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CONFORMAL HEMI-SLANT SUBMERSION FROM KENMOTSU MANIFOLD

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Abstract. As a generalization of conformal semi-invariant submersion, conformal slant submersion and conformal semi-slant submersion, in this paper we study conformal hemi-slant submersion from Kenmotsu manifold onto a Riemannian manifold. The necessary and sufficient conditions for the integrability and totally geodesicness of distributions are discussed. Moreover, we have obtained sufficient condition for a conformal hemi-slant submersion to be a homothetic map. The condition for a total manifold of the submersion to be twisted product is studied, followed by other decomposition theorems.

1. Introduction

Riemannian submersions between Riemannian manifolds were studied by O' Neill and Gray ([11], [20]). After this kind of submersions were studied between manifolds endowed with differentiable structure, many authors studied different geometric properties of the Riemannian submersion, anti-invariant submersion ([31], [33]), semi-invariant submersion ([2], [32]), slant submersion ([8], [9], [12], [21]), semi-slant submersion ([13], [22], [23]), conformal slant submersion ([3], [14]) and conformal semi-slant submersion [1]. A step forward, R Parsad et. al. studied Quasi-bi-slant submersions, hemi-slant submersions, conformal semi-invariant submersions, conformal semi-slant submersions and conformal anti-invariant submersions for almost contact metric manifolds and almost Hermitian manifolds ([25], [26], [27], [28], [29], [30]). On the other hand, Riemannian submersions have some applications in physics and in mathematics. More precisely, Riemannian submersions have applications in Kaluza-Klein theory ([7], [15]) and Yang-Mills theory ([6], [35]).

As a generalization of anti-invariant, semi-invariant and slant submersion, Tastan et al. defined the notion of hemi-slant Riemannian submersion in [34]. As special, horizontally conformal maps which were introduced independently

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by Fuglede [10] and Ishihara [16], horizontally conformal submersions are defined as follows: Let (M_1, g_1) and (M_2, g_2) are Riemannian manifolds of dimension m_1 and m_2 respectively. A smooth submersion $\pi : (M_1, g_1) \to (M_2, g_2)$ is called a horizontally conformal submersion if there is a positive function λ such that

$$\lambda^2 g_1(X_1, X_2) = g_2(\pi_* X_1, \pi_* X_2)$$

for all $X_1, X_2 \in \Gamma(\ker \pi_*)$. Here, a horizontally conformal submersion π is called horizontally homothetic if the $grad\lambda$ is vertical, i.e., $H(grad\lambda) = 0$. We denote by V and H the projections on the vertical distributions $(\ker \pi_*)$ and horizontal distributions $((\ker \pi_*)^{\perp})$. It can be said that Riemannian submersion is a special horizontally conformal submersion with $\lambda = 1$. Recently, Akyol and Sahin have introduced conformal anti-invariant submersions [4], conformal semi-invariant submersion [5], conformal slant submersion [3], and conformal semi-slant submersions [1]. Also, the geometry of conformal submersions have been studied by several authors ([14], [19]). Our motivation is to fill a gap in the geometry of conformal hemi-slant Riemannian submersions in contact geometry.

Our motivation is to further study the conformal hemi-slant submersions in contact geometry, specifically we study conformal hemi-slant submersions from Kenmotsu manifold onto a Riemannian manifold. The organization of the presented article is as follows: Section 2 is provided with pre-requisite of almost contact metric manifold and it also enriches the article with the basic fact of Riemannian and horizontally conformal submersions, which makes it self contained. In Section 3, we have discussed the geometry of the foliations whereas Section 4 focusses on the product theorems for the total manifold of the submersion.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field ξ, a vector field and η, a 1- form satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0. \tag{1}$$

On an almost contact manifold, there exists a Riemannian metric g which is compatible with the almost contact structure (M, ϕ, ξ, η) in the sense that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2}$$

from which it can be observed that

$$g(U, \xi) = \eta(U), \tag{3}$$

for any $U, \ V \in \Gamma(TM)$ and the manifold (M, ϕ, ξ, η, g) is called an *almost contact metric manifold*. If $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ, then the almost contact structure is normal if and only if the torsion tensor $[\phi, \phi] +$

$2d\eta \otimes \xi$ vanishes. An almost contact metric structure is called a *contact metric structure* if $d\eta = \Phi$, where Φ is the fundamental 2-form defined by $\Phi(U, V) = g(U, \phi V)$. Almost contact metric structure (ϕ, ξ, η, g) are said to define a *Kenmotsu structure* on M if the following characterizing tensorial equation is satisfied (cf.[17])

$$(\bar{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U. \tag{4}$$

One can deduce from the above relations that

$$\bar{\nabla}_U \xi = U - \eta(U)\xi. \tag{5}$$

It is also seen that

$$g(\phi U, V) = -g(U, \phi V). \tag{6}$$

The covariant derivative is defined by

$$(\nabla_U \phi)V = \nabla_U \phi V - \phi \nabla_U V. \tag{7}$$

Now, we recall the notion of Riemannian submersion and horizontally conformal submersion followed by some basic results those will be useful throughout the text.

Definition 2.1. *Let (M, g_1) and (N, g_2) be two Riemannian manifolds and $\pi : M \to N$ be a smooth Riemannian submersion. Then π is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(X, Y) = \frac{1}{\lambda^2} g_2(\pi_* X, \pi_* Y), \tag{8}$$

for any $X, Y \in \Gamma(\ker \pi_*)^\perp$. It is obvious that every Riemannian submersion is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $\pi : M \to N$ be a conformal submersion. A vector field E on M is called projectable if there exists a vector field $\bar{E}$ on N such that $\pi_*(E_p) = \bar{E}$ for any $p \in M$. In this case E and $\bar{E}$ are called π-related. A horizontal vector field Y on M is called basic, if it is projectable. It is a well known fact that if $\bar{Z}$ is a vector field on N, then there exists a unique basic vector field Z which is called the horizontal lift of $\bar{Z}$ [20].

The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ defined by O Neill for vector field E and F on M such that

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{9}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{10}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections. On the other hand, from equations (9) and (10), we have

$$\nabla_U V = \mathcal{T}_U V + \bar{\nabla}_U V \tag{11}$$

$$\nabla_U X = \mathcal{T}_U X + \mathcal{H}\nabla_U X \tag{12}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U \tag{13}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{14}$$

for any $U, V \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^{\perp}$, where $\bar{\nabla}_U V = \mathcal{V}\nabla_U V$.

It is easily seen that for $p \in \Gamma(TM)$, $V \in \mathcal{V}_p, X \in \mathcal{H}_p$ the linear operators $\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$ are skew-symmetric, that is

$$g(\mathcal{A}_X E, F) = -g(E, \mathcal{A}_X F) \tag{15}$$

$$g(\mathcal{T}_V E, F) = -g(E, \mathcal{T}_V F) \tag{16}$$

for any $E, F \in \Gamma(T_pM)$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution acts as a second fundamental form of the fibres of the submersion π. Since $\mathcal{T}_V$ is skew-symmetric, hence, π has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Definition 2.2. *A horizontally conformal submersion $\pi : M \to N$ is called horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,*

$$H(grad\lambda) = 0 \tag{17}$$

at $p \in \Gamma(TM)$, where H is the complement orthogonal distribution to $\nu = \ker \pi_*$ in $\Gamma(T_pM)$.

Let (M, g_1) and (N, g_2) be two Riemannian manifolds. Let $\varphi : M \to N$ be a smooth map. Then, the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_* Y - \varphi_*(\nabla_X Y), \tag{18}$$

for all $X, Y \in \Gamma(T_pM)$, where we denote conveniently by ∇ the Levi-Civita connection of the metrics g_1 and g_2 and ∇^{φ} is the pullback connection. We also know that φ is said to be totally geodesic map if $(\nabla\varphi_*)(X, Y) = 0$ for any $X, Y \in \Gamma(T_pM)$.

Lemma 2.3. *Let $\pi : M \to N$ be a horizontal conformal submersion. Then, for any horizontal vector fields X, Y and vertical vector fields U, V*

(i) $(\nabla\pi_*)(X, Y) = X(ln\lambda)\pi_*(Y) + Y(ln\lambda)\pi_*(X) - g_1(X, Y)\pi_*(grad\ ln\lambda)$,
(ii) $(\nabla\pi_*)(U, V) = -\pi_*(\mathcal{T}_U V)$,
(iii) $(\nabla\pi_*)(X, U) = -\pi_*(\nabla_X U) = -\pi_*(\mathcal{A}_X U)$.

3. Conformal Hemi-slant Submersion

A horizontal conformal submersion $\pi : M \to N$ from almost contact metric manifold $(M, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N, g_2), is called a conformal hemi-slant submersion if the vertical distribution $\ker \pi_*$ of π admits two orthogonal complementary distributions D and $\bar{D}$ such that D is slant with angle θ and $\bar{D}$ is anti-invariant, i.e.,

$$\ker \pi_* = D \oplus \bar{D} \oplus < \xi > \tag{19}$$

where $< \xi >$ is 1-dimensional distribution spanned by ξ and angle θ is called the hemi-slant angle of submersion. It is known that the distribution $\ker \pi_*$ is integrable. Hence, above definition implies that the integral manifold (no matter fibers $\pi^{-1}(n), n \in N$) of $\ker \pi_*$ is a hemi-slant submanifold.

We observe that the notion of conformal hemi-slant submersion is natural generalization of the notions of conformal anti-invariant [4], conformal semi-invariant submersion [5] and conformal slant submersion [3]. More precisely, if we denote the dimension of $\bar{D}$ and D by m_1 and m_2, respectively, then we have the following:

1. If $m_2 = 0$, then M is a conformal anti-invariant submersion,
2. If $m_1 = 0$ and $\theta = 0$, then M is a conformal invariant submersion.
3. If $m_1 = 0$ and $\theta \neq 0$=, then M is a proper conformal slant submersion with slant angle θ.
4. If $\theta = \frac{\pi}{2}$, then M is a conformal anti-invariant submersion.

We say that the conformal hemi-slant submersion is proper if $\bar{D} \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$. Let π be a conformal hemi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N, g_2). Then, for any $V \in \Gamma(\ker \pi_*)$, we have

$$V = PV + QV + \eta(V)\xi \tag{20}$$

$PV \in \Gamma(D), QV \in \Gamma(\bar{D})$. For all $X \in \Gamma(\ker \pi_*)^\perp$

$$\phi X = BX + CX \tag{21}$$

where $BX \in \Gamma(\ker \pi_*)$ and $CX \in \Gamma(\mu)$. For $V \in \Gamma(\ker \pi_*)$, we have

$$\phi V = \psi V + \omega V \tag{22}$$

where $\psi V \in \Gamma(\ker \pi_*), \omega V \in \Gamma(\ker \pi_*)^\perp$. Then the horizontal distribution $\Gamma(\ker \pi_*)^\perp$ decomposed as

$$\Gamma(\ker \pi_*)^\perp = \omega D \oplus \phi \bar{D} \oplus \mu \tag{23}$$

where μ is the orthogonal complement of $\omega D \oplus \phi \bar{D}$ in $\Gamma(\ker \pi_*)^\perp$.

Theorem 3.1. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant Riemannian submersion. Then*

$$\psi^2 = -\lambda(I - \eta \otimes \xi). \tag{24}$$

Furthermore, if π is Riemannian submersion then it satisfies $\lambda = \cos^2 \theta$.

Proof. Let π be a hemi-slant Riemannian submersion from almost contact metric manifold $(M, \phi, \xi, \eta, g_1)$ into a Riemannian manifold (N, g_2) with the hemi-slant angle θ. Then for any $U \in \Gamma(ker\pi_*)$, we have

$$\cos\theta = \frac{|\psi U|}{|\phi U|} \tag{25}$$

$$\cos\theta = \frac{g_1(\phi U, \psi U)}{|\phi U||\psi U|}.$$

On using equation (22), we have

$$\cos\theta = -\frac{g_1(U, \psi^2 U)}{|\phi U||\psi U|}. \tag{26}$$

On using equations (1), (25) and (26), we get

$$\psi^2 U = -\cos^2\theta(U - \eta(U)\xi).$$

Here $\lambda = \cos^2\theta$, then finally, we have

$$\psi^2 = -\lambda(I - \eta \otimes \xi). \tag{27}$$

□

The proof of above Theorem is similar to the proof of the Theorem (3.5) of [26].

Lemma 3.2. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant Riemannian submersion. Then we have*

$$g_1(\psi U, \psi V) = \cos^2\theta\{g_1(U, V) - \eta(U)\eta(V)\} \tag{28}$$

$$g_1(\omega U, \omega V) = \sin^2\theta\{g_1(U, V) - \eta(U)\eta(V)\} \tag{29}$$

$U, V \in \Gamma(Ker\pi_*)$.

Proof. On using equation (22), we have

$$g_1(\psi U, \psi V) = g_1(\phi U, \psi V).$$

Again from equation (22) and using above Theorem, we get

$$g_1(\psi U, \psi V) = \cos^2\theta\{g_1(U, V) - \eta(U)\eta(V)\}.$$

With using equations (22), we have

$$g_1(\omega U, \omega V) = g_1(\phi U, \phi V) - g_1(\phi U, \psi V).$$

On using equations (22) and (28), we have

$$g_1(\omega U, \omega V) = \sin^2\theta\{g_1(U, V) - \eta(U)\eta(V)\}.$$

□

Lemma 3.3. *Let $(M, \phi, \xi, \eta, g_1)$ be Kenmotsu manifold and (N, g_2) be a Riemannian manifold. If $\pi : M \to N$ is a conformal hemi slant submersion, then we have*

$$\omega BX + C^2 X = -X, \ \psi BX + BCX = 0$$

$$\psi^2 U + B\omega U = -U + \eta(U)\xi, \ \omega\psi U + C\omega U = 0$$

for $U \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^\perp)$.

Lemma 3.4. *Let $(M, \phi, \xi, \eta, g_1)$ be Kenmotsu manifold and (N, g_2) be a Riemannian manifold. If $\pi : M \to N$ is a conformal hemi slant submersion, then we have*

(i)

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \psi\mathcal{H}\nabla_X Y + B\mathcal{A}_X Y - g_1(X, Y)\xi$$

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \omega\mathcal{H}\nabla_X Y + C\mathcal{A}_X Y.$$

(ii)

$$\mathcal{V}\nabla_X \psi V + \mathcal{A}_X \omega V + B\mathcal{A}_X V = \omega\mathcal{V}\nabla_X V + g_1(BX, V) - \eta(V)BX$$

$$\mathcal{A}_X \psi V + \mathcal{H}\nabla_X \omega V + C\mathcal{A}_X V = \omega\mathcal{V}\nabla_X V + \eta(CX).$$

(iii)

$$\mathcal{V}\nabla_V BX + \nabla_V CX = B\mathcal{T}_V BX + \psi\nabla_V CX - g_1(\omega V, X)$$

$$\mathcal{T}_V BX + \mathcal{T}_V CX = C\mathcal{T}_V CX + \omega\nabla_V CX.$$

(iv)

$$\mathcal{V}\nabla_V \psi U + \nabla_U \omega V + \eta(V)\psi U = B\mathcal{T}_U V + \psi\mathcal{V}\nabla_U V + g_1(\psi U, V)\xi$$

$$\mathcal{T}_U \psi V + \mathcal{T}_U \omega V = C\mathcal{T}_U V + \omega\mathcal{V}\nabla_U V + \eta(V)\omega U.$$

for $U, V \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma((\ker\pi_*)^\perp)$

Theorem 3.5. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then slant distribution D is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2\{(\nabla\pi_*)(U, \omega V) - (\nabla\pi_*)(V, \omega U), \pi_*(\phi X)\} \\
&= g_1(\mathcal{T}_U \omega\psi V, Z) - g_1(\mathcal{T}_V \omega\psi U, Z) \\
&\ - 2\eta(Z)\cos^2\theta\{g_1(U, V) - \eta(U)\eta(V)\}
\end{aligned}$$

for any $U, V \in \Gamma(D), X \in \Gamma(\ker \pi_*)^\perp$ and $Z \in \Gamma(\bar{D})$

Proof. Taking $g([U, V], Z)$ and $g([U, V], X)$, since $g_1([U, V], X) = 0$ as $X \in \Gamma(\ker \pi_*)^\perp$ and $U, V \in \Gamma(D)$. Now, using (2) - (6), (21) and Theorem 3.1, we have

$$\begin{aligned}
g_1([U, V], Z) = {} & cos^2\theta g_1([U, V], Z) - g_1(\nabla_U \omega\psi V, Z) + g_1(\nabla_V \omega\psi U, Z) \\
& - g_1(\nabla_U \omega V, \phi Z) + g_1(\nabla_V \omega U, \phi Z)
\end{aligned}$$

Now using equation (12), we have

$$\begin{aligned}
& sin^2\theta g_1([U, V], Z) \\
& = -g_1(\mathcal{T}_U \omega\psi V, Z) + g_1(\mathcal{T}_V \omega\psi U, Z) - g_1(\mathcal{H}\bar{\nabla}_U \omega V, \phi Z) \\
& \ + g_1(\mathcal{H}\nabla_V \omega U, \phi Z) + 2\eta(Z)\cos^2\theta\{g_1(U, V) - \eta(U)\eta(V)\}
\end{aligned}$$

Considering definition of Conformal submersion from Lemma (2.3), we have

$$\begin{aligned}
&sin^2\theta g_1([U,V],Z)\\
&= -g_1(\mathcal{T}_U\omega\psi V,Z)+g_1(\mathcal{T}_V\omega\psi U,Z)-\frac{1}{\lambda^2}g_2((\nabla\pi_*)(U,\omega V),\pi_*(\phi Z))\\
&-\frac{1}{\lambda^2}g_2(\nabla^\pi_U\pi_*\omega U,\pi_*(\phi Z))-\frac{1}{\lambda^2}g_2((\nabla\pi_*)(V,\omega U),\pi_*(\phi Z))\\
&+\frac{1}{\lambda^2}g_2(\nabla^\pi_V\pi_*\omega U,\pi_*(\phi Z))+2\eta(Z)\cos^2\theta\{g_1(U,V)-\eta(U)\eta(V)\}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&sin^2\theta g_1([U,V],Z)\\
&= -g_1(\mathcal{T}_U\omega\psi V,Z)+g_1(\mathcal{T}_V\omega\psi U,Z)+\frac{1}{\lambda^2}\{g_2(\nabla\pi_*)(U,\omega V)\\
&-(\nabla\pi_*)(V,\omega U),\pi_*(\phi Z)\}+2\eta(Z)\cos^2\theta\{g_1(U,V)-\eta(U)\eta(V)\}.
\end{aligned} \tag{30}$$

From above equation, we get the result. □

Theorem 3.6. *Let $\pi:(M,\phi,\xi,\eta,g_1)\to(N,g_2)$ be a conformal hemi-slant submersion from a Kenmotsu manifold to Riemannian manifold N. Then anti-invariant distribution $\bar{D}$ is always integrable.*

Theorem 3.7. *Let $\pi:(M,\phi,\xi,\eta,g_1)\to(N,g_2)$ be a conformal hemi slant submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then $(\ker\pi_*)^\perp$ is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2(\nabla^\pi_X\pi_*CY+\nabla^\pi_Y\pi_*CX,\pi_*(\omega V))\\
&= g_1(\mathcal{H}grad\ln\lambda,X)g_1(X,\omega V)\\
&+g_1(\mathcal{H}grad\ln\lambda,CX)g_1(Y,\omega Y)\\
&-2g_1(X,CY)g_1(\mathcal{H}grad\ln\lambda,\omega V)\\
&-g_1(\mathcal{V}\nabla_XBY+\mathcal{V}\nabla_YBX+\mathcal{A}_YCX+\mathcal{A}_XCY,\psi V)\\
&-g_1(\mathcal{A}_YBX-\mathcal{A}_XBY,\omega V),
\end{aligned} \tag{31}$$

for any $U,V\in\Gamma(\ker\pi_*)$ and $X,Y\in\Gamma((\ker\pi_*)^\perp)$.

Proof. Taking $U,V\in\Gamma(\ker\pi_*)$ and $X,Y\in\Gamma((\ker\pi_*)^\perp)$ with using (2), (4) and (22), we have

$$\begin{aligned}
g_1([X,Y],V) &= -g_1(\nabla_XBY,\phi V)-g_1(\nabla_XCY,\phi V)\\
&+g_1(\nabla_YBX,\phi V)+g_1(\nabla_YCX,\phi V).
\end{aligned}$$

On using equations(13), (14), (21) and Lemma 2.3, we have

$$\begin{aligned} g_1([X,Y],V) &= -g_1(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) - g_1(\mathcal{H}\nabla_X CY + \mathcal{A}_X CY, \phi V) \\ &\quad + g_1(\mathcal{A}_Y BX + \mathcal{V}\nabla_Y BX, \phi V) + g_1(\mathcal{H}\nabla_Y CX + \mathcal{A}_Y CX, \phi V) \\ &= g_1(\mathcal{V}\nabla_X BY + \mathcal{V}\nabla_Y BX + \mathcal{A}_Y CX - \mathcal{A}_X CY, \psi V) + g_1(\mathcal{A}_Y BX \\ &\quad - \mathcal{A}_X BY, \omega V) + \frac{1}{\lambda^2} g_2(\pi_*(-\mathcal{H}\nabla_X CY + \mathcal{H}\nabla_Y CX), \pi_*(\omega V)). \end{aligned}$$

Since π is conformal hemi-slant submersion, using Lemma 2.3 and equation (18), we get

$$\begin{aligned} g_1([X,Y],V) = \frac{1}{\lambda^2} & g_2(\nabla^\pi_X \pi_* CY + \nabla^\pi_Y \pi_* CX, \pi_*(\omega V)) \\ & - g_1(\mathcal{H} grad \ln\lambda, X) g_1(X, \omega V) \\ & - g_1(\mathcal{H} grad \ln\lambda, CX) g_1(Y, \omega Y) \\ & + 2 g_1(X, CY) g_1(\mathcal{H} grad \ln\lambda, \omega V) \\ & - g_1(\mathcal{V}\nabla_X BY + \mathcal{V}\nabla_Y BX + \mathcal{A}_Y CX + \mathcal{A}_X CY, \psi V) \\ & - g_1(\mathcal{A}_Y BX - \mathcal{A}_X BY, \omega V). \end{aligned}$$

From the above equation, we get the desired result. □

Theorem 3.8. *Let π is conformal hemi-slant submersion from a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N, g_2). If $(\ker \pi_*)^\perp$ is integrable and the equation*

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* CY + \nabla^\pi_Y \pi_* CX, \pi_*(\omega V)) \\ & = -g_1(\mathcal{V}\nabla_X BY + \mathcal{V}\nabla_Y BX + \mathcal{A}_Y CX + \mathcal{A}_X CY, \psi V) \\ & \quad - g_1(\mathcal{A}_Y BX - \mathcal{A}_X BY, \omega V), \end{aligned} \tag{32}$$

for $X, Y \in \Gamma(\ker \pi_*)^\perp$ and $V, W \in \Gamma(\ker \pi_*)$ holds, then π is horizontally homothetic.

Proof. For $X, Y \in \Gamma(\ker \pi_*)^\perp$ and $V, W \in \Gamma(\ker \pi_*)$, from Theorem 3.7, we have

$$\begin{aligned} & g_1([X,Y],V) \\ & = \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* CY + \nabla^\pi_Y \pi_* CX, \pi_*(\omega V)) \\ & \quad - g_1(\mathcal{H} grad \ln\lambda, X) g_1(X, \omega V) \\ & \quad - g_1(\mathcal{H} grad \ln\lambda, CX) g_1(Y, \omega Y) \\ & \quad + 2 g_1(X, CY) g_1(\mathcal{H} grad \ln\lambda, \omega V) \\ & \quad - g_1(\mathcal{V}\nabla_X BY + \mathcal{V}\nabla_Y BX + \mathcal{A}_Y CX + \mathcal{A}_X CY, \psi V) \\ & \quad - g_1(\mathcal{A}_Y BX - \mathcal{A}_X BY, \omega V) \end{aligned}$$

Now, If $(\ker \pi_*)^\perp$ is integrable and equation (32) holds, we have

$$\begin{aligned}&- g_1(\mathcal{H}grad \ln \lambda, CY)g_1(X, \omega V)\\&- g_1(\mathcal{H}grad \ln \lambda, CX)g_1(Y, \omega V)\\&+ 2g_1(X, CY)g_1(\mathcal{H}grad \ln \lambda, \omega V) = 0.\end{aligned}$$

Now taking $X = \phi V$ in above equation and using the fact $g_1(\phi V, CY) = 0$, for $Y \in \Gamma(\ker \pi_*)^\perp, V \in \Gamma(\ker \pi_*)$. we have

$$-g_1(\mathcal{H}grad \ln \lambda, CY)g_1(\phi V, \phi V) = 0.$$

Here λ is constant on $\Gamma(\mu)$. On the other hand taking $X = CY$, we arrived at

$$2g_1(CY, CY)g_1(\mathcal{H}grad \ln \lambda, \omega V) = 0.$$

From above equation, λ is a constant on $\Gamma(\phi(\ker \pi_*))$. Similarly, one can obtain the other assertions. □

Theorem 3.9. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant submersion from a Kenmotsu manifold M to Riemannian manifold N. Then slant distribution D defines totally geodesic foliation on M if and only if*

$$g_1(\mathcal{T}_Z \omega\psi W, X) = -\frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z, \omega W), \pi_*(\phi X)), \tag{33}$$

and

$$\begin{aligned}&\frac{1}{\lambda^2}[g_2((\nabla \pi_*)(\omega W, \phi CV), \pi_*(\omega Z)) - g_2(\nabla_{\omega W} \pi_*(\phi CV), \pi_*(\omega Z))]\\&= g_1(\mathcal{T}_{Z\omega W}, BV) + g_1(\mathcal{A}_\omega W \phi CV, \psi Z) - g_1(\nabla_Z \omega\psi W, V)\end{aligned} \tag{34}$$

for any $Z, W \in \Gamma(D), V \in \Gamma(\ker \pi_*)^\perp$ and $X \in \Gamma(\bar{D})$.

Proof. Taking $Z, W \in \Gamma(D), V \in (\ker \pi_*)^\perp$ and $X \in \Gamma(\bar{D})$ with using equations (2), (4), (21) and Theorem 3.1, we have

$$\begin{aligned}&g_1(\nabla_Z W, X)\\&= cos^2\theta g_1(\nabla_Z W, X) - g_1(\nabla_Z \omega\psi W, X) + g_1(\nabla_Z \omega W, \phi X).\end{aligned}$$

Now, from equations (8) and (12), we have

$$\begin{aligned}&sin^2\theta g_1(\nabla_Z W, X)\\&= -g_1(\mathcal{H}\nabla_Z \omega\psi W, X) - g_1(\mathcal{T}_{XZ} \omega\psi W, X) + g_1(\mathcal{H}\nabla_Z \omega W, \phi X) + g_1(\mathcal{T}_Z \omega W, \phi X)\\&= -g_1(\mathcal{T}_Z \omega\psi W, X) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z, \omega W), \pi_*(\phi X)).\end{aligned}$$

On the other hand by using (2), (4), (21) and Theorem 3.1, we have

$$\begin{aligned}g_1(\nabla_Z W, V) = &cos^2\theta g_1(\nabla_Z W, V) - g_1(\nabla_Z \omega\psi W, V)\\&+ g_1(\mathcal{T}_Z \omega W + \mathcal{H}\nabla_Z \omega W, BV + CV).\end{aligned}$$

We have on using equations (2), (4), (8), (14) and (21)

$$\begin{aligned}
&sin^2\theta g_1(\nabla_Z W, V)\\
&= -g_1(\nabla_Z \omega\psi W, V) + g_1(\mathcal{T}_Z \omega W, BV + g_1(\mathcal{H}\nabla_Z \omega W, CV)\\
&= -g_1(\nabla_Z \omega\psi W, V) + g_1(\mathcal{T}_Z \omega W, BV) - g_1(\mathcal{A}_{\omega W}\phi CV, \omega Z)\\
&- \frac{1}{\lambda^2} g_2(\nabla_{\omega W}\pi_*(\phi CV), \pi_*(\omega Z)) + \frac{1}{\lambda^2} g_2((\nabla\pi_*)(\omega W, \phi CV), \pi_*(\omega Z)).
\end{aligned}$$

This complete the proof of the theorem. □

Theorem 3.10. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant submersion from a Kenmotsu manifold M onto a Riemannian manifold N. Then $(\ker\pi_*)^\perp$ is defined totally geodesic on M if and only if*

$$\begin{aligned}
&- \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* Y, \pi_*(\omega\psi PV)) - \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* CY, \pi_*(\omega V))\\
&= -cos^2\theta g_1(\nabla_X Y, PV) + g_1(\mathcal{A}_X BY, \omega V)\\
&- g_1(X, grad\ln\lambda) g_1(Y, \omega\psi PV)\\
&+ g_1(Y, grad\ln\lambda) g_1(X, \omega\psi PV)\\
&- g_1(X, Y) g_1(grad\ln\lambda, \omega\psi PV)\\
&+ g_1(CY, grad\ln\lambda) g_1(X, \omega V)\\
&- g_1(X, CY) g_1(grad\ln\lambda, \omega V)
\end{aligned}$$

for any $X, Y \in \Gamma(\ker\pi_*)^\perp$ and $V \in \Gamma(\ker\pi_*)$.

Proof. On Using equations (2), (20) and (21) with taking $X, Y \in \Gamma(\ker\pi_*)^\perp$ and $V \in \Gamma(\ker\pi_*)$, we have

$$\begin{aligned}
&g_1(\nabla_X Y, V)\\
&= g_1(\nabla_X \phi^2 Y, \psi^2 PV) + g_1(\nabla_X \phi^2 Y, \omega\psi PV)\\
&+ g_1(\nabla_X \phi Y, \omega PV) + g_1(\nabla_X \phi Y, \phi QV).
\end{aligned}$$

From (1), (14) and Theorem 3.1, we have

$$\begin{aligned}
g_1(\nabla_X Y, V) == -cos^2\theta g_1(\nabla_X Y, PV) - g_1(\mathcal{H}\nabla_X Y, \omega\psi PV)\\
+ g_1(\nabla_X BY, \omega V) + g_1(\nabla_X CY, \omega V)
\end{aligned}$$

as $\omega PV + \phi QV = \omega V$. From equations (8), (13), (18) and Lemma 2.3, we have

$$\begin{aligned}
&g_1(\nabla_X Y, V)\\
&= -cos^2\theta g_1(\nabla_X Y, PV) + g_1(\mathcal{A}_X BY, \omega V) + \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* Y, \pi_*(\omega\psi PV))\\
&- g_1(X, grad\ln\lambda) g_1(Y, \omega\psi PV) + g_1(Y, grad\ln\lambda) g_1(X, \omega\psi PV)\\
&- g_1(X, Y) g_1(grad\ln\lambda, \omega\psi PV) + \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* CY, \pi_*(\omega V))\\
&+ g_1(CY, grad\ln\lambda) g_1(X, \omega V) - g_1(X, CY) g_1(grad\ln\lambda, \omega V).
\end{aligned}$$

This proves the theorem completely. □

Theorem 3.11. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant submersion from a Kenmotsu manifold M onto a Riemannian manifold N. Then $(\ker \pi_*)$ defines totally geodesic on M if and only if*

$$\begin{aligned} &- cos^2\theta g_1(\nabla_U PV, X) - g_1(\mathcal{T}_U \omega V, BX) \\ &= \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{\omega V} \pi_* \phi CX, \pi_*(\omega V)) \\ &- \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{U} \pi_* \omega\psi PV, \pi_*(X)) \\ &- \frac{1}{\lambda^2} g_2((\nabla \pi_*)(U, \omega\psi PV), \pi_*(X)) \\ &- \frac{1}{\lambda^2} g_2((\nabla \pi_*)(\omega V, \phi CX), \pi_*(\omega V)) \\ &+ g_1(\mathcal{A}_{\omega V} \phi CX, \phi U) + \eta(QV) g_1(U, BX) \end{aligned} \tag{35}$$

for any $X \in \Gamma(\ker \pi_*)^{\perp}$ and $U, V \in \Gamma(\ker \pi_*)$.

Proof. On taking $X \in \Gamma(\ker \pi_*)^{\perp}$ and $U, V \in \Gamma(\ker \pi_*)$ with using decomposition (20), equations (2), (4), (21) and Theorem 3.1 we have

$$\begin{aligned} g_1(\nabla_U V, X) &= g_1(\nabla_U PV, X) + g_1(\nabla_U QV, X) + \eta(V) g_1(\nabla_U \xi, X) \\ &= g_1(\phi \nabla_U PV, \phi X) + g_1(\phi \nabla_U QV, \phi X) \\ &= cos^2\theta g_1(\nabla_U PV, X) - g_1(\nabla_\omega \psi PV, X) \\ &\quad + g_1(\nabla_U \omega PV, \phi X) + g_1(\nabla_U \phi QV, \phi X). \end{aligned}$$

On using equations (12), (14) and (22), we have

$$\begin{aligned} &g_1(\nabla_U V, X) \\ &= cos^2\theta g_1(\nabla_U PV, X) - g_1(\mathcal{H} \nabla_U \omega\psi PV, X) + g_1(\mathcal{T}_U \omega V, BX) \\ &\quad + g_1(\mathcal{H} \nabla_{\omega V} \phi CX, \omega U) + g_1(\mathcal{A}_{\omega V} \phi CX, \phi U) + \eta(QV) g_1(U, BX). \end{aligned}$$

Furthermore. using equation (8) and (18), we have

$$\begin{aligned} &g_1(\nabla_U V, X) \\ &= cos^2\theta g_1(\nabla_U PV, X) + g_1(\mathcal{T}_U \omega V, BX) + \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{\omega V} \pi_* \phi CX, \pi_*(\omega V)) \\ &- \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{U} \pi_*(\omega\psi PV), \pi_(*X)) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(U, \omega\psi PV), \pi_*(X)) \\ &- \frac{1}{\lambda^2} g_2((\nabla \pi_*)(\omega V, \phi CX), \pi_*(\omega V)) + g_1(\mathcal{A}_{\omega V} \phi CX, \phi U) + \eta(QV) g_1(U, BX). \end{aligned}$$

This proves the theorem completely. □

Theorem 3.12. *Let $\pi : (M, \phi, \xi, \eta, g_1) \to (N, g_2)$ be conformal hemi slant submersion from a Kenmotsu manifold M to Riemannian manifold N. Then* *anti-invariant distribution $\bar{D}$ defines totally geodesic on M if and only if*

$$g_1(\mathcal{T}_W\bar{W},\omega\psi Z)-\eta(V)g_1(\phi W,\omega\psi Z)=\frac{1}{\lambda^2}g_2(\nabla^{\pi}_{W}\pi_*(\phi\bar{W}),\pi_*(\omega Z))-\frac{1}{\lambda^2}g_2((\nabla\pi_*)(W,\phi Y),\pi_*(\omega Z)) \tag{36}$$

and

$$\begin{aligned}g_1(\nabla_W\phi\bar{W},BX)&=\frac{1}{\lambda^2}g_2(\nabla^{\pi}_{\phi\bar{W}}\pi_*(\phi CX),\pi_*(\phi W))\\&-\frac{1}{\lambda^2}[g_2((\phi\bar{W}\ln\lambda)\pi_*(\phi CX)+(\phi CX\ln\lambda)\pi_*(\phi\bar{W})\\&-g_1(\phi\bar{W},\phi CV)\pi_*(grad\ln\lambda),\pi_*(\phi W))]\end{aligned}$$

for any $W,\bar{W}\in\Gamma(\bar{D}),Z\in\Gamma(D)$ and $X\in\Gamma(\ker\pi_*)^{\perp}$.

Proof. Taking $W,\bar{W}\in\Gamma(\bar{D}),Z\in\Gamma(D)$ and $X\in\Gamma(\ker\pi_*)^{\perp}$ and from equations (2), (3), (4), and (7), (21), we have

$$\begin{aligned}g_1(\nabla_W\bar{W},Z)&=g_1(\nabla_W\bar{W},\psi^2Z)-\eta(\bar{W})g_1(\nabla_W\xi,\psi^2Z)+g_1(\nabla_W\bar{W},\omega Z)\\&-\eta(\bar{W})g_1(\nabla_W\xi,\omega\psi Z)+g_1(\nabla_W\phi Y,\omega Z).\end{aligned}$$

By using Theorem 3.1 and from equation (11), we arrived at

$$\begin{aligned}g_1(\nabla_W\bar{W},Z)&=cos^2\theta g_1(\nabla_W\bar{W},Z)-g_1(\mathcal{V}\nabla_W\bar{W},\omega\psi Z)-g_1(\mathcal{T}_W\bar{W},\omega\psi Z)\\&+\eta(V)g_1(\phi W,\omega\psi Z)+g_1(\nabla_W\phi\bar{W},\omega Z).\end{aligned}$$

Considering equations (8) and (12), we have

$$\begin{aligned}sin^2\theta g_1(\nabla_W\bar{W},Z)&=-g_1(\mathcal{T}_W\bar{W},\omega\psi Z)+\frac{1}{\lambda^2}g_2(\nabla_W\pi_*(\phi\bar{W}),\pi_*(\omega Z))\\&-\frac{1}{\lambda^2}g_2((\nabla\pi_*)(W,\phi\bar{W}),\pi_*(\omega Z))+\eta(V)g_1(\phi W,\omega\psi Z).\end{aligned}$$

On the other hand from (2), (7) and (22), we have

$$\begin{aligned}g_1(\nabla_W\bar{W},X)&=g_1(\phi\nabla_W\bar{W},\phi X)\\&=g_1(\nabla_W\phi\bar{W},BX)-g_1(\nabla_{\phi\bar{W}}\phi CX,\phi W).\end{aligned}$$

From equations (8), (18) and Lemma 2.3, we obtained

$$\begin{aligned}&g_1(\nabla_W\bar{W},X)\\&=g_1(\nabla_W\phi\bar{W},BX)+\frac{1}{\lambda^2}g_2((\nabla\pi_*)(\phi\bar{W},\phi CX),\pi_*(\phi X))\\&-\frac{1}{\lambda^2}g_2(\nabla_{\phi\bar{W}}\pi_*(\phi CX),\pi_*(\phi W))\\&=g_1(\nabla_W\phi\bar{W},BX)-\frac{1}{\lambda^2}g_2(\nabla_{\phi\bar{W}}\pi_*(\phi CX),\pi_*(\phi W))\\&+\frac{1}{\lambda^2}[g_2((\phi\bar{W}\ln\lambda)\pi_*(\phi CX)+(\phi CX\ln\lambda)\pi_*(\phi\bar{W})\\&-g_1(\phi\bar{W},\phi CX)\pi_*(grad\ln\lambda),\pi_*(\phi W))].\end{aligned}$$

From the above equations, we get the desired result. □

Theorem 3.13. *Let π be a conformal hemi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (N, g_2). Then π is a totally geodesic map on M if and only if*

(a)

$$\frac{1}{\lambda^2} g_2(\nabla^{\pi}_{U} \pi_* \omega\psi V, \pi_*(X)) + \frac{1}{\lambda^2} g_2((\nabla\pi_*)(U, \omega\psi V), \pi_*(X)) + \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{\omega V} \pi_* \phi CX, \pi_*(\omega U)) = -g_1(A_{\omega V} \phi CX, \psi U) - \frac{1}{\lambda^2} g_2(\omega V(ln\lambda)\pi_*\phi CZ + \phi CX(ln\lambda)\pi_*(\omega V) + g_1(\omega V, \phi CX)\pi_*(gradln\lambda), \pi_*(\omega U))$$

(b)

$$\frac{1}{\lambda^2} g_2(Z, W) g_1(grad(ln\lambda), \pi_*(\phi CX)) - \frac{1}{\lambda^2} g_2(\nabla^{\pi}_{\phi W} \pi_* \phi CX, \pi_*(\phi Z)) = -g_1(\mathcal{T}_Z \phi W, BX)$$

(c)

$$\begin{aligned} &-\frac{1}{\lambda^2} g_2(\nabla^{\pi}_{CY} \pi_*(\phi CX), \omega U_1) \\ &= g_1(\mathcal{T}_{U_1} \psi BY, U) - g_1(\mathcal{T}_{U_1} CY, BX) \\ &+ g_1(\mathcal{H}\nabla_{U_1} \omega BY, X) + g_1(\mathcal{A}_{CY} \phi CX, \psi U_1) \\ &- g_1(CY, grad \ln\lambda) g_1(\omega U_1, \phi CX) \\ &- g_1(\phi CX, grad \ln\lambda) g_1(CY, \omega U_1) \\ &+ g_1(CY, \phi CX) g_1(gradln\lambda, \omega U_1) \end{aligned}$$

for any $U, V \in \Gamma(D), Z, W \in \Gamma(\bar{D})$ and $X, Y \in \Gamma(\ker\pi_*)^{\perp}$ with $U_1 \in \Gamma(\ker\pi_*)$.

Proof. (a) For any $U, V \in \Gamma(D)$, and $X \in \Gamma(\ker\pi_*)^{\perp}$ and using equations (8) and (18), we have

$$\begin{aligned} g_2((\nabla\pi_*)(U, V), \pi_* X) &= g_1(\pi_*(\nabla_U V), \pi_* X) \\ &= \lambda^2 g_1(\nabla_U V, X). \end{aligned}$$

From equations (2) and (21), we obtained

$$\begin{aligned} \frac{1}{\lambda^2} g_2((\nabla\pi_*)(U, V), \pi_* X) &= g_M(\nabla_U V, X) \\ &= -g_1(\big(\nabla_U \psi^2 V, Z\big) - g_1(\nabla_U \omega\psi V, X) + g_1((\nabla_U \phi)\psi V, X) \\ &+ g_1(\nabla_U \omega V, \phi X) - g_1(g(U, V)\xi - \eta(V)U, \phi X) + g_1(\phi U, V)\eta(X). \end{aligned}$$

By using Theorem 3.1 and from equations (2), (4) and (22), we have

$$\begin{aligned}&g_1((\nabla\pi_*)(U,V),\pi_*X)\\&=-cos^2\theta g_1(\nabla_U V,X)-g_1(\nabla_U\omega\psi V,X)-g_1(g(\nabla_{\omega V}\phi)BX,\phi U)\\&\ +g_1((\nabla_{\omega V}\phi)BX,\phi U)-g_1(\nabla_{\omega V}\phi CX,\phi U)+g_1(\nabla_{\omega V}\phi CX,\phi U).\end{aligned}$$

By using equations (12), (18) and Lemma (2.3), we got the result as

$$\begin{aligned}&sin^2\theta\frac{1}{\lambda^2}g_2((\nabla\pi_*)(U,V),\pi_*(X))\\&=-g_1(\mathcal{H}\nabla_U\omega\psi V,X)+g_1(\nabla_{\omega V}\phi BX,\phi U)+g_1(\nabla_{\omega V}\phi CX,\omega U)\\&\ +g_1(\nabla_{\omega V}\phi CX,\psi U)\\&=-\frac{1}{\lambda^2}g_2(\nabla^\pi_U\pi_*\omega\psi V,\pi_*(X))-\frac{1}{\lambda^2}g_2(\nabla\pi_*)(U,\omega\psi V),\pi_*X)\\&\ -g_1(A_{\omega U}\phi CX,\psi U)-\frac{1}{\lambda^2}g_2(\nabla^\pi_{\omega U}\pi_*\phi CX,\pi_*(\omega UV))\\&\ -\frac{1}{\lambda^2}g_2(\omega U(ln\lambda)\pi_*\phi CX+\phi CX(ln\lambda)\pi_*\omega U\\&\ -g_2(\omega U,\phi CX)\pi_*(gradln\lambda),\pi_*(\omega U)).\end{aligned}$$

(b) Take $Z,W\in\Gamma(\bar{D})$ and $X\in\Gamma(\ker\pi_*^\perp)$ and using equation (8) and equation (18), we have

$$\begin{aligned}g_2((\nabla\pi_*)(Z,W),\pi_*(X))&=g_1(\pi_*(\nabla_Z W),\pi_*(X))\\&=\lambda^2 g_1(\nabla_Z W,X)\end{aligned}$$

Now from (2), (4) and (22),we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z,W),\pi_*(X))&=g_1(\pi_*(\nabla_Z W),\pi_*(X))\\&=g_1(\nabla_Z\phi W,BX)-g_1(\nabla_{\phi W}CX,Z).\end{aligned}$$

By using equations (12),(1) and (4), we have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\pi_*)(Z,W),\pi_*(X))\\&=g_1(\mathcal{T}_Z\phi W,BX)+\eta(Z)\eta(\nabla_{\phi W}CX)-g_1(\phi\nabla_{\phi W}CX,\phi Z)\\&=g_1(\mathcal{T}_Z\phi W,BX)-g_1(\nabla_{\phi W}\phi CX,\phi Z).\end{aligned}$$

From equations (8), (18) and using definition of Conformal submersion from Lemma 2.3, we got

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z,W), \pi_*(X)) \\
&= g_1(\mathcal{T}_Z \phi W, BX) - \frac{1}{\lambda^2} g_2(\pi_*(\nabla_{\phi W} \phi CX), \pi_*(\phi Z)) \\
&= g_1(\mathcal{T}_Z \phi W, BX) + \frac{1}{\lambda^2} g_2(Z,W) g_1(grad(ln\lambda), \pi_*(\phi CX)) \\
&- \frac{1}{\lambda^2} g_2(\nabla_{\phi W} \pi_* \phi CX, \pi_*(\phi Z)).
\end{aligned}$$

(c) Take $X, Y \in \Gamma(\ker \pi_*)^{\perp}, U_1 \in \Gamma(\ker \pi_*)$ and on using equations (8) and (18), we have

$$\begin{aligned}
g_2((\nabla \pi_*)(U_1, Y), \pi_*(X)) &= g_1(\nabla^{\pi}_{U_1} \pi_* Y - \pi_* \nabla_{U_1} Y, \pi_*(X)) \\
&= \lambda^2 g_1(\nabla_{U_1} Y, X).
\end{aligned}$$

From equations (2), (4) and equation (22), we have

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2((\nabla \pi_*)(U_1, Z), \pi_*(X)) \\
&= -g_1((\nabla_{U_1} \phi) BY, \phi X \; - g_1(\mathcal{H} \nabla_{U_1} CY + \mathcal{T}_{U_1} CY, BX + CX).
\end{aligned}$$

From equations (2), (3), (18) and Lemma 2.3, we have the final result

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2((\nabla \pi_*)(U_1, Z), \pi_* X) \\
&= -g_1(g(U_1, BY)\xi - \eta(BY) U_1, X) + g_1(\nabla_{U_1} \psi BY, X) \\
&+ g_1(\nabla_{U_1} \omega BY, X) - g_1(\mathcal{H} \nabla_{U_1} CY, CX) - g_1(\mathcal{T}_{U_1} CY, BX) \\
&= g_1(\mathcal{T}_{U_1} \psi BY, X) - g_1(\mathcal{T}_{U_1} CY, BX) + g_1(\mathcal{H} \nabla_{U_1} \omega BY, X) + g_1(\mathcal{A}_{CY} \phi CX, \psi U_1) \\
&+ \frac{1}{\lambda^2} g_2(\nabla_{CY} \pi_* \phi CX, \omega U_1) - g_1(CY, grad \ln \lambda) g_1(\omega U_1, \phi CX) \\
&- g_1(\phi CX, grad \ln \lambda) g_1(CY, \omega U_1) + g_1(CY, \phi CX) g_1(gradln\lambda, \omega U_1).
\end{aligned}$$

If π is horizontal homothetic, it follows that $(\nabla \pi_*)(X, Y) = 0$, for any $X, Y \in \Gamma(\mu)$. Conversely, if $(\nabla \pi_*)(X, Y) = 0$, taking $Y = \phi X$ in the above equation, we have

$$X(\ln \lambda) \pi_*(\phi X) + \phi X(\ln \lambda) \pi_*(X) = 0 \tag{37}$$

Taking inner product in above equation with $\pi_*(\phi X)$, we get

$$g_M(\mathcal{H} grad \ln \lambda, X) g_M(\phi X, \phi X) + g_M(\mathcal{H} grad \ln \lambda, \phi X) g_M(X, \phi X) = 0. \tag{38}$$

From the above equation λ is constant on $\Gamma(\mu)$. In a similar way $V_1, V_2 \in \Gamma(\ker\ \pi_*)$, using Lemma 2.3 (i) we have

$$\begin{aligned}
(\nabla \pi_*)(\omega V_1, \omega V_2) &= \psi V_1(\ln \lambda) \pi_*(\psi V_2) + \psi V_2(\ln \lambda) \pi_*(\psi V_1) \\
&- g_M(\psi V_1, \psi V_2) \phi_*(\mathcal{H} grad \ln \lambda).
\end{aligned} \tag{39}$$

Again π is horizontal homothetic, it follows that $(\nabla\pi_*)(\psi V_1, \psi V_2) = 0$. Conversely, if $(\nabla\pi_*)(\psi V_1, \psi V_2) = 0$, taking $V_1 = V_2$ in above equation. we obtained

$$2\psi V_1(\ln\lambda)\pi_*(\psi V_1) - g_M(\psi V_1, \psi V_1)\pi_*(\mathcal{H}grad\ln\lambda) = 0. \tag{40}$$

Taking inner product with Lemma (2.3) (i) with $\pi_*\psi V_1$ and since π is conformal submersion, we get

$$g_M(\psi V_1, \psi V_1)g_M(\mathcal{H}grad\ln\lambda) = 0. \tag{41}$$

From above equation, it follows that λ is constant on $\Gamma(\beta(\ker\ \pi_*))$. So λ is constant on $\Gamma(\ker\ \pi_*)^\perp$. On the other hand, if π is horizontally homothetic map, it follows that $(\nabla\pi_*)(X, Y) = 0$. This proves the theorem completely. □

4. Decomposition Theorems

In this section, we obtained decomposition theorems by using the existence of conformal hemi- slant submersions. First, we recall the following results from [24]. Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and assume that the canonical foliations D_{M_1} and D_{M_2} intersect perpendicularly everywhere. Then g is the metric tensor of

(i) a twisted product $M_1 \times_f M_2$ if and only if D_{M_1} is a totally geodesic foliation and D_{M_2} is a totally umbilic foliation,

(ii) a warped product $M_1 \times_f M_2$ if and only if D_{M_1} is a totally geodesic foliation and D_{M_2} is a spherics foliation, i.e., it is umbilic and its mean curvature vector field is parallel.

(iii) a usual product of Riemannian manifolds if and only if D_{M_1} and D_{M_2} are totally geodesic foliations.

Our first decomposition theorem for a conformal hemi slant submersion comes from Theorem (3.10) and Theorem (3.11) in terms of the second fundamental forms of such submersions.

Theorem 4.1. *Let π be a conformal hemi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_1)$ to a Riemannian manifold (N, g_2). Then M is a locally product manifold if and only if*

$$\begin{aligned}
&- cos^2\theta g_1(\nabla_U PV, X) - g_1(\mathcal{T}_U\omega V, BX)\\
&= \frac{1}{\lambda^2}g_2(\nabla^\pi_{\omega V}\pi_*\phi CX, \pi_*(\omega V)) - \frac{1}{\lambda^2}g_2(\nabla^\pi_U\pi_*\omega\psi PV, \pi_*(X))\\
&- \frac{1}{\lambda^2}g_2((\nabla\pi_*)(U, \omega\psi PV), \pi_*(X)) - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(\omega V, \phi CX), \pi_*(\omega V))\\
&+ g_1(\mathcal{A}_{\omega V}\phi CX, \phi U)
\end{aligned}$$

and

$$-\frac{1}{\lambda^2}g_2(\nabla^{\pi}_X\pi_*Y,\pi_*(\omega\psi PV))-\frac{1}{\lambda^2}g_2(\nabla^{\pi}_X\pi_*CY,\pi_*(\omega V))$$
$$=-cos^2\theta g_1(\nabla_XY,PV)+g_1(\mathcal{A}_XBY,\omega V))$$
$$-g_1(X,grad\ln\lambda)g_1(Y,\omega\psi PV)$$
$$+g_1(Y,grad\ln\lambda)g_1(X,\omega\psi PV)-g_1(X,Y)g_1(grad\ln\lambda,\omega\psi PV)$$
$$+g_1(CY,grad\ln\lambda)g_1(X,\omega V)-g_1(X,CY)g_1(grad\ln\lambda,\omega V)$$

for any $X,Y\in\Gamma(\ker\pi_*)^{\perp}$ *and* $U,V\in\Gamma(\ker\pi_*)$.

Theorem 4.2. *Let π be a conformal hemi-slant submersion from a Kenmotsu manifold (M,ϕ,ξ,η,g_1) to a Riemannian manifold (N,g_2). Then M is a locally twisted product manifold of the form $M_{(\ker\pi_*)}\times M_{(\ker\pi_*)^{\perp}}$ if and only if*

$$\begin{aligned}g_1(\mathcal{A}_XBY,\phi V)=&\frac{1}{\lambda^2}g_2(\mathcal{H}grad\ln\lambda,CY)g_1(\pi_*X,\pi_*(\phi V))\\&-\frac{1}{\lambda^2}g_2(X,CY)g_1(\pi_*grad\ln\lambda,\pi_*(\phi V))\\&-\frac{1}{\lambda^2}g_2(\nabla^{\pi}_X\pi_*CY,\pi_*(\phi V)).\end{aligned}\tag{42}$$

and

$$\begin{aligned}g_1(V,W)H=&B(\nabla_V\phi W)+\phi\pi_*(\nabla^{\pi}_{\phi W}\pi_*\phi V)\\&-B(Hgrad\ln\lambda)g_1(\phi V,\phi W).\end{aligned}\tag{43}$$

for any $V,W\in\Gamma(\ker\pi_*)$ *and* $X,Y\in\Gamma(\ker\pi_*)^{\perp}$

Proof. for $X,Y\in\Gamma(\ker\pi_*)^{\perp},V\in\Gamma(\ker\pi_*)$, we arrived at

$$\begin{aligned}g_1(\nabla_XY,V)=&g_1(\mathcal{A}_XBY+\mathcal{V}\nabla_XBY,\phi V)\\&+g_1(\mathcal{A}_XCY+\mathcal{H}\nabla_XCY,\phi V)-g_1(CX,Y)\end{aligned}$$

From above equation, we get

$$g_1(\nabla_XY,V)=g_1(\mathcal{A}_XBY,\phi V)+g_1(\mathcal{H}\nabla_XCY,\phi V)-g_1(CX,Y).$$

Since π is conformal hemi-slant submersion, on using equation (18) and Lemma 2.3, we arrived at

$$\begin{aligned}g_1(\nabla_XY,V)=&g_1(\mathcal{A}_XBY,\phi V)-\frac{1}{\lambda^2}g_2(\mathcal{H}grad\ln\lambda,X)g_1(\pi_*CY,\pi_*(\phi V))\\&-\frac{1}{\lambda^2}g_2(\mathcal{H}grad\ln\lambda,CY)g_1(\pi_*X,\pi_*(\phi V))\\&+\frac{1}{\lambda^2}g_2(X,CY)g_1(\pi_*grad\ln\lambda,\pi_*(\phi V))\\&+\frac{1}{\lambda^2}g_2(\nabla^{\pi}_X\pi_*CY,\pi_*(\phi V))-g_1(CX,Y).\end{aligned}$$

Moreover using the fact that $g_1(CX, \phi V) = 0$, for $X \in \Gamma(\ker \pi_*)^\perp, V \in \Gamma(\ker \pi_*)$, we arrived at

$$\begin{aligned} g_1(\nabla_X Y, V) = g_1(\mathcal{A}_X BY, \phi V) & \\ & - \frac{1}{\lambda^2} g_2(\mathcal{H} grad \ln \lambda, CY) g_1(\pi_* X, \pi_*(\phi V)) \\ & + \frac{1}{\lambda^2} g_2(X, CY) g_1(\pi_* grad \ln \lambda, \pi_*(\phi V)) \\ & + \frac{1}{\lambda^2} g_2(\nabla^\pi_X \pi_* CY, \pi_*(\phi V)). \end{aligned}$$

It follows that $M_{(\ker \pi_*)}$ is totally geodesic if and only if the equation (42) is satisfied. On the other hand, For any $V, W \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^\perp$, we have

$$g_1(\nabla_V W, X) = g_1(\mathcal{H} \nabla_V \phi W, CX) + g_1(\mathcal{T}_V \phi W, BX).$$

Since ∇ is torsion free, $[V, \phi W] \in \Gamma(\ker \pi_*)$, we have

$$g_1(\nabla_V W, X) = g_1(\nabla_V \phi W, BX) + g_1(\nabla_{\phi W} \phi V, \phi CX).$$

Since π is conformal hemi-slant submersion, by using Lemma 2.3 and from the fact that $g_1(CX, \phi V) = 0$ for $X \in (\ker \pi_*)^\perp$ and $V \in (\ker \pi_*)$, we have

$$\begin{aligned} g_1(\nabla_V W, X) = g_1(\nabla_V \phi W, BX) + \frac{1}{\lambda^2} g_2(\nabla^\pi_{\phi W} \pi_* \phi V, \pi_*(\phi CX)) \\ - g_1(\phi V, \phi W) g_1(grad \ln \lambda, \pi_*(\phi CV)). \end{aligned}$$

From the above result, we conclude that $M_{(\ker \pi_*)^\perp}$ is totally umbilical if and only if the equation (43) satisfied. □

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