (a, b, c, d) runs over the points in (5.3), β takes on the values -1, -1, 1, 1, ∞, 0. As (a, b, c, d) runs over the other points on the line $a - id = c - ib = 0$, β takes on every value in $\Bbbk - \{0, \pm 1\}$.

By Lemma 2.3, $A(\beta, 1, -1) \cong A(-1, \beta, 1) \cong A(1, -1, \beta)$ so to prove (3) it suffices to show that $A(\alpha, 1, -1)$ has generators $Y_\pm, K, K'$ satisfying the stated relations. We do this in Proposition 5.3 below. □

Proposition 5.3. *Suppose $\alpha \in \Bbbk - \{0, \pm 1\}$. Let i be a square root of -1.*

(1) *$A(\alpha, 1, -1)$ is a Sklyanin algebra $A(E, \tau)$ with τ being translation by a 4-torsion point.*
(2) *There is a basis $Y_\pm$, K, K' for $A(\alpha, 1, -1)_1$ such that*

$$KY_\pm = \mp i Y_\pm K, \qquad K' Y_\pm = \pm i Y_\pm K',$$

$$[Y_+, Y_-] = i(K'^2 - K^2), \qquad [K, K'] = i\alpha(Y_+^2 - Y_-^2).$$

Proof. Let $A = A(\alpha, 1, -1)$. By Lemma 2.3, $A(\alpha, 1, -1) \cong A(1, -1, \alpha) \cong A(-1, \alpha, 1)$. Algebras of the form $A(1, -1, \alpha)$ are those identified in equation (1.9.4) of [21] so the results in [21] apply to A. The zero locus of the 4×4 minors in the proof of [21, Prop. 2.4] is the curve E given by the equations

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 - x_1^2 + \alpha x_2^2 - \alpha x_3^2 = 0. \tag{5.4}$$

The restrictions on α imply that the Jacobian matrix has rank 2 at all points of E so E is an elliptic curve. (The description of E, in particular, the formula for the polynomial g_2, in [21, Prop. 2.4] does not make sense when $\beta = 1$.) The formula for the automorphism $\sigma : E \to E$ in [21, Cor. 2.8] is

$$\sigma\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2\alpha x_1x_2x_3 \ - \ x_0(-x_0^2 - x_1^2 - \alpha x_2^2 + \alpha x_3^2) \\ 2\alpha x_0x_2x_3 \ + \ x_1(\ x_0^2 + x_1^2 - \alpha x_2^2 + \alpha x_3^2) \\ 2x_0x_1x_3 \ + x_2(\ x_0^2 - x_1^2 + \alpha x_2^2 + \alpha x_3^2) \\ -2x_0x_1x_2 \ + x_3(\ x_0^2 - x_1^2 - \alpha x_2^2 - \alpha x_3^2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_0 \\ x_3 \\ -x_2 \end{pmatrix}, \tag{5.5}$$

where the last equality uses the fact that $x_0^2 - x_1^2 + \alpha x_2^2 - \alpha x_3^2 = 0$ on E. The formula for σ can also be verified by observing that

$$\begin{pmatrix} -x_1 & x_0 & -\alpha x_3 & -\alpha x_2 \\ -x_2 & -x_3 & x_0 & -x_1 \\ -x_3 & x_2 & x_1 & x_0 \\ -x_3 & -x_2 & x_1 & -x_0 \\ -x_1 & -x_0 & -x_3 & x_2 \\ -x_2 & x_3 & -x_0 & -x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_0 \\ x_3 \\ -x_2 \end{pmatrix} = 0$$

for all $(x_0, x_1, x_2, x_3) \in E$; the 6×4 matrix in the previous equation is the 6×4 matrix in (3.1).

Corollary 2.11 in [4] involves elements $a, b, c \in \Bbbk$ such that $a^2 = \alpha$, $b^2 = 1$, and $c^2 = -1$; let $(a, b, c) = (a, 1, i)$; [4, Cor. 2.11] then says there is a 4-torsion point $\varepsilon_1 \in E$ such that if $p = (x_0, x_1, x_2, x_3) \in E$, then

$$p + \varepsilon_1 = (ix_1, -ix_0, ix_3, ix_2) = (x_1, -x_0, x_3, x_2);$$

by [4, §2.6], there is a 2-torsion point $\gamma_2 \in E$ such that $p + \varepsilon_1 + \gamma_2 = (x_1, x_0, x_3, -x_2)$; thus

$$\sigma(p) = p + \varepsilon_1 + \gamma_2.$$

Calculations like those in [21, §1.2] show that $Y_\pm := x_0 \pm x_1$, $K := x_0 + x_1$, $K' := x_0 - x_1$, satisfy the relations in (2). ☐

Part (2) of Proposition 5.3 remains true when $A = A(0, 1, -1)$ and in that case, $A(0, 1, -1)$ is a homogenization of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ with $q = -i$. See [7, §2.4] for details.

5.3. Parameter spaces and modular curves. After Proposition 5.3(1), it is natural to ask which pairs (E, τ) have the property that $A(E, \tau) \cong R(a, b, c, d)$ for some point (a, b, c, d) in the "Sklyanin locus" $\ell_1^\circ \cup \ell_2^\circ$.

Similarly, we can ask how much redundancy there is in this parametrization: how many $(a, b, c, d) \in \ell_1^\circ \cup \ell_2^\circ$ lead to the same pair (E, τ)?

Since the transformation

$$a \longleftrightarrow -c, \qquad b \longleftrightarrow -d$$

interchanges ℓ_1 and ℓ_2 and intertwines the respective transformations

$$(a, b, c, d) \mapsto \beta,$$

it suffices to consider what happens for ℓ_1.

Note first that the map

$$\varphi : (a,b,c,d) \mapsto \frac{(b+d+ib-id)^2}{(b+d-ib+id)^2} \tag{5.6}$$

recovering the parameter α of the Sklyanin algebra $A(\alpha,1,-1)$ from $(a,b,c,d) \in \ell_1^\circ$ is a two-fold cover of

$$X := \mathbb{P}^1 - \{\pm 1, 0, \infty\}.$$

Now, to each $\alpha \in X$ associate the elliptic curve E_α of point modules of $A(\alpha,1,-1)$, defined by (5.4). Since furthermore the point $(1,1,i,i)$ belongs to all E_α, the α-indexed family $E \to X$ of elliptic curves E_α over X has a section.

Finally, (5.5) defines an automorphism of order 4 of the family $E \to X$. Since the section $(1,1,i,i)$ puts on E a unique structure of an abelian curve over X [13, Theorem 2.1.2], we can identify said automorphism with a point of E of order (precisely) 4. In other words, we obtain a family of abelian curves over X with marked order-4 points. This moduli problem is represented by the modular curve $Y_1(4)$ classifying such data (see e.g. [9, Theorem 8.2.1] and references therein), and hence we obtain a morphism

$$\psi : X \to Y_1(4).$$

The following results give the full picture of the parametrization of the Cho-Hong-Lau algebras.

Proposition 5.4. *The map* $\psi : X = \mathbb{P}^1 - \{\pm 1, 0, \infty\} \longrightarrow Y_1(4)$ *defined above as*

$$X \ni \alpha \mapsto (E,\tau) \in Y_1(4) \text{ for } A(\alpha,1,-1) \cong A(E,\tau)$$

is a two-fold cover, identifying $\pm\alpha$.

Proof. Note first that the automorphism

$$(x_0,x_1,x_2,x_3) \mapsto (x_0,x_1,x_3,-x_2)$$

of $\mathbb{P}^3$ interchanges the elliptic curves E_α and $E_{-\alpha}$, and moreover it intertwines their respective order-4 automorphisms defining them as points of $Y_1(4)$. This implies that ψ factors through a morphism

$$\psi' : X/\pm \to Y_1(4).$$

Since $Y_1(4)$ is known to have three cusps and the left-hand side is a thrice-punctured projective line, ψ' extends to a endomorphism $\overline{\psi'}$ of $\mathbb{P}^1$. It follows from the fact that three distinct points have singleton preimages that $\overline{\psi'}$ is an isomorphism, and hence so is ψ'. □

In conclusion, we have:

Corollary 5.5. *The maps* $\ell_i^\circ \longrightarrow Y_1(4)$, $i = 1,2$, *that send* (a,b,c,d) *to the underlying elliptic curve and automorphism of the Sklyanin algebra* $R(a,b,c,d)$ *are fourfold covers.*

Proof. Simply compose ψ and its analogue for ℓ_2 (which are double covers by Proposition 5.4) with the two-fold covers of the form (5.6). □

6. Central elements

6.1. Central elements in $A(\alpha,\beta,\gamma)$. The next result is often asserted but we could not find a proof in the literature so we include one here.

Proposition 6.1. *Let* $\Bbbk$ *be any field. If* $\{\alpha,\beta,\gamma\}\cap\{0,\pm 1\}=\varnothing$ *and* $\alpha+\beta+\gamma+\alpha\beta\gamma=0$, *then* $-x_0^2+x_1^2+x_2^2+x_3^2$ *and* $x_0^2+\beta\gamma x_1^2-\gamma x_2^2+\beta x_3^2$ *belong to the center of* $A(\alpha,\beta,\gamma)$.

Proof. Let's simplify the notation by omitting the x's and just retaining the subscripts, so kji denotes $x_kx_jx_i$, $ii0$ denotes $x_ix_ix_0$, and so on. We also write $\{i,j\}$ for $\{x_i,x_j\}$, $[i,j]$ for $[x_i,x_j]$, etc.

For each cyclic permutation (i,j,k) of $(1,2,3)$, we define

$$c_i := [x_0,x_i]-\alpha_i\{x_j,x_k\} \quad \text{and} \quad a_i := \{x_0,x_i\}-[x_j,x_k].$$

Straightforward computations in the free algebra $\Bbbk\langle x_0,x_1,x_2,x_3\rangle$ show that

$$\begin{aligned}\{x_i,c_i\} &= \{i,[0,i]\}-\alpha_i\{i,\{j,k\}\}\\ &= [0,ii]-\alpha_i(ijk+kji)-\alpha_i(ikj+jki), \qquad \text{and}\\ [x_i,a_i] &= [i,\{0,i\}]-[i,[j,k]]\\ &= [ii,0]-(ijk+kji)+(ikj+jki).\end{aligned}$$

When $\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2\alpha_3=0$, error-prone calculations[3] show that

$$\begin{aligned}&(1+\alpha_2\alpha_3)\{x_1,c_1\}+\alpha_2\alpha_3[x_1,a_1]\\ &+(1+\alpha_3)\{x_2,c_2\}+\alpha_3[x_2,a_2]\\ &+(1-\alpha_2)\{x_3,c_3\}-\alpha_2[x_3,a_3]\end{aligned}$$

equals $[x_0,x_1^2+x_2^2+x_3^2]$. Hence $[x_0,-x_0^2+x_1^2+x_2^2+x_3^2]=0$ in $A(\alpha_1,\alpha_2,a_3)$.

A similar calculation shows that

$$\begin{aligned}&(1+\alpha_2)\big(\{x_0,c_1\}+\alpha_1[x_0,a_1]\big)+(1-\alpha_1)[x_3,c_2]\\ &+(1+\alpha_1+2\alpha_1\alpha_2)\{x_3,a_2\}-(1+\alpha_1\alpha_2)\big([x_2,c_3]+\{x_2,a_3\}\big)\\ =\ &(1+\alpha_1)(1+\alpha_2)[x_1,-x_0^2+x_1^2+x_2^2+x_3^2].\end{aligned}$$

Hence $[x_1,-x_0^2+x_1^2+x_2^2+x_3^2]=0$. The transformation $x_0\mapsto x_0$, $x_i\mapsto x_{i+1}$ for $i=1,2,3$, and $\alpha_i\mapsto\alpha_{i+1}$ for $i=1,2,3$, leaves $-x_0^2+x_1^2+x_2^2+x_3^2$ fixed; it follows that $[x_2,-x_0^2+x_1^2+x_2^2+x_3^2]=0$ and then that $[x_3,-x_0^2+x_1^2+x_2^2+x_3^2]=0$. This completes the proof that $-x_0^2+x_1^2+x_2^2+x_3^2$ belongs to the center of $A(\alpha_1,\alpha_2,\alpha_3)$ when $\{\alpha_1,\alpha_2,\alpha_3\}\cap\{0,\pm1\}=\varnothing$ and $\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2\alpha_3=0$.

The automorphism ψ_1 in Table 1 sends $-x_0^2+x_1^2+x_2^2+x_3^2$ to $x_0^2+\alpha_2\alpha_3x_1^2-\alpha_3x_2^2+\alpha_2x_3^2$ so the latter also belongs to the center of $A(\alpha_1,\alpha_2,\alpha_3)$. □

[3]In carrying out these calculations one should not attempt to "simplify" the expressions $ijk+kji$ and $ikj+jki$.

Proposition 6.2. *Let* $\Bbbk$ *be any field. If* $\alpha\beta\gamma \neq 0$ *and* $\alpha + \beta + \gamma + \alpha\beta\gamma \neq 0$, *then* x_0^2, x_1^2, x_2^2, *and* x_3^2, *belong to the center of* $A(\alpha, \beta, \gamma)$.

Proof. We use the same notation as that in the proof of Proposition 6.1. Calculations in $\Bbbk\langle x_0, x_1, x_2, x_3\rangle$ show that if (i, j, k) is a cyclic permutation of $(1, 2, 3)$, then

$$\begin{aligned}
&(\alpha_j + \alpha_k)\{x_i, c_i\} - \alpha_i(\alpha_j\alpha_k + 1)[x_i, a_i] \\
&\quad - \alpha_i(\alpha_k + 1)\big(\{x_j, c_j\} + [x_j, a_j]\big) + \alpha_i(\alpha_j - 1)\big(\{x_k, c_k\} + [x_k, a_k]\big) \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3)[x_0, x_i^2], \\
&(\alpha_j + \alpha_k)\{x_j, a_i\} + (\alpha_j\alpha_k + 1)[x_j, c_i] \\
&\quad - (\alpha_k + 1)\big(\alpha_j\{x_i, a_j\} + [x_i, c_j]\big) + (\alpha_j - 1)\big([x_0, a_k] - \{x_0, c_k\}\big) \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3)[x_k, x_j^2], \\
&(\alpha_i + \alpha_k)\{x_i, a_j\} - \alpha_i(\alpha_j\alpha_k + 1)[x_i, c_j] \\
&\quad + (\alpha_i + 1)\big(\{x_0, c_k\} - [x_0, a_k]\big) - (\alpha_k - 1)\big(\alpha_i\{x_j, a_i\} - [x_j, c_i]\big) \\
&= -\, (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3)[x_k, x_i^2],
\end{aligned}$$

and

$$\begin{aligned}
&-(\alpha_j + \alpha_k)\{x_0, c_i\} - \alpha_i(\alpha_j\alpha_k + 1)[x_0, a_i] \\
&\quad + \alpha_i(\alpha_k + 1)\big(\alpha_j\{x_k, a_j\} - [x_k, c_j]\big) + \alpha_i(\alpha_j - 1)\big(\alpha_k\{x_j, a_k\} + [x_j, c_k]\big) \\
&= -\, (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3)[x_i, x_0^2].
\end{aligned}$$

Since the images of a_i and c_i in $A(\alpha_1, \alpha_2, \alpha_3)$ are zero, if $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 \neq 0$, then $[x_0, x_1^2] = [x_0, x_2^2] = [x_0, x_3^2] = 0$ and $[x_1, x_3^2] = [x_2, x_1^2] = [x_3, x_2^2] = 0$ and $[x_1, x_2^2] = [x_2, x_3^2] = [x_3, x_1^2] = 0$ in $A(\alpha_1, \alpha_2, \alpha_3)$. □

6.2. Degree-two central elements in $R(a, b, c, d)$. It is conjectured at [8, p. 47] that the elements

$$C_1 := ax_1x_3 + bx_2x_4 + cx_2^2 + dx_1^2$$

and

$$C_2 := a'(v)(x_1x_3 + x_3x_1) + b'(v)(x_2x_4 + x_4x_2) + c'(v)(x_2^2 + x_4^2) + d'(v)(x_1^2 + x_3^2)$$

generate the center of $R(a, b, c, d)$ (we have suppressed an irrelevant scaling constant from the original expression of C_2 in [8]). In order to have a little more symmetry, and to emphasize the parallels with the Sklyanin algebras, we will replace C_1 by

$$Z_1 := a(x_1x_3 + x_3x_1) + b(x_2x_4 + x_4x_2) + c(x_2^2 + x_4^2) + d(x_1^2 + x_3^2),$$

which is equal to $2C_1$, and replace C_2 by the element Z_2 in Corollary 6.5 below, and show that Z_1 and Z_2 belong to the center of $R(a, b, c, d)$.

Proposition 6.3. *For all* $a, b, c, d \in \Bbbk$, *the element* Z_1 *is central in* $R(a, b, c, d)$.

Proof. In terms of the generators z_i in Proposition 5.1,

$$Z_1 = 2(b+c)z_0^2 + 2(a+d)z_1^2 + 2(-a+d)z_2^2 + 2(-b+c)z_3^2. \tag{6.1}$$

Using the expression for Z_1 in (6.1), we get

$$[z_0, Z_1] = 2(a+d)[z_0, z_1^2] + 2(-a+d)[z_0, z_2^2] + 2(-b+c)[z_0, z_3^2]. \tag{6.2}$$

We now label the relations in the statement of Proposition 5.1 (in the form LHS $-$ RHS) according to which commutator or anticommutator involving z_0 they contain. For example, the first and third relations in Proposition 5.1 are

$$c_1 = (a-b-c+d)[z_0, z_1] - (-a-b+c+d)\{z_2, z_3\} = 0$$

and

$$a_2 = (a+b+c-d)\{z_0, z_2\} - (-a+b-c-d)[z_3, z_1] = 0.$$

With this in place, we leave the reader to check that (6.2) equals

$$\{z_1, c_1\} - [z_1, a_1] + \{z_2, c_2\} + [z_2, a_2] - 2\{z_3, c_3\} - 2[z_3, a_3],$$

which obviously belongs to the ideal generated by the relations c_i and a_i. Thus $[z_0, Z_1] = 0$.

We now prove that $[Z_1, z_i] = 0$ for $i = 1, 2, 3$ by changing the labels of the z_i and the structure constants a, b, etc. so that both Z_1 and the space of relations in Proposition 5.1 are preserved. The transformation

$$z_0 \longleftrightarrow z_1, \quad z_2 \longleftrightarrow z_3, \quad a \longleftrightarrow b, \quad c \longleftrightarrow d$$

is such a relabeling so the fact that $[Z_1, z_0] = 0$ implies $[Z_1, z_1] = 0$. The transformation

$$z_0 \longleftrightarrow z_3, \quad z_1 \longleftrightarrow z_2, \quad a \longleftrightarrow -a, \quad b \longleftrightarrow -b$$

(while c and d are fixed) is another such transformation, so the fact that $[Z_1, z_0] = 0$ implies $[Z_1, z_3] = 0$. Finally, composing the two transformations will prove that $[Z_1, z_2] = 0$. □

Proposition 6.4. *Let*

$$\rho_2 = \frac{(a+b-c+d)(-a+b-c-d)}{(-a+b+c+d)(a+b+c-d)} \quad \textit{and} \quad \rho_3 = \frac{da}{bc}.$$

Assume that the denominators in the expressions for ρ_2 and ρ_3 are non-zero. Fix q_2 and q_3 such that $q_2^4 = \rho_2$ and $q_3^4 = \rho_3$, and define

$$\tau_0 := -q_2 q_3, \quad \tau_1 := 1/q_2 q_3, \quad \tau_2 := q_2/q_3, \quad \tau_3 := q_3/q_2.$$

The linear map $\psi : R_1 \to R_1$ given by the formula

$$\psi(z_0) = \tau_0 z_1, \quad \psi(z_1) = \tau_1 z_0, \quad \psi(z_2) = \tau_2 z_3, \quad \psi(z_3) = \tau_3 z_2,$$

extends to an algebra automorphism of $R(a, b, c, d)$.

Proof. By Proposition 5.1, R is $\Bbbk\langle z_0, z_1, z_2, z_3\rangle$ modulo the relations:

$$c_i = \kappa_i[z_0, z_i] - \mu_i\{z_j, z_k\} \quad \text{and} \quad a_i = \lambda_i\{z_0, z_i\} - \nu_i[z_j, z_k],$$

where (i, j, k) runs over the cyclic permutation of $(1, 2, 3)$ and

$$\begin{aligned} \kappa_1 &= a - b - c + d, & \mu_1 &= -a - b + c + d, \\ \kappa_2 &= -a + b + c + d, & \mu_2 &= a + b - c + d, \\ \kappa_3 &= b, & \mu_3 &= d, \\ \lambda_1 &= a + b + c + d, & \nu_1 &= a - b + c - d, \\ \lambda_2 &= a + b + c - d, & \nu_2 &= -a + b - c - d, \\ \lambda_3 &= c, & \nu_3 &= a. \end{aligned}$$

Furthermore,

$$\frac{\tau_0\tau_1}{\tau_2\tau_3} = -1, \qquad \frac{\tau_0\tau_2}{\tau_3\tau_1} = -\rho_2 = -\frac{\mu_2\nu_2}{\kappa_2\lambda_2}, \qquad \frac{\tau_0\tau_3}{\tau_1\tau_2} = \rho_3 = \frac{\mu_3\nu_3}{\kappa_3\lambda_3}.$$

Since

$$\begin{aligned} \psi(c_1) &= \kappa_1\tau_0\tau_1[z_1, z_0] - \mu_1\tau_2\tau_3\{z_3, z_2\}, & \psi(a_1) &= \lambda_1\tau_0\tau_1\{z_1, z_0\} - \nu_1\tau_2\tau_3[z_3, z_2], \\ \psi(c_2) &= \kappa_2\tau_0\tau_2[z_1, z_3] - \mu_2\tau_3\tau_1\{z_2, z_0\}, & \psi(a_2) &= \lambda_2\tau_0\tau_2\{z_1, z_3\} - \nu_2\tau_3\tau_1[z_2, z_0], \\ \psi(c_3) &= \kappa_3\tau_0\tau_3[z_1, z_2] - \mu_3\tau_1\tau_2\{z_0, z_3\}, & \psi(a_3) &= \lambda_3\tau_0\tau_3\{z_1, z_2\} - \nu_3\tau_1\tau_2[z_0, z_3], \end{aligned}$$

we have

$$\begin{aligned} \tau_2^{-1}\tau_3^{-1}\psi(c_1) &= -\kappa_1[z_1, z_0] - \mu_1\{z_3, z_2\} = c_1, \\ \tau_2^{-1}\tau_3^{-1}\psi(a_1) &= -\lambda_1\{z_1, z_0\} - \nu_1[z_3, z_2] = -a_1, \\ \tau_3^{-1}\tau_1^{-1}\psi(c_2) &= -\kappa_2\rho_2[z_1, z_3] - \mu_2\{z_2, z_0\} = -\frac{\mu_2}{\lambda_2}a_2, \\ \tau_3^{-1}\tau_1^{-1}\psi(a_2) &= -\lambda_2\rho_2\{z_1, z_3\} - \nu_2[z_2, z_0] = \frac{\nu_2}{\kappa_2}c_2, \\ \tau_1^{-1}\tau_2^{-1}\psi(c_3) &= \kappa_3\rho_3[z_1, z_2] - \mu_3\{z_0, z_3\} = -\frac{\mu_3}{\lambda_3}a_3, \\ \tau_1^{-1}\tau_2^{-1}\psi(a_3) &= \lambda_3\rho_3\{z_1, z_2\} - \nu_3[z_0, z_3] = -\frac{\nu_3}{\kappa_3}c_3. \end{aligned}$$

Hence ψ extends to an algebra automorphism, as claimed.

Since $\psi^2(z_0) = \tau_0\tau_1 z_0 = -z_0$, $\psi^2 \neq \mathrm{id}_R$. Since $(\tau_0\tau_1)^2 = (\tau_2\tau_3)^2 = 1$, $\psi^4 = \mathrm{id}_R$. □

Corollary 6.5. *With the notation and hypotheses in Proposition 6.4, The element*

$$Z_2 := (a+d)(q_2q_3)^{-2}z_0^2 + (b+c)(q_2q_3)^2 z_1^2 + (c-b)(q_2/q_3)^2 z_2^2 + (d-a)(q_3/q_2)^2 z_3^2$$

belongs to the center of R.

Proof. Let ψ be the automorphism in Proposition 6.4. Since $Z_2 = \psi\big(\frac{1}{2}Z_1\big)$, the result follows from Proposition 6.3. □

References

[1] M. Artin, J. T. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, in The Grothendieck Festschrift, Vol. I, 33–85, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[2] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics, 1327, Springer, Berlin, 1988. https://doi.org/10.1007/BFb0080378

[3] R. G. Chandler and M. Vancliff, *The one-dimensional line scheme of a certain family of quantum* $\mathbb{P}^3$s, J. Algebra **439** (2015), 316–333. https://doi.org/10.1016/j.jalgebra.2015.04.036

[4] A. Chirvasitu and S. P. Smith, *Exotic elliptic algebras of dimension* 4, Adv. Math. **309** (2017), 558–623. https://doi.org/10.1016/j.aim.2017.01.010

[5] A. Chirvasitu and S. P. Smith, *Exotic elliptic algebras*, Trans. Amer. Math. Soc. **371** (2019), no. 1, 279–333. https://doi.org/10.1090/tran/7341

[6] A. Chirvasitu, S. P. Smith, and M. Vancliff, *A geometric invariant of* 6*-dimensional subspaces of* 4×4 *matrices*, Proc. Amer. Math. Soc. **148** (2020), no. 3, 915–928. https://doi.org/10.1090/proc/14294

[7] A. Chirvasitu, S. P. Smith, and L. Z. Wong, *Noncommutative geometry of homogenized quantum* $\mathfrak{sl}(2,\mathbb{C})$, Pacific J. Math. **292** (2018), no. 2, 305–354. https://doi.org/10.2140/pjm.2018.292.305

[8] C.-H. Cho, H. Hong, and S.-C. Lau, *Noncommutative homological mirror functor*, Mem. Amer. Math. Soc. **271** (2021), no. 1326, v+116 pp. https://doi.org/10.1090/memo/1326

[9] F. I. Diamond and J. Im, *Modular forms and modular curves*, in Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994), 39–133, CMS Conf. Proc., 17, Amer. Math. Soc., Providence, RI, 1995.

[10] D. Eisenbud, *Commutative Algebra*, Graduate Texts in Mathematics, 150, Springer, New York, 1995. https://doi.org/10.1007/978-1-4612-5350-1

[11] B. L. Feigin and A. Odesskii, *Vector bundles on an elliptic curve and Sklyanin algebras*, in Topics in quantum groups and finite-type invariants, 65–84, Amer. Math. Soc. Transl. Ser. 2, 185, Adv. Math. Sci., 38, Amer. Math. Soc., Providence, RI, 1998. https://doi.org/10.1090/trans2/185/04

[12] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer, New York, 1977.

[13] N. M. Katz and B. C. Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, 108, Princeton Univ. Press, Princeton, NJ, 1985. https://doi.org/10.1515/9781400881710

[14] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, in Integrable quantum field theories (Tvärminne, 1981), 61–119, Lecture Notes in Phys., 151, Springer, Berlin, 1982.

[15] T. Levasseur and S. P. Smith, *Modules over the* 4*-dimensional Sklyanin algebra*, Bull. Soc. Math. France **121** (1993), no. 1, 35–90.

[16] B. Shelton and M. Vancliff, *Some quantum* $\mathbf{P}^3$*s with one point*, Comm. Algebra **27** (1999), no. 3, 1429–1443. https://doi.org/10.1080/00927879908826504

[17] B. Shelton and M. Vancliff, *Schemes of line modules. I*, J. London Math. Soc. (2) **65** (2002), no. 3, 575–590. https://doi.org/10.1112/S0024610702003186

[18] E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation*, Funktsional. Anal. i Prilozhen. **16** (1982), no. 4, 27–34, 96.

[19] E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation. Representations of a quantum algebra*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 4, 34–48.

[20] S. P. Smith, *The four-dimensional Sklyanin algebras*, *K*-Theory **8** (1994), no. 1, 65–80. https://doi.org/10.1007/BF00962090

[21] S. P. Smith and J. T. Stafford, *Regularity of the four-dimensional Sklyanin algebra*, Compositio Math. **83** (1992), no. 3, 259–289.
[22] S. P. Smith and J. M. Staniszkis, *Irreducible representations of the* 4*-dimensional Sklyanin algebra at points of infinite order*, J. Algebra **160** (1993), no. 1, 57–86. https://doi.org/10.1006/jabr.1993.1178
[23] J. T. Tate and M. Van den Bergh, *Homological properties of Sklyanin algebras*, Invent. Math. **124** (1996), no. 1-3, 619–647. https://doi.org/10.1007/s002220050065
[24] M. Van den Bergh, *An example with* 20 *point modules*, Circulated privately, 1988.
[25] M. Van den Bergh, *A translation principle for the four-dimensional Sklyanin algebras*, J. Algebra **184** (1996), no. 2, 435–490. https://doi.org/10.1006/jabr.1996.0269
[26] M. Vancliff, K. Van Rompay, and L. Willaert, *Some quantum* $\mathbf{P}^3$*s with finitely many points*, Comm. Algebra **26** (1998), no. 4, 1193–1208. https://doi.org/10.1080/00927879808826193

Alexandru Chirvasitu
Department of Mathematics
University at Buffalo
Buffalo, NY 14260-2900, USA
Email address: achirvas@buffalo.edu

S. Paul Smith
Department of Mathematics
University of Washington
Box 354350, Seattle, WA 98195, USA
Email address: smith@math.washington.edu