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ANALYSIS OF AN EXTENDED WHITTAKER FUNCTION AND ITS PROPERTIES

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Abstract. For the numerous uses and significance of the Whittaker function in the diverse research areas of mathematical sciences and engineering sciences, This paper aims to introduce an extension of the Whittaker function by using a new extended confluent hypergeometric function of the first kind in terms of the Mittag-Leffler function. We also drive various valuable results like integral representation, integral transform and derivative formula. Further, we also analyze specific known results as a particular case of the main result.

1. Introduction

The Whittaker function is a solution of the Whittaker equation which is a modified version of a confluent hypergeometric function of the first class introduced by E.T. Whittaker (see [22]), and it has numerous applications in many areas such as mathematical physics and many research areas that are investigated by various mathematicians (see [1, 7, 14, 23]). Many authors give an extension and generalization of several special functions like Beta function, Gamma function, Gauss hypergeometric function, confluent hypergeometric function and Whittaker function (see [1, 3, 4, 7, 8, 9, 10, 11, 14, 18, 22, 21, 17]). Recently, N.U. Khan et al. (see [9]) investigated a new form of Beta function using an extended Mittag-Leffler function and its properties and different applications in statistical sciences. Motivated by the above mentioned works, in this paper, we introduced a new generalization of Whittaker function in terms of extended Mittag-Leffler function by using extended confluent hypergeometric function.
The symbol $\mathbb{R}$ is used all over the paper to denote the set of all real numbers, $\mathbb{N}$ has been used to denote the set of natural numbers, $\mathbb{C}$ can be used to denote the set of complex numbers, $\mathbb{Z}$ is used to denote the set of integers numbers

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and $\mathbb{N}_0$ is being used to denote the set of all positive integers. For our present study, we recall here the following basic definition and their generalization as follows:

The classical beta function $B(u,v)$ is defined as (see [2])

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \tag{1}$$

where

$$(\Re(u)>0,\ \Re(v)>0).$$

The classical Gauss hypergeometric function $F(u,v;w;\omega)$ and confluent hypergeometric function $\Phi(v;w;\omega)$ are defined as (see [16, 20])

$$F\,(u,v;w;\omega)=\frac{1}{B(v,w-v)}\int_0^1 t^{v-1}(1-t)^{w-v-1}(1-\omega t)^{-u}\,dt \tag{2}$$

$$(\ |\arg(1-\omega)|<\pi;\ \Re(w)>\Re(v)>0\ ),$$

$$\Phi\,(v;w;\omega)=\frac{1}{B(v,w-v)}\int_0^1 t^{v-1}(1-t)^{w-v-1}\,exp\,(\omega t)\,dt \tag{3}$$

$$(\ \Re(w)>\Re(v)>0\ ).$$

By using the series expansion of $(1-\omega t)^{-u}$ and $\exp(\omega t)$ in (2) and (3) respectively, the hypergeometric and confluent hypergeometric functions are written in terms of beta function as

$$F\,(u,v;w;\omega)=\sum_{n=0}^{\infty}(u)_n\,\frac{B(v+n,w-v)}{B(v,w-v)}\,\frac{\omega^n}{n!} \tag{4}$$

$$(\ |\omega|<1,\ \Re(w)>\Re(v)>0),$$

and

$$\Phi\,(v;w;\omega)=\sum_{n=0}^{\infty}\frac{B(v+n,w-v)}{B(v,w-v)}\,\frac{\omega^n}{n!} \tag{5}$$

$$(\ \Re(w)>\Re(v)>0).$$

In 1997, Chaudhary et al. (see [3]) give an extension of beta function defined as

$$B_\rho(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\exp\left(-\frac{\rho}{t(1-t)}\right)dt, \tag{6}$$

where

$$(\Re(\rho) > 0,\, \Re(u) > 0,\, \Re(v) > 0).$$

Remark 1.1. *If $\rho = 0$, then extended beta function* (6) *reduced to a familiar classical beta function* (1).

In 2004, Chaudhary et al. (see [4]) introduced the extended hypergeometric and confluent hypergeomeric functions by using extended beta function (6) as follow:

$$F_\rho\,(u,v;w;\omega) = \sum_{n=0}^{\infty} (u)_n\, \frac{B_\rho(v+n, w-v)}{B(v, w-v)}\, \frac{\omega^n}{n!} \tag{7}$$

$$(\rho \geq 0,\; |\omega| < 1,\; \Re(w) > \Re(v) > 0),$$

and

$$\Phi_\rho\,(v;w;\omega) = \sum_{n=0}^{\infty} \frac{B_\rho(v+n, w-v)}{B(v,w-v)}\, \frac{\omega^n}{n!} \tag{8}$$

$$(\rho \geq 0,\; \Re(w) > \Re(v) > 0),$$

and their integral representation:

(9)

$$F_\rho\,(u,v;w;\omega) = \frac{1}{B(v,w-v)} \int_0^1 t^{v-1}\,(1-t)^{w-v-1}\,(1-\omega t)^{-u}\, exp\left(-\frac{\rho}{t(1-t)}\right) dt$$

$$(\rho > 0;\; \rho = 0 \;\; and \;\; |\arg(1-\omega)| < \pi;\; \Re(w) > \Re(v) > 0),$$

$$\Phi_\rho\,(v;w;\omega) = \frac{1}{B(v,w-v)} \int_0^1 t^{v-1}\,(1-t)^{w-v-1}\, exp\left(\omega t - \frac{\rho}{t(1-t)}\right) dt \tag{10}$$

$$(\rho > 0;\; \rho = 0 \;\; and \;\; \Re(w) > \Re(v) > 0).$$

Shadab et al. (see [18]) introduced an extension of beta function using generalized Mittag-Leffler function as follow:

$$B_\alpha^\rho(u,v) = \int_0^1 t^{u-1}\,(1-t)^{v-1}\, E_\alpha\left(-\frac{\rho}{t(1-t)}\right) dt \tag{11}$$

$$(\Re(u) > 0,\, \Re(v) > 0, \Re(\rho) > 0\,;\, \alpha \in \mathbb{R}_0^+),$$

where $E_\alpha(.)$ is the classical Mittag-Leffler [12, 13] function defined by

$$E_\alpha\,(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + 1)},$$

where

$$(\omega \in \mathbb{C}, \alpha \in \mathbb{R}_0^+).$$

In 2018, Shadab et al. [18] expressed the extended hypergeometric and confluent hypergeometric functions in terms of extended beta function (11) is as follow:

$$F_{\rho,\alpha}(u,v;w;\omega) = \sum_{n=0}^{\infty} (u)_n \, \frac{B_\alpha^\rho(v+n, w-v)}{B(v,w-v)} \, \frac{\omega^n}{n!} \tag{12}$$

$$(\alpha \in \mathbb{R}^+, \ \rho \in \mathbb{R}_0^+, \ |\omega| < 1, \ \Re(w) > \Re(v) > 0),$$

and

$$\Phi_{\rho,\alpha}(v;w;\omega) = \sum_{n=0}^{\infty} \frac{B_\alpha^\rho(v+n, w-v)}{B(v,w-v)} \, \frac{\omega^n}{n!} \tag{13}$$

$$(\alpha \in \mathbb{R}^+, \ \rho \in \mathbb{R}_0^+, \ |\omega| < 1, \ \Re(w) > \Re(v) > 0),$$

and their integral representation:

$$F_{\rho,\alpha}(u,v;w;\omega) = \frac{1}{B(v,w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-\omega t)^{-u} E_\alpha \left(-\frac{\rho}{t(1-t)} \right) dt \tag{14}$$

$$(\alpha \in \mathbb{R}^+, \ \rho > 0; \ \rho = 0 \ \textit{and} \ |\arg(1-\omega)| < \pi; \ \Re(w) > \Re(v) > 0),$$

$$\Phi_{\rho,\alpha}(v;w;\omega) = \frac{1}{B(v,w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} E_\alpha \left(\omega t - \frac{\rho}{t(1-t)} \right) dt \tag{15}$$

$$(\alpha \in \mathbb{R}^+, \ \rho > 0; \rho = 0 \ \textit{and} \ \Re(w) > \Re(v) > 0).$$

Recently, N.U. Khan et al. [9] introduced an extension of beta function using generalized Mittag-Leffler function as follow:

$$B_{\alpha,\beta}^{\rho,\eta,\nu}(u,v) = \int_0^1 t^{u-1} (1-t)^{v-1} E_{\alpha,\beta} \left(-\frac{\rho}{t^\eta (1-t)^\nu} \right) dt \tag{16}$$

$$(\Re(u) > 0, \, \Re(v) > 0; \alpha, \beta \in \mathbb{R}_0^+; \, \eta, \nu \in \mathbb{R}^+, \, \rho \geq 0),$$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function define as

$$E_{\alpha,\beta}(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + \beta)},$$

where

$$(\alpha, \beta \in \mathbb{R}_0^+, \omega \in \mathbb{C}).$$

By using (16) we further extended the Gauss hypergeometric and confluent hypergeometric functions (12), (13) and their integral representation (14), (15) are defined as:

$$F^{\rho,\eta,\nu}_{\alpha,\beta}\,(u,v;w;\omega) = \sum_{n=0}^{\infty} (u)_n \, \frac{B^{\rho,\eta,\nu}_{\alpha,\beta}(v+n,w-v)}{B(v,w-v)} \, \frac{\omega^n}{n!} \tag{17}$$

$$(\rho \geq 0,\ |\omega| < 1,\ \alpha,\beta,\eta,\nu \in \mathbb{R}^+,\ \Re(w) > \Re(v) > 0),$$

$$\Phi^{\rho,\eta,\nu}_{\alpha,\beta}\,(v;w;\omega) = \sum_{n=0}^{\infty} \frac{B^{\rho,\eta,\nu}_{\alpha,\beta}(v+n,w-v)}{B(v,w-v)} \, \frac{\omega^n}{n!} \tag{18}$$

$$(\rho \geq 0,\ \alpha,\beta,\eta,\nu \in \mathbb{R}^+,\ \Re(w) > \Re(v) > 0).$$

(19)

$$\begin{aligned} F^{\rho,\eta,\nu}_{\alpha,\beta}\,(u,v;w;\omega) =& \frac{1}{B(v,w-v)} \\ & \times \int_0^1 t^{v-1}\,(1-t)^{w-v-1}\,(1-\omega t)^{-u}\,E_{\alpha,\beta}\left(-\frac{\rho}{t^{\eta}(1-t)^{\nu}}\right) dt \end{aligned}$$

$$(\rho \in \mathbb{R}_0^+, \alpha,\beta,\eta,\nu \in \mathbb{R}^+ \ and \ |\arg(1-\omega)| < \pi;\ \Re(w) > \Re(v) > 0),$$

and

$$\begin{aligned} \Phi^{\rho,\eta,\nu}_{\alpha,\beta}\,(v;w;\omega) =& \frac{1}{B(v,w-v)} \\ & \times \int_0^1 t^{v-1}\,(1-t)^{w-v-1}\,e^{\omega t}\,E_{\alpha,\beta}\left(-\frac{\rho}{t^{\eta}(1-t)^{\nu}}\right) dt \end{aligned} \tag{20}$$

$$(\rho \in \mathbb{R}_0^+, \alpha,\beta,\eta,\nu \in \mathbb{R}^+ \ and \ \Re(w) > \Re(v) > 0).$$

The extension of Kummer's relation to the generalized extended confluent hypergeometric function of the first kind as follows:

$$\Phi^{\rho,\eta,\nu}_{\alpha,\beta}\,(v,w;\,\omega) = e^{\omega}\,\Phi^{\rho,\eta,\nu}_{\alpha,\beta}\,(w-v;w;\,-\omega)\,. \tag{21}$$

For $\alpha = \beta = \eta = \nu = 1$ and $\rho = 0$, (21) reduce to the Kummer's first kind formula for the classical confluent hypergeometric function [16].

The Whittaker function $M_{\kappa,\zeta}(\omega)$ in terms of confluent hypergeometric function of first kind [22, 23] define as:

$$M_{\kappa,\zeta}(\omega) = \omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\Phi(\zeta-\kappa+\frac{1}{2};2\zeta+1;\omega), \tag{22}$$

$$\left(\Re(\zeta) > \frac{-1}{2}\ \ and\ \ \Re(\zeta\pm\kappa) > \frac{-1}{2}\right).$$

In 2013, Nagar et al. [14] generalized the Whittaker function by using extended confluent hypergeometric function Φ_ρ define as:

$$M_{\rho,\kappa,\zeta}(\omega) = \omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\Phi_\rho(\zeta-\kappa+\frac{1}{2};2\zeta+1;\omega), \tag{23}$$

$$\left(\Re(\zeta) > \frac{-1}{2}\ \ and\ \ \Re(\zeta\pm\kappa) > \frac{-1}{2}\right).$$

Motivated by the research article (see [9]) introduced by N.U. Khan et al. an extended Beta function involving extended Mittag-Leffler function utilizing this result. This paper investigates the extended Whittaker function for its numerous applications in various research fields in the following ways. Section 2 discusses extended Whittaker functions, integral representation, and special cases. Section 3 looked at their integral transforms, such as the Mellin transform and the Hankel transform. Section 4 looks at the n^{th} derivative formula of an extended Whittaker function.

2. Extended Whittaker function

In this section, we give a new generalization of Whittaker function of the first kind by applying the extended confluent hypergeometric function (18) defined as:

$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega) = \omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\Phi^{\rho,\eta,\nu}_{\alpha,\beta}(\zeta-\kappa+\frac{1}{2};2\zeta+1;\omega) \tag{24}$$

$$\left(\rho\geq 0,\ \alpha,\beta,\eta,\nu\in\mathbb{R}^+,\ \Re(\zeta) > \frac{-1}{2}\ \ and\ \ \Re(\zeta\pm\kappa) > \frac{-1}{2}\right)$$

Remark 2.1. *We get some well-known results when we put different particular values in* (24) *(see* [7, 14, 22, 23]*). It is mentioned in the following points:*

(*i*) *For* $\rho = 0,\ \alpha=\beta=\eta=\nu=1$, (24) *reduced to the classical Whittaker function (see* [22, 23]*).*

$$M^{0,1,1}_{1,1,\kappa,\zeta}(\omega) = M_{\kappa,\zeta}(\omega)$$

(*ii*) *For* $\alpha = \beta = \nu = 1$, (24) *reduced to the generalized Whittaker function* [7].

$$M^{\rho,\eta,1}_{1,1,\kappa,\zeta}(\omega) = M^{\eta}_{\rho,\kappa,\zeta}(\omega)$$

(*iii*) *For* $\alpha = \beta = \eta = \nu = 1$, (24) *reduced to the generalized Whittaker function* [14].

$$M^{\rho,1,1}_{1,1,\kappa,\zeta}(\omega) = M_{\rho,\kappa,\zeta}(\omega)$$

2.1. Integral Representation

We define integral representation of the new extended Whittaker function by using (20) and (21) as:

(25)
$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega) = \frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_0^1 t^{\zeta-\kappa+\frac{1}{2}}\,(1-t)^{\zeta+\kappa-\frac{1}{2}}\,e^{\omega t}\,E_{\alpha,\beta}\left(-\frac{\rho}{t^{\eta}(1-t)^{\nu}}\right)dt$$

$$\left(\rho \geq 0,\ \alpha,\beta,\eta,\nu \in \mathbb{R}^{+},\ \Re(\zeta) > \frac{-1}{2}\ \ and\ \ \Re(\zeta \pm \kappa) > \frac{-1}{2}\right).$$

Substituting $t = \frac{x-a}{b-a}$ in (25), we get the generalized extended Whittaker function as:

(26)
$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega) = \frac{(b-a)^{-2\zeta}\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_a^b (x-a)^{\zeta-\kappa-\frac{1}{2}}\,(b-x)^{\zeta+\kappa-\frac{1}{2}}\,e^{\omega\left(\frac{x-a}{b-a}\right)}\,E_{\alpha,\beta}\left(-\frac{\rho(b-a)^{\eta+\nu}}{(x-a)^{\eta}(b-x)^{\nu}}\right)dx,$$

where a and b are scalars such that $(b-a) > 0$.

If we take $a = -1$ *and* $b = 1$ in (26), we obtain another representation of generalized extended Whittaker function:

(27)
$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega) = \frac{2^{-2\zeta}\omega^{\zeta+\frac{1}{2}}}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_a^b (x+1)^{\zeta-\kappa-\frac{1}{2}}\,(1-x)^{\zeta+\kappa-\frac{1}{2}}\,e^{\left(\frac{\omega x}{2}\right)}\,E_{\alpha,\beta}\left(-\frac{\rho(2)^{\eta+\nu}}{(x+1)^{\eta}(1-x)^{\nu}}\right)dx.$$

In (25) substitute $t = \frac{x}{1+x}$, we get another integral representation of generalized extended Whittaker function as

(28)
$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega) = \frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_0^\infty x^{\zeta-\kappa-\frac{1}{2}}\,(1+x)^{-(2\zeta+1)}\,e^{\frac{\omega x}{1+x}}\,E_{\alpha,\beta}\left(-\frac{\rho(1+x)^{\eta+\nu}}{x^\eta}\right)dx.$$

Remark 2.2. *Below we mention some special cases:*

(i) *If $\alpha=\beta=1$ and $\eta=\nu=m$ in (26), (27) and (28), we obtain integral representation of extended Whittaker function defined by N.U. Khan et al.* [7].

(ii) *If $\alpha=\beta=\eta=\nu=1$ in (26), (27) and (28), we obtain integral representation of extended Whittaker function defined by Nagar et al.* [14].

(iii) *If $\alpha=\beta=\eta=\nu=1$ and $\rho=0$ in (26), (27) and (28), we obtain integral representation of Classical Whittaker function.*

Theorem 2.3. *The following relation holds true:*

$$M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(-\omega) = (-1)^{\zeta+\frac{1}{2}} M^{\rho,\eta,\nu}_{\alpha,\beta,-\kappa,\zeta}(\omega). \quad (\rho \geq 0) \tag{29}$$

Proof. Using the relation (21) in (24), we have

$$M^{\rho,\eta,\zeta}_{\alpha,\beta,\kappa,\zeta}(-\omega) = (-1)^{\zeta+\frac{1}{2}}\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\Phi^{\rho,\eta,\zeta}_{\alpha,\beta}(\zeta-\kappa+\frac{1}{2};2\zeta+1;\omega)$$

Now, writing the right hand side of above representation by using (24), we get desired result. □

3. Integral Transform of $M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)$

In the following theorems, we deal with the integral transform of the extended Whittaker function, i.e. Mellin transform and Hankel transform. In addition, we investigate at many important results obtained from the theorems mentioned below:

Theorem 3.1. *The following Millen transform formula hold true:*

(30)
$$\int_0^\infty \rho^{s-1} M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)d\rho = \frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\Gamma(s)\Gamma(1-s)B\left(\zeta+\eta s-\kappa+\frac{1}{2},\zeta+\nu s+\kappa+\frac{1}{2}\right)}{\Gamma(\beta-s\alpha)B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times\phi\left(\zeta+\eta s-\kappa+\frac{1}{2};2\zeta+(\eta+\nu)s+1;\omega\right)$$

Proof. Using extended generalized Whittaker function in terms of extended confluent hypergeometric function (24), we get

$$\begin{aligned}\int_0^\infty \rho^{s-1} M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)d\rho =&\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)\\ &\times\int_0^\infty \rho^{s-1}\Phi^{\rho,\eta,\nu}_{\alpha,\beta}\left(\zeta-\kappa+\frac{1}{2};2\zeta+1;\,\omega\right)d\rho\end{aligned}\tag{31}$$

Again, applying extended confluent hypergeometric function (20), we get

$$\begin{aligned}=&\frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})}\\ &\times\int_0^\infty \rho^{s-1}\int_0^1 t^{\zeta-\kappa-\frac{1}{2}}\,(1-t)^{\zeta+\kappa-\frac{1}{2}}\,e^{\omega t}\,E_{\alpha,\beta}\left(-\frac{\rho}{t^\eta(1-t)^\nu}\right)dt\,d\rho,\end{aligned}$$

changing the order of integration, we get

$$\begin{aligned}=&\frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})}\\ &\times\int_0^1 t^{\zeta-\kappa-\frac{1}{2}}\,(1-t)^{\zeta+\kappa-\frac{1}{2}}\,e^{\omega t}\int_0^\infty \rho^{s-1}E_{\alpha,\beta}\left(-\frac{\rho}{t^\eta(1-t)^\nu}\right)d\rho\,dt.\end{aligned}\tag{32}$$

Substituting $u=\frac{\rho}{t^\eta(t-1)^\nu}$ in integral (32), we get

$$\begin{aligned}=&\frac{\omega^{\zeta+\frac{1}{2}}\ exp\left(\frac{-\omega}{2}\right)}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})}\\ &\times\int_0^1 t^{\zeta+\eta s-\kappa-\frac{1}{2}}\,(1-t)^{\zeta+\nu s+\kappa-\frac{1}{2}}\,e^{\omega t}\int_0^\infty u^{s-1}E_{\alpha,\beta}\left(-u\right)du\,dt\end{aligned}\tag{33}$$

we know well known result (see [6], p.102)

$$\int_0^\infty u^{s-1}E^\gamma_{\alpha,\beta}\left(-u\right)du\,dt=\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-s\alpha)}\tag{34}$$

By using (34) for $\gamma=1$ and definition of confluent hypergeometric function (3) in (33), we get the desired result. □

Corollary 3.2. *If we take* $\eta=\nu=a$ *in* (30) *we get the following special case:*

$$\int_0^\infty \rho^{s-1}M^{\rho,a}_{\alpha,\beta,\kappa,\zeta}(\omega)d\rho=\frac{\Gamma(s)\Gamma(1-s)B\left(\zeta+as-\kappa+\frac{1}{2},2(\zeta+as)+1\right)}{\Gamma(\beta-s\alpha)\omega^{as}\ B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})}\,M_{\kappa,\zeta+as}(\omega)$$

Theorem 3.3. *The following formula holds true:*

(35)

$$\int_0^\infty \omega^{a-1}\, e^{-b\omega}\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(c\omega)d\omega = \frac{c^{\zeta+\frac{1}{2}}\ \Gamma(a+\zeta+\frac{1}{2})}{\left(b+\frac{c}{2}\right)^{a+\zeta+\frac{1}{2}}} \times F^{\rho,\eta,\zeta}_{\alpha,\beta}(a+\zeta+\frac{1}{2},\zeta-\kappa+\frac{1}{2};2\zeta+1;\frac{2c}{2b+c})$$

$$\left(\rho \geq 0,\ 2b > c > 0,\ \Re(a+\zeta) > \frac{-1}{2}\right).$$

Proof. Using integral representation of $M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)$, we get

$$\int_0^\infty \omega^{a-1}\, e^{-b\omega}\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(c\omega)d\omega = \frac{1}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_0^\infty \omega^{a-1}\, e^{-b\omega}\, (c\omega)^{\zeta+\frac{1}{2}}\ exp\left(\frac{-c\omega}{2}\right) \times \int_0^1 t^{\zeta-\kappa-\frac{1}{2}}\,(1-t)^{\zeta+\kappa-\frac{1}{2}}\, e^{c\omega t}\, E_{\alpha,\beta}\left(-\frac{\rho}{t^\eta(1-t)^\nu}\right) dt\, d\omega$$

Now interchange the order of integration and using the definition of gamma function, we get

(36)

$$\int_0^\infty \omega^{a-1}\, e^{-b\omega}\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(c\omega)d\omega = \frac{c^{\zeta+\frac{1}{2}}\ \Gamma(a+\zeta+\frac{1}{2})}{\left(b+\frac{c}{2}\right)^{a+\zeta+\frac{1}{2}} B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})} \times \int_0^1 t^{\zeta-\kappa-\frac{1}{2}}\,(1-t)^{\zeta+\kappa-\frac{1}{2}} \left(1-\frac{2ct}{2b+c}\right)^{-(a+\zeta+\frac{1}{2})} \times E_{\alpha,\beta}\left(-\frac{\rho}{t^\eta(1-t)^\nu}\right) dt\, d\omega$$

Using (19) in (36), we get the desired result (35). □

Corollary 3.4. *If we take $a = c = 1$ in* (35)*, we get the following special cases:*

$$\int_0^\infty e^{-b\omega}\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)d\omega = \frac{2^{\zeta+\frac{3}{2}}\ \Gamma(\zeta+\frac{3}{2})}{(2b+1)^{\zeta+\frac{3}{2}}} \times F^{\rho,\eta,\nu}_{\alpha,\beta}\left(\zeta+\frac{3}{2},\zeta-\kappa+\frac{1}{2};2\zeta+1;\frac{2}{2b+1}\right)$$

Theorem 3.5. *The following Hankel transform formula hold true:*

(37)
$$\int_0^\infty \omega\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)\; J_m(a\omega)d\omega = \frac{\Gamma(\zeta+\kappa+\frac{5}{2})}{\left(a^2+\frac{1}{4}\right)^{\frac{\zeta}{2}+\frac{5}{4}}} \times \sum_{n=o}^{\infty} \frac{B^{\rho,\eta,\nu}_{\alpha,\beta}(\zeta-\kappa+n+\frac{1}{2},\zeta+\kappa+\frac{1}{2})\left(\zeta+m+\frac{5}{2}\right)_n}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})\left(a^2+\frac{1}{4}\right)^{\frac{n}{2}} n!} \times P^{-m}_{\zeta+n+\frac{3}{2}}\left(\frac{1}{\sqrt{4a^2+1}}\right)$$

$$\left(\Re(\zeta\pm\kappa) > \frac{-1}{2} \;\; and \;\; \Re(\zeta+m) > \frac{-5}{2}\right),$$

where $P^m_\zeta(z)$ is the Legendre function [19], p.34.

Proof. Using (24) and (18), expanding $M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)$ in terms of generating extended beta function an changing the order of integration and summation, we get

$$\int_0^\infty z\, M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)\; J_m(a\omega)d\omega = \sum_{n=o}^{\infty} \frac{B^{\rho,\eta,\nu}_{\alpha,\beta}(\zeta-\kappa+n+\frac{1}{2},\zeta+\kappa+\frac{1}{2})}{B(\zeta-\kappa+\frac{1}{2},\zeta+\kappa+\frac{1}{2})n!} \times \int_0^\infty \omega^{\zeta+n+\frac{3}{2}}\; e^{\frac{-\omega}{2}}\; J_m(a\omega)d\omega. \tag{38}$$

On using the know result [5] p.182(9):

$$\int_0^\infty e^{-pt}\; t^\zeta\, J_m(at)dt = \Gamma(\zeta+m+1)\, r^{-\zeta-1} P^{-m}_\zeta\left(\frac{p}{r}\right), \tag{39}$$

where $\Re(\zeta+m) > -1$, $r = (p^2+a^2)^{\frac{1}{2}}$ and $P^m_\zeta(z)$ is the Legendre function [19]. By using (39) in (38) and after some simplification, we get the desired result. □

Corollary 3.6. *If we take* $\alpha = \beta = \zeta = 1$ *in* (37)*, we obtain the transformation for extended Whittaker function defined by N.U. Khan at el.* [7].
If we take $\alpha = \beta = \zeta = \eta = 1$ *in* (37)*, we obtain the transformation for extended Whittaker function defined by Nagar at el.* [14].

4. Derivative of $M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)$

Theorem 4.1. *The following differential formula holds true:*

(40)
$$\frac{d^n}{d\omega^n}\left[e^{\omega/2}\omega^{-\zeta-\frac{1}{2}} M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)\right] = \frac{(\zeta-\kappa+\frac{1}{2})_n}{(2\zeta+1)_n}\, e^{\omega/2}\, \omega^{-\zeta-\frac{n}{2}-\frac{1}{2}}\; M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa-\frac{n}{2},\zeta+\frac{n}{2}}(\omega)$$

Proof. The n^{th} order derivative of generalized extended confluent hypergeometric function is given by [11].

$$\frac{d^n}{d\omega^n}\left[\Phi^{\rho,\eta,\nu}_{\alpha,\beta}(v;w;\omega)\right]=\frac{(v)_n}{(w)_n}\Phi^{\rho,\eta,\nu}_{\alpha,\beta}(v+n;w+n;\omega)$$

Now using the (24) on the left hand side of (40), we obtain

$$\begin{aligned}\frac{d^n}{d\omega^n}\left[e^{\omega/2}z^{-\zeta-\frac{1}{2}}\ M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa,\zeta}(\omega)\right]=&\frac{d^n}{d\omega^n}\left[\Phi^{\rho,\eta,\nu}_{\alpha,\beta}(\zeta-\kappa+\frac{1}{2};2\zeta+1;\omega)\right]\\ =&\frac{(\zeta-\kappa+\frac{1}{2})_n}{(2\zeta+1)_n}\\ &\times\Phi^{\rho,\eta,\nu}_{\alpha,\beta}\left[(\zeta+\frac{n}{2})-(\kappa-\frac{n}{2})+\frac{1}{2};2(\zeta+\frac{1}{2})+1;\omega\right]\\ =&\frac{(\zeta-\kappa+\frac{1}{2})_n}{(2\zeta+1)_n}\ e^{\omega/2}\ \omega^{-\zeta-\frac{n}{2}-\frac{1}{2}}\ M^{\rho,\eta,\nu}_{\alpha,\beta,\kappa-\frac{n}{2},\zeta+\frac{n}{2}}(\omega)\end{aligned}$$

we get the desired result (40). □

5. Conclusion

In the present investigation, we have attempted here to introduce generalized extended Whittaker function in terms of an extended confluent hypergeometric function using extended beta function. We have provided an elegant of the usual properties of Whittaker function such as integral representation, integral transform and derivative formula. It is noticed that most of the special functions of mathematical physics i.e. modified Bessel function, Laguerre and Hermite polynomial etc. can be express in terms of Whittaker function. Therefore numerous results involving extensions and generalization of the Whittaker function are capable of playing important roles in the theory of the special function of applied mathematics and mathematical physics.

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